15-251: Great Theoretical Ideas In Computer Science

Recitation 3

- Two sets have the same cardinality (size) if there is a bijection between them. $|A| \leq |B|$ if there is an injection from A to B (or a surjection from B to A).
- A set S is **countable** if $|S| \leq |\mathbb{N}|$ (the natural numbers). A set S is **uncountable** if $|S| > \mathbb{N}$.
  Examples of countable sets are $\mathbb{N}, Primes, Squares, \mathbb{Z}, \mathbb{Z}^2, \mathbb{N}^2, \mathbb{Q}, \{0,1\}^*$
  Examples of uncountable sets are $\{0,1\}^\infty, [0,1], \mathbb{R}$
- For $A, B \subseteq \Sigma^*, A \leq_T B$ (read "A Turing reduces to B") iff it is possible to decide A with a TM which has access to a magical black box function $f(x)$ (or a subroutine if you're boring) which outputs true if $x \in B$ and false if $x \notin B$. To show a reduction we describe such a TM at a high level, using pseudocode. (That is, we use the Church Turing Thesis)
- If you are struggling with the course, PLEASE come to office hours/small groups/set up an individual meeting with a TA. We are happy to talk and here to help!

Counting sheep

For each set below, determine if it is countable or not. Prove your answers.

(a) $S = \{a_1 a_2 a_3 \ldots \in \{0,1\}^\infty \mid \forall n \geq 1$ the string $a_1 \ldots a_n$ contains more 1's than 0's.$\}$.

(b) Σ^*, where Σ is an alphabet that is allowed to be countably infinite (e.g., $\Sigma = \mathbb{N}$).

Indecisive

Prove that the following languages are undecidable:

- FINITE = { M | M is a TM which accepts a finite language }
- TOTAL = { M | M is a decider (M halts on all inputs) }
- REGULAR = { M | M is a TM which accepts a regular language }

Proof of Existance

A real number is called trancendental if it is not a root (a value of x which makes the polynomial zero) of any polynomial with integer coefficients. An important theorem of algebra, which you may use for this problem, states that a polynomial of degree d (a polynomial in which x^d is the highest power of x which occurs) is has at most d real roots. One motivation of