CMU 15-251
Game theory

Teachers:
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Ariel Procaccia (this time)
Normal-Form Game

• A game in normal form consists of:
  o Set of players $N = \{1, \ldots, n\}$
  o Strategy set $S$
  o For each $i \in N$, utility function $u_i : S^n \to \mathbb{R}$: if each $j \in N$ plays the strategy $s_j \in S$, the utility of player $i$ is $u_i(s_1, \ldots, s_n)$

• Next example created by taking screenshots of
  http://youtu.be/jILgxeNBK_8
Selling ice cream at the beach.

One day your cousin Ted shows up.

His ice cream is identical!

You split the beach in half; you set up at 1/4.

50% of the customers buy from you.

50% buy from Teddy.

One day Teddy sets up at the 1/2 point!

Now you serve only 37.5%!
The Ice Cream Wars

- $N = \{1, 2\}$
- $S = [0, 1]$
- $u_i(s_i, s_j) = \begin{cases} 
\frac{s_i + s_j}{2}, & s_i < s_j \\
1 - \frac{s_i + s_j}{2}, & s_i > s_j \\
\frac{1}{2}, & s_i = s_j
\end{cases}$

- To be continued...
The prisoner’s dilemma

- Two men are charged with a crime
- They are told that:
  - If one rats out and the other does not, the rat will be freed, other jailed for nine years
  - If both rat out, both will be jailed for six years
- They also know that if neither rats out, both will be jailed for one year
# The prisoner’s dilemma

<table>
<thead>
<tr>
<th></th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>-1,-1</td>
<td>-9,0</td>
</tr>
<tr>
<td>Defect</td>
<td>0,-9</td>
<td>-6,-6</td>
</tr>
</tbody>
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What would you do?
In real life

- Presidential elections
  - Cooperate = positive ads
  - Defect = negative ads
- Nuclear arms race
  - Cooperate = destroy arsenal
  - Defect = build arsenal
- Climate change
  - Cooperate = curb CO$_2$ emissions
  - Defect = do not curb
On TV

http://youtu.be/S0qjK3TWZE8
### The Professor's Dilemma

<table>
<thead>
<tr>
<th>Class</th>
<th>Listen</th>
<th>Sleep</th>
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<tbody>
<tr>
<td>Make effort</td>
<td>$10^6,10^6$</td>
<td>$-10,0$</td>
</tr>
<tr>
<td>Slack off</td>
<td>$0,-10$</td>
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**Dominant strategies?**
Nash equilibrium

• Each player’s strategy is a best response to strategies of others

• Formally, a Nash equilibrium is a vector of strategies $s = (s_1, \ldots, s_n) \in S^n$ such that
  \[ \forall i \in N, \forall s_i' \in S, u_i(s) \geq u_i(s_i', s_{-i}), \]
  where $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. 
Nash equilibrium

**Poll 1:** How many Nash equilibria does the Professor’s Dilemma have?

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<tr>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$10^6,10^6$</td>
<td>-10,0</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>Make effort</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>Slack off</td>
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15-251 Fall 2015: Lecture 26
Nash equilibrium

http://youtu.be/CemLiSI5ox8
Russel Crowe was wrong
End of the Ice Cream Wars

Day 3 of the ice cream wars...

The PLAN

75% 

Teddy sets up south of you!

You go south of Teddy.

Eventually...

Nash Equilibrium
This is why competitors open their stores next to one another!
Does NE make sense?

