CMU 15-251
Game theory

Teachers:
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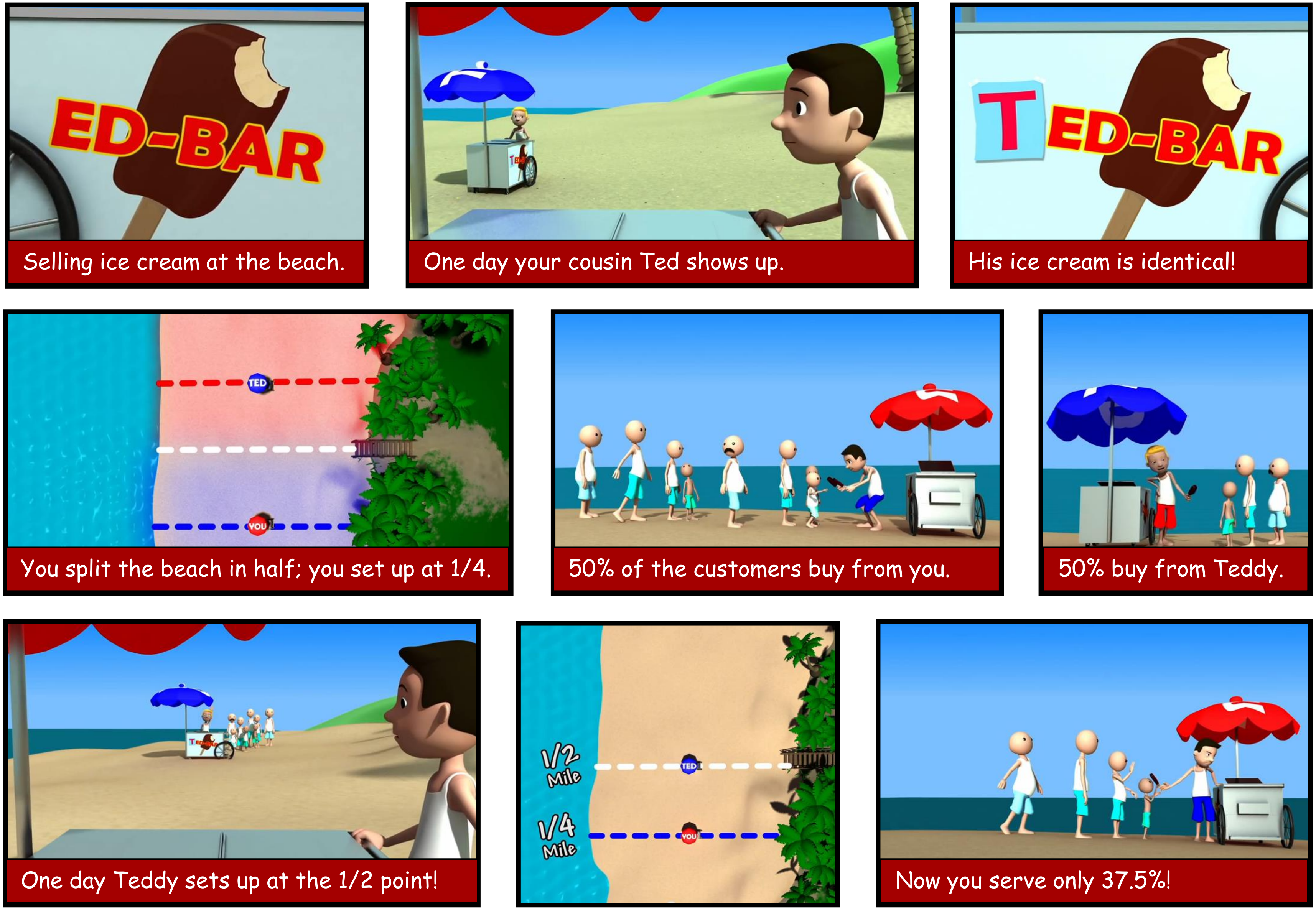
Normal-Form Game

- A game in normal form consists of:
  - Set of players $N = \{1, \dots, n\}$
  - Strategy set S
  - For each $i \in N$, utility function $u_i: S^n \to \mathbb{R}$: if each $j \in N$ plays the strategy $s_j \in S$, the utility of player i is $u_i(s_1, \dots, s_n)$
- Next example created by taking screenshots of http://youtu.be/jILgxeNBK_8


ED-BAR
Selling ice cream at the beach.
One day your cousin Ted shows up.
TED-BAR
His ice cream is identical!
TED
YOU
You split the beach in half; you set up at 1/4.
50% of the customers buy from you.
50% buy from Teddy.
One day Teddy sets up at the 1/2 point!
1/2 Mile
TED
1/4 Mile
YOU
Now you serve only 37.5%!

The Ice Cream Wars

- $N = \{1,2\}$
- $S = [0,1]$
- $u_i(s_i, s_j) = \begin{cases} \frac{s_i+s_j}{2}, & s_i < s_j \\ 1 - \frac{s_i+s_j}{2}, & s_i > s_j \\ \frac{1}{2}, & s_i = s_j \end{cases}$

- To be continued…

The prisoner's dilemma

- Two men are charged with a crime
- They are told that:
  - If one rats out and the other does not, the rat will be freed, other jailed for nine years
  - If both rat out, both will be jailed for six years
- They also know that if neither rats out, both will be jailed for one year

The prisoner's dilemma

	Cooperate	Defect
Cooperate	-1,-1	-9,0
Defect	0,-9	-6,-6

What would you do?

In real life

- Presidential elections
  - Cooperate = positive ads
  - Defect = negative ads
- Nuclear arms race
  - Cooperate = destroy arsenal
  - Defect = build arsenal
- Climate change
  - Cooperate = curb CO_2 emissions
  - Defect = do not curb

On TV

http://youtu.be/S0qjK3TWZE8

The professor's dilemma

Professor \ Class	Listen	Sleep
Make effort	$10^6, 10^6$	-10,0
Slack off	0,-10	0,0

Dominant strategies?

Nash equilibrium

- Each player's strategy is a best response to strategies of others
- Formally, a Nash equilibrium is a vector of strategies $\boldsymbol{s} = (s_1 \ldots, s_n) \in S^n$ such that $\forall i \in N, \forall s_i' \in S, u_i(\boldsymbol{s}) \geq u_i(s_i', \boldsymbol{s}_{-i})$, where $\boldsymbol{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$

Nash equilibrium

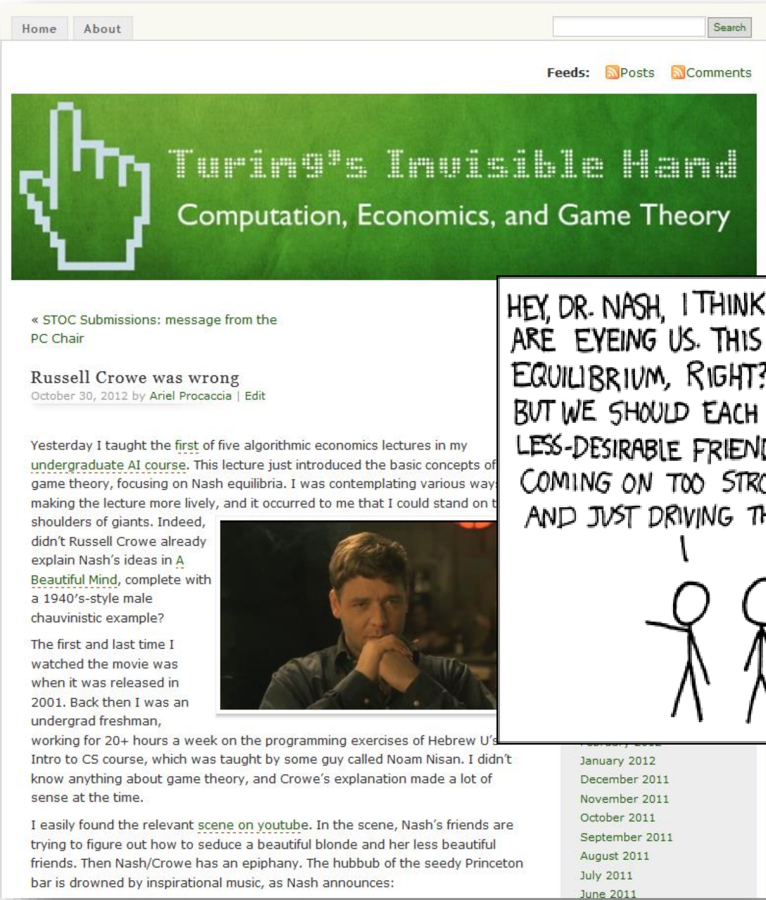
- Poll 1: How many Nash equilibria does the Professor's Dilemma have?
  1. 0
  2. 1
  3. 2
  4. 3

	Listen	Sleep
Make effort	$10^6, 10^6$	-10,0
Slack off	0,-10	0,0

Nash equilibrium

http://youtu.be/CemLiSI5ox8

Russel Crowe was wrong



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Russell Crowe was wrong

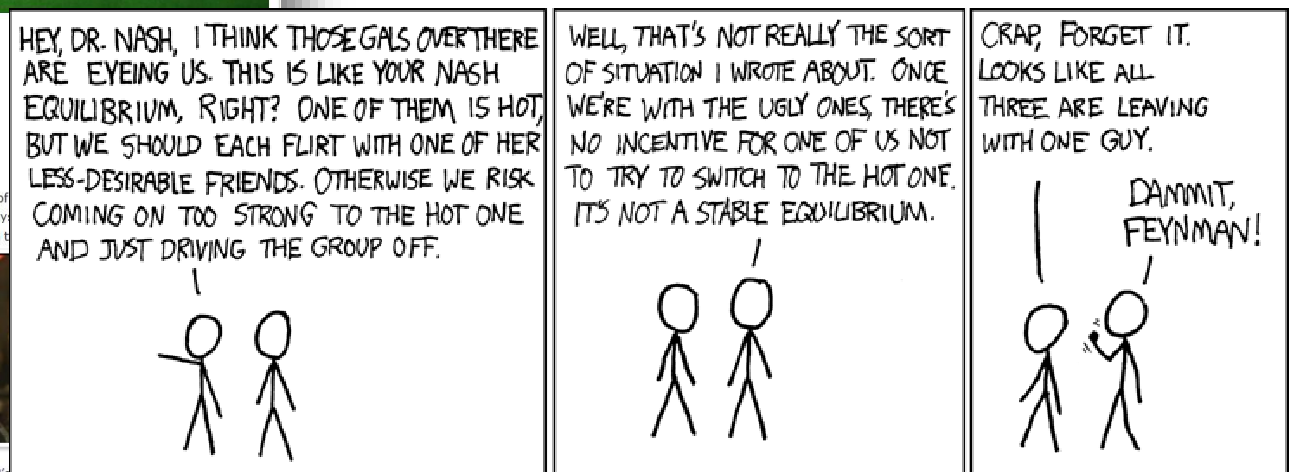
October 30, 2012 by Ariel Procaccia | Edit

Yesterday I taught the first of five algorithmic economics lectures in my undergraduate AI course. This lecture just introduced the basic concepts of game theory, focusing on Nash equilibria. I was contemplating various way making the lecture more lively, and it occurred to me that I could stand on t shoulders of giants. Indeed, didn't Russell Crowe already explain Nash's ideas in A Beautiful Mind, complete with a 1940's-style male chauvinistic example?

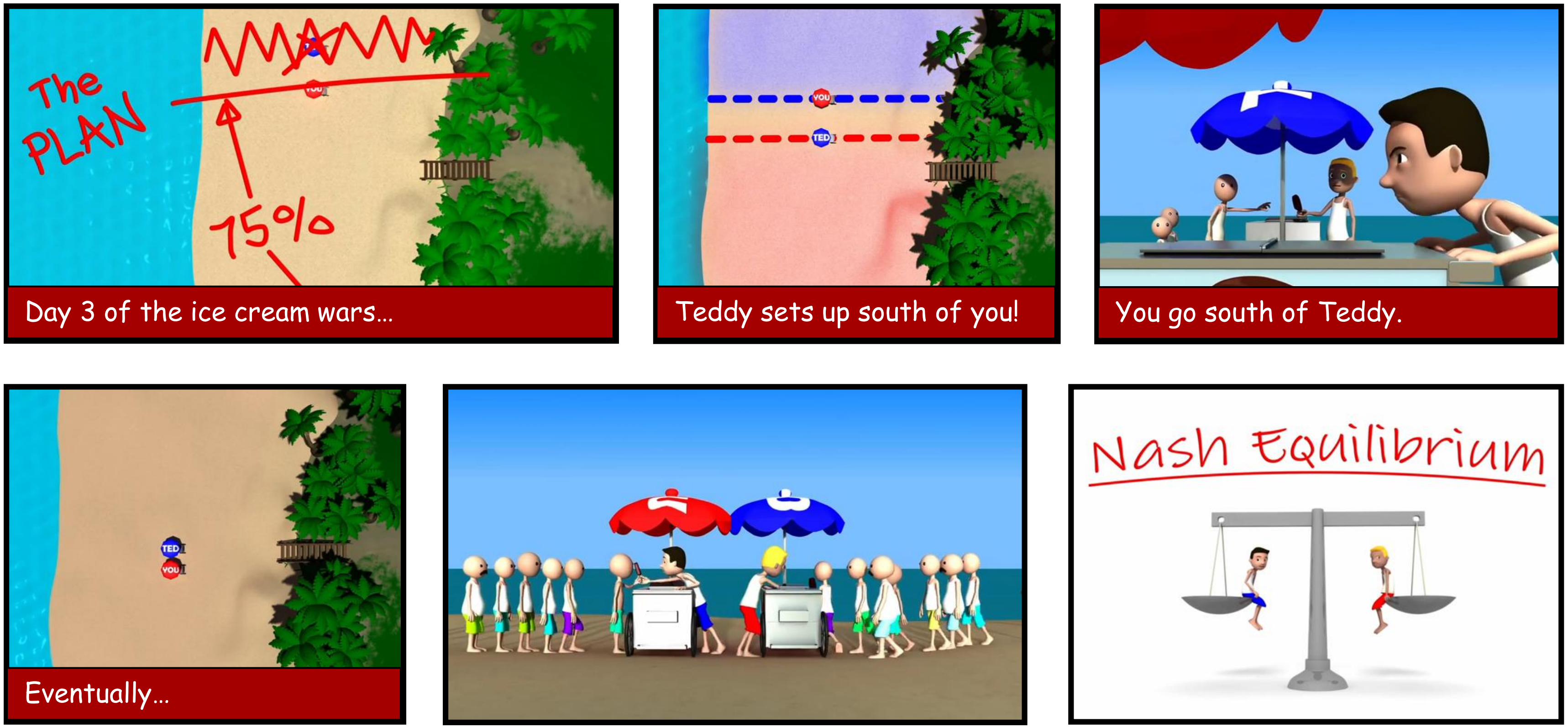
The first and last time I watched the movie was when it was released in 2001. Back then I was an undergrad freshman, working for 20+ hours a week on the programming exercises of Hebrew U's Intro to CS course, which was taught by some guy called Noam Nisan. I didn't know anything about game theory, and Crowe's explanation made a lot of sense at the time.

I easily found the relevant scene on youtube. In the scene, Nash's friends are trying to figure out how to seduce a beautiful blonde and her less beautiful friends. Then Nash/Crowe has an epiphany. The hubbub of the seedy Princeton bar is drowned by inspirational music, as Nash announces:

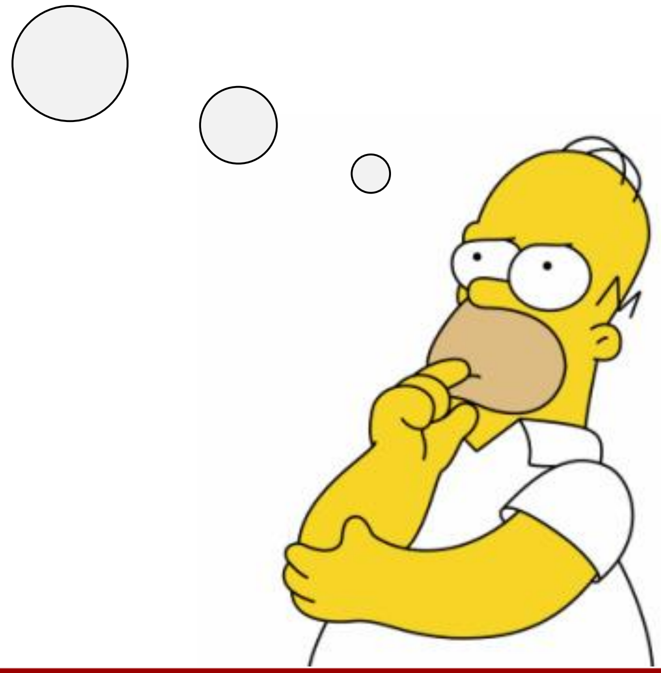
January 2012
December 2011
November 2011
October 2011
September 2011
August 2011
July 2011
June 2011



End of the Ice Cream Wars

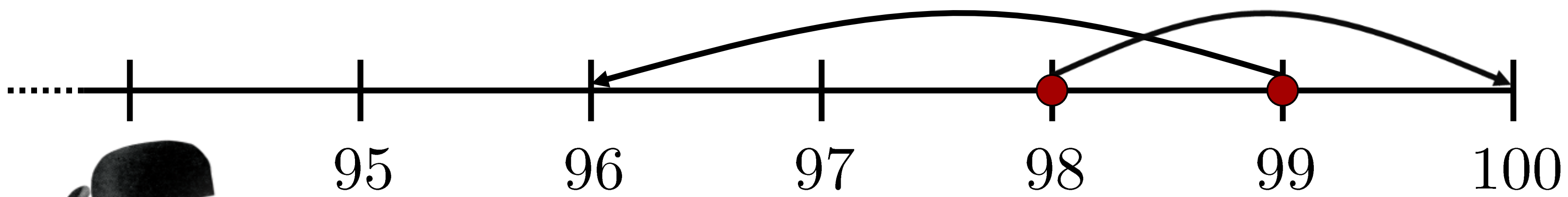


This is why competitors open their stores next to one another!



Does NE make sense?

- Two players, strategies are $\{2, \ldots, 100\}$
- If both choose the same number, that is what they get
- If one chooses s, the other t, and $s < t$, the former player gets $s + 2$, and the latter gets $s - 2$
- Poll 2: what would you choose?



Back to prison

- The only Nash equilibrium in Prisoner's dilemma is bad; but how bad is it?
- Objective function: social cost = sum of costs
- NE is six times worse than the optimum

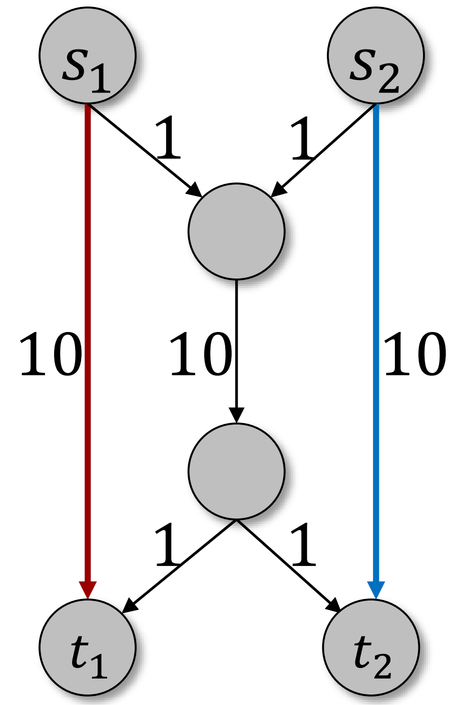
	Cooperate	Defect
Cooperate	-1,-1	-9,0
Defect	0,-9	-6,-6

Anarchy and Stability

- Fix a class of games, an objective function, and an equilibrium concept
- The price of anarchy (stability) is the worst-case ratio between the worst (best) objective function value of an equilibrium of the game, and that of the optimal solution
- In this lecture:
  - Objective function = social cost
  - Equilibrium concept = Nash equilibrium

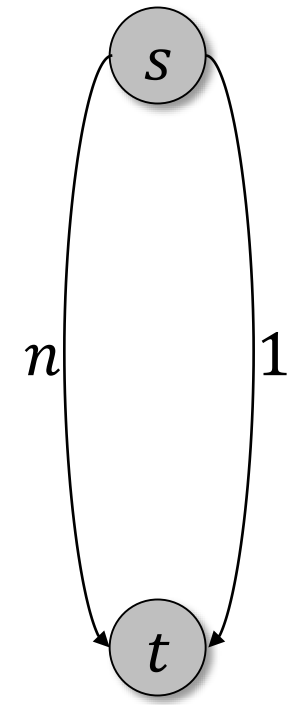
Example: Cost sharing

- n players in weighted directed graph G
- Player i wants to get from s_i to t_i; strategy space is $s_i \rightarrow t_i$ paths
- Each edge e has cost c_e
- Cost of edge is split between all players using edge
- Cost of player is sum of costs over edges on path



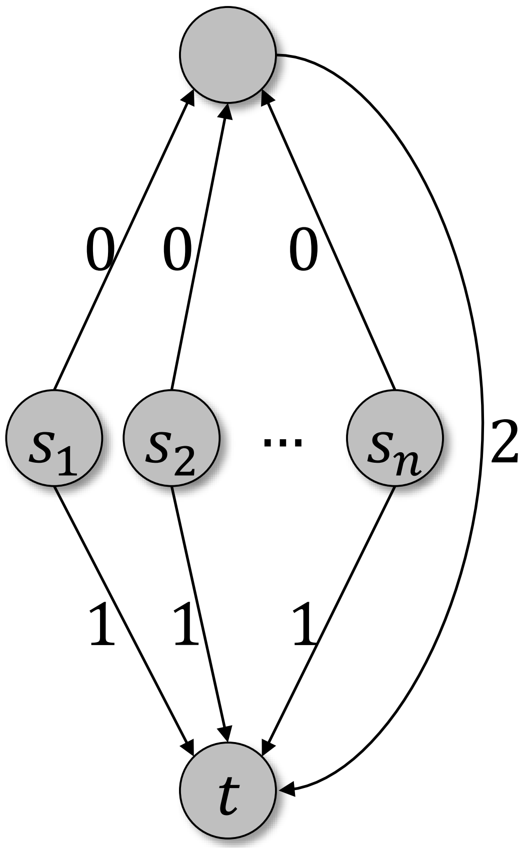
Example: Cost sharing

- With n players, the example on the right has an NE with social cost n
- Optimal social cost is 1
- $\Rightarrow$ Price of anarchy $\geq n$
- Price of anarchy is also $\leq n$
  - Each player can always deviate to his strategy at the optimal solution, and pay for it alone; the cost is at most OPT
  - At equilibrium, no player wants to deviate, so each player pays at most OPT



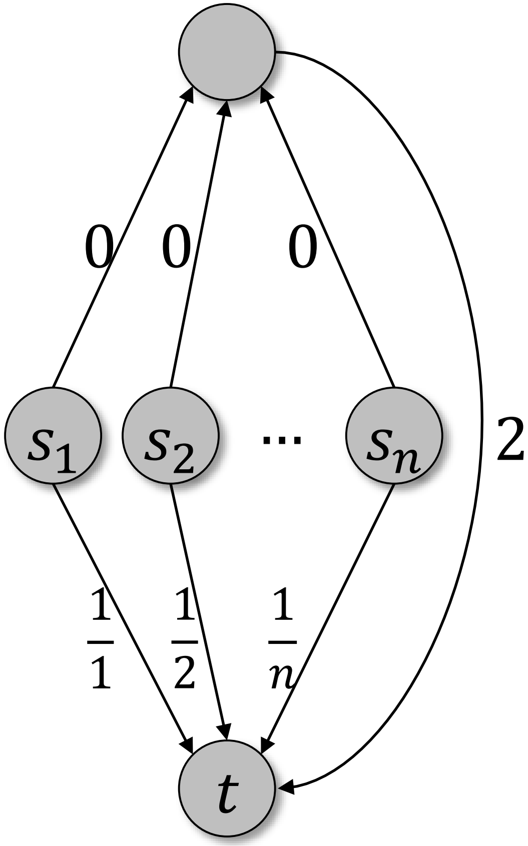
Example: Cost sharing

- Think of the 1 edges as cars, and the 2 edge as mass transit
- Bad Nash equilibrium with cost n
- Good Nash equilibrium with cost 2
- Now let's modify the example...



Example: Cost sharing

- OPT = 2
- Poll 3: What is the social cost at Nash equilibrium?
- ⇒ price of stability is at least this cost
- We will show a matching upper bound on the price of stability



Potential games

- A game is an exact potential game if there exists a function $\Phi: \prod_{i=1}^{n} S_i \to \mathbb{R}$ such that for all $i \in N$, for all $\boldsymbol{s} \in \prod_{i=1}^{n} S_i$, and for all $s_i' \in S_i$,
$\text{cost}_i(s_i', \boldsymbol{s}_{-i}) - \text{cost}_i(\boldsymbol{s}) = \Phi(s_i', \boldsymbol{s}_{-i}) - \Phi(\boldsymbol{s})$
- Existence of an exact potential function implies the existence of a Nash equilibrium (why?)

Potential games *

- Theorem: the cost sharing game is an exact potential game
- Proof:
  - Let $n_e(\boldsymbol{s})$ be the number of players using $\boldsymbol{e}$ under $\boldsymbol{s}$
  - Define the potential function

$$\Phi(\boldsymbol{s}) = \sum_{e} \sum_{k=1}^{n_e(\boldsymbol{s})} \frac{c_e}{k}$$

  - If player changes paths, pays $\frac{c_e}{n_e(\boldsymbol{s})+1}$ for each new edge, gets $\frac{c_e}{n_e(\boldsymbol{s})}$ for each old edge, so $\Delta\mathrm{cost}_i = \Delta\Phi$ ■

* Just for fun

POTENTIAL GAMES *

- Theorem: The cost of stability of cost sharing games is $O(\log n)$
- Proof:
  - It holds that
    $$\text{cost}(\boldsymbol{s}) \leq \Phi(\boldsymbol{s}) \leq H(n) \cdot \text{cost}(\boldsymbol{s})$$
  - Take a strategy profile $\boldsymbol{s}^*$ that minimizes Φ
  - $\boldsymbol{s}^*$ is an NE
  - $\text{cost}(\boldsymbol{s}^*) \leq \Phi(\boldsymbol{s}^*) \leq \Phi(\text{OPT})$
    $\leq H(n) \cdot \text{cost}(\text{OPT})$ ■

* Just for fun

Cost sharing summary

- In every cost sharing game
  - $\forall$NE $\boldsymbol{s}$, $\text{cost}(\boldsymbol{s}) \le n \cdot \text{cost}(\text{OPT})$
  - $\exists$NE $\boldsymbol{s}$ such that $\text{cost}(\boldsymbol{s}) \le H(n) \cdot \text{cost}(\text{OPT})$
- There exist cost sharing games s.t.
  - $\exists$NE $\boldsymbol{s}$ such that $\text{cost}(\boldsymbol{s}) \ge n \cdot \text{cost}(\text{OPT})$
  - $\forall$NE $\boldsymbol{s}$, $\text{cost}(\boldsymbol{s}) \ge H(n) \cdot \text{cost}(\text{OPT})$

What we have learned

- Terminology:
  - Normal-form game
  - Nash equilibrium
  - Price of anarchy/stability
  - Cost sharing games
  - Potential games
- Nobel-prize-winning ideas:
  - Nash equilibrium ☺