15-251: Great Theoretical Ideas In Computer Science

Recitation 1

As you prepare to do the writeups next week, remember the 10 styles of proof that you should avoid (adapted from Luis Von Ahn's lecture notes):

- Proof by Stating Every Theorem in the Relevant Subject Area and Hoping for Partial Credit
- Proof by Example ("Here is a proof for $n = 2$. The general case is basically the same idea.")
- Proof by Obscurity (using enough cumbersome and complex notation that your proof is impossible to decipher)
- Proof by Lengthiness (especially powerful combined with Proof by Obscurity and Proof by Stating Every Theorem)
- Proof by Switcheroo ($p \implies q$ is true, and q is true, so p must be true, right?)
- Proof by "It is clear that..." ("Clearly, this is the worst case for our algorithm.")
- Proof by Generalization (This specific algorithm doesn't work to solve this problem, and it seems optimal, therefore no algorithm can solve it! - a particularly common and pernicious flawed proof)
- Proof by Missing Cases (Your induction hypothesis doesn't work for all cases, or your base cases are incomplete)
- Proof "by definition" ("By definition, our algorithm is $\mathrm{O}(n)$")
- Proof by OMGWTFBBQ (enough said.)

Note that the proofs on this handout are bad or incorrect. Do not use them as a template.

Clearly false

Prove or disprove: for $n \in \mathbb{N}^+$, any n people all have the same hair color.

Solution: We prove the claim via induction on n.
Base case ($n = 1$): trivial.
Induction hypothesis: Suppose that for some $n \in \mathbb{N}^+$, all groups of n people have the same hair color.
Induction step: Consider a group of $n+1$ people and take the entire group except one. By the IH, these n have the same hair color. Now take another n-sized subgroup and exclude a different person. Again, these n share a hair color. Now note that the two people who were excluded both have the same hair color as the $n - 1$ people who were picked twice - by transitivity, they must have the same hair color too. So all $n + 1$ people have the same hair color.

Arrangement of $\clubsuit$ and $\diamondsuit$

How many ways are there to arrange $c \geq 0\clubsuit$s and $d \geq 0\diamondsuit$s so that all $\clubsuit$s are consecutive?

Solution: You can have any number between 0 and d $\diamondsuit$s, then a string of $\clubsuit$s; then the remainder of the $\diamondsuit$s. Hence, there are $d + 1$ possibilities.

A useful identity

Prove $2^n > n$ for $n \geq 1$.

Solution:

$$\begin{aligned} F_n &= \text{“}2^n > n\text{”} \\ F_1 &= \text{“}2 > 1\text{”}\checkmark \\ F_n &\Rightarrow F_{n+1} \\ 2^{n+1} &= 2(2^n) > 2 \times n(\text{induction}) \geq n+1 \end{aligned}$$

because $n \geq 1$.

Therefore proved.

Induction

Prove $n^2 \geq n$ for all integers n.

Solution: We prove $F_n =$ “$n^2 \geq n$” by induction on n. The base case is $n = 0$: indeed, $0^2 \geq 0$. For the induction step, assume F_k holds for all k. We now show that F_{k+1} holds...

Chips in a circle

There is a circle of 15,251 chips, green on one side, red on the other. Initially, all show the green side. In one step, you may take any four consecutive chips and flip them. Is it possible to get all of the chips showing red?

Solution: No it is not possible. Let's assume for contradiction we converted all 15,251 chips to red. But this means in the very last step there must be 4 consecutive green chips and the remaining 15,247 must be red. Repeating this k times for $1 \leq k \leq 3812$, we get three consecutive red chips, with the rest green. But we started from all green, contradiction.

Sneaky structures (bonus problem)

Suppose that everyone in your recitation knows at least one other person in the recitation. We say that two students are *connected* if there exists a chain of students, each consecutive pair of which know each other, spanning between the two. For example, if Xavier knows Yvonne and Yvonne knows Zachary, then Xavier and Zachary are connected even if they don't know each other. Prove or disprove: every student is connected to every other student.

Solution: We prove the claim via induction on n, the number of students.
Base case ($n = 2$): Since every student knows at least one other, the two students must know each other and are therefore connected.
Induction hypothesis: Suppose for some k that this works for all groups of k students. Induction step: Consider $k + 1$. We know the k-people recitation is connected. The $(k + 1)$th person cannot know no one, so they are connected to at least one other person in the recitation, who, by the induction hypothesis, is connected to everyone else. Thus, the whole party is still connected.