• Two players, strategies are \{2, ..., 100\}
• If both choose the same number, that is what they get
• If one chooses \(s\), the other \(t\), and \(s < t\), the former player gets \(s + 2\), and the latter gets \(s - 2\)
• Poll 2: what would you choose?
**Back to prison**

- The only Nash equilibrium in Prisoner’s dilemma is bad; but how bad is it?
- **Objective function**: social cost = sum of costs
- NE is six times worse than the optimum

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Anarchy and stability

• Fix a class of games, an objective function, and an equilibrium concept
• The price of anarchy (stability) is the worst-case ratio between the worst (best) objective function value of an equilibrium of the game, and that of the optimal solution
• In this lecture:
  o Objective function = social cost
  o Equilibrium concept = Nash equilibrium
Example: Cost sharing

• $n$ players in weighted directed graph $G$
• Player $i$ wants to get from $s_i$ to $t_i$; strategy space is $s_i \rightarrow t_i$ paths
• Each edge $e$ has cost $c_e$
• Cost of edge is split between all players using edge
• Cost of player is sum of costs over edges on path
Example: Cost sharing

- With \( n \) players, the example on the right has an NE with social cost \( n \)
- Optimal social cost is 1
- \( \Rightarrow \) Price of anarchy \( \geq n \)
- Price of anarchy is also \( \leq n \)
  - Each player can always deviate to his strategy at the optimal solution, and pay for it alone; the cost is at most OPT
  - At equilibrium, no player wants to deviate, so each player pays at most OPT
Example: Cost sharing

- Think of the 1 edges as cars, and the 2 edge as mass transit
- Bad Nash equilibrium with cost $n$
- Good Nash equilibrium with cost 2
- Now let’s modify the example...
**Example: Cost sharing**

- **OPT = 2**
- **Poll 3:** What is the social cost at Nash equilibrium?
- $\Rightarrow$ price of stability is at least this cost
- We will show a matching upper bound on the price of stability
Potential games

• A game is an exact potential game if there exists a function $\Phi: \prod_{i=1}^{n} S_i \to \mathbb{R}$ such that for all $i \in N$, for all $s \in \prod_{i=1}^{n} S_i$, and for all $s'_i \in S_i$,
  
  \[ \text{cost}_i(s'_i, s_{-i}) - \text{cost}_i(s) = \Phi(s'_i, s_{-i}) - \Phi(s) \]

• Existence of an exact potential function implies the existence of a Nash equilibrium (why?)
Potential games *

• **Theorem:** the cost sharing game is an exact potential game

• **Proof:**
  - Let $n_e(s)$ be the number of players using $e$ under $s$
  - Define the potential function
    $$\Phi(s) = \sum_e \sum_{k=1}^{n_e(s)} \frac{c_e}{k}$$
  - If player changes paths, pays $\frac{c_e}{n_e(s)+1}$ for each new edge, gets $\frac{c_e}{n_e(s)}$ for each old edge, so $\Delta \text{cost}_i = \Delta \Phi$ ■

* Just for fun
**Potential games** *

- **Theorem:** The cost of stability of cost sharing games is $O(\log n)$
- **Proof:**
  - It holds that
    $$\text{cost}(s) \leq \Phi(s) \leq H(n) \cdot \text{cost}(s)$$
  - Take a strategy profile $s^*$ that minimizes $\Phi$
  - $s^*$ is an NE
  - $\text{cost}(s^*) \leq \Phi(s^*) \leq \Phi(\text{OPT})$
    $$\leq H(n) \cdot \text{cost}(\text{OPT})$$

*Just for fun*
Cost sharing summary

- In every cost sharing game
  - ∀NE $s$, $\text{cost}(s) \leq n \cdot \text{cost(OPT)}$
  - ∃NE $s$ such that $\text{cost}(s) \leq H(n) \cdot \text{cost(OPT)}$

- There exist cost sharing games s.t.
  - ∃NE $s$ such that $\text{cost}(s) \geq n \cdot \text{cost(OPT)}$
  - ∀NE $s$, $\text{cost}(s) \geq H(n) \cdot \text{cost(OPT)}$
What we have learned

• Terminology:
  o Normal-form game
  o Nash equilibrium
  o Price of anarchy/stability
  o Cost sharing games
  o Potential games

• Nobel-prize-winning ideas:
  o Nash equilibrium 😊