15-251: Great Theoretical Ideas In Computer Science

Recitation 5

Gates has 3 floors

Show that any boolean function $f : \{0,1\}^n \to \{0,1\}$ (i.e. a function that takes in n input bits and outputs 1 bit) can be computed by a circuit of depth at most 3 (This means that the longest path from any input bit to the output should have at most 3 gates). Your gates may have any number of inputs. What is the size (in big-O) of such a circuit in the worst case?

Bounds on circuit size

Let $x_1, x_2, \ldots, x_n$ be input bits ($n \geq 2$). We use the convention that truth assignments are either 0 or 1. We are interested in computing the following Boolean function:

$$H(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if at least } 2 \text{ of the } x_i\text{'s are assigned } 1, \\ 0 & \text{if fewer than } 2 \text{ of the } x_i\text{'s are assigned } 1. \end{cases}$$

Prove there is a circuit computing H that uses at most $C \cdot n$ gates. Here C should be some fixed positive number, like 3 or 4 or 10. (Your C should work for every choice of n.) Your circuit can use any type of gate with fan-in at most 2 (though perhaps you will only need AND and OR gates?). If it helps you, you may assume that n is a power of 2.

[(**Bonus.** Lower bounds are hard. Prove that any circuit computing H must have at least $2n-3$ gates]

Degrees and Paths

Suppose that a graph G has minimal degree d (so the vertex with the smallest degree has degree d). Show that G has a path of length d.

Useful tree facts

(a) Let T be a tree with at least two vertices. Prove that T has at least two leaves.

(b) Let $G = (V, E)$ be a graph. Show that G is a tree if and only if for every pair of distinct vertices $u, v \in V$ there is a unique path in G from u to v.