15-251: Great Theoretical Ideas In Computer Science

Recitation 1

Announcements

- Office hours officially start on Saturday.
- Monday office hours will still be held despite Labor Day.

Rules of Games

- **P-position**: A losing position, one where the **P**revious player to play wins the game. Any move from this position leads to an N-position.
- **N-position**: A Wi**NN**ing position, one where the **N**ext player to play wins the game. There exists a move from this position to a P-position
- **Terminal Position**: A position where no player can make any more moves.
- **MEX (Minimum EXcluded element)**: The smallest non-negative number not included in a set S.
- **Nimber**: The Nimber of a game, denoted $N(G)$, is defined as follows:
  - If G is in a terminal position, then $N(G) = 0$.
  - If G is a game and $G_1, G_2, \ldots, G_k$ are possible successor positions, then $N(G) = MEX\{N(G_1), N(G_2), \ldots,$

Inductio Ad Absurdum

It is well known that $\ln 2$ is an irrational number that is equal to the infinite sum

$$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$$

.

However, Leonhard claims to have a proof that shows otherwise:

He claims that $\ln 2$ is rational and will prove this by showing $\sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}$ is rational for all $n > 0$ via induction.

Base Case: $n = 1$: $\sum_{i=1}^{1} \frac{(-1)^{i+1}}{i} = 1$ is indeed rational.

Induction Hypothesis: Suppose that $\sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}$ is rational for $0 < n < k + 1$ for some $k \in \mathbb{N}$.

Induction Step: It now suffices to show that $\sum_{i=1}^{k+1} \frac{(-1)^{i+1}}{i}$ is rational. We have that

$$\sum_{i=1}^{k+1} \frac{(-1)^{i+1}}{i} = \sum_{i=1}^{k} \frac{(-1)^{i+1}}{i} + \frac{(-1)^{k+2}}{k+1}$$

and by induction hypothesis, the first term is rational, and clearly the second term is also rational, and since the sum of two rationals is rational, we are done! ■

Where did Leonhard go wrong?

Chomp away

(a) Consider a stack of n coins. Each turn, a player may take 1, 3, or 6 coins. The game is played under normal rules (last player to make a valid move wins). Define the P and N positions of this game.

(b) Professor Ada challenges Professor Guruswami to a game. There is a stack of 21127 coins on the $(1, 1)$ square of an 251×251 Chomp board. Each turn, a player may either take 1, 3, or 6 coins or make a move in the chomp board. Who will win if Professor Ada goes first?

Infinite Curling

Disappointed at not making the Rio Olympics, Apoorva and Calvin decide to partake in the great and extreme sport of Infinite Curling. The game is played on a field with an end on the left and stretching infinitely to the right, with markers at every meter interval. On this field, there are n 1-meter diameter stones spaced out at various points. A turn in this game consists of selecting a stone with at least one empty space to its left, and sliding it at least one square, up to the number of spaces between it and the next stone to its left or the left end of the field. Note that these stones are magically physics defying and ignore conservation of momentum, so stones that collide instantly stop moving. The game ends when no move is possible. Find and prove the P-positions of this game.

Bonus: Super Nim

Consider the game of Nim with the following variant: Instead of taking sticks from just one pile, on a player's turn, they can take arbitrarily many sticks from up to k piles. As usual, the player who takes the last stick wins. Find and prove the N and P-positions in this game.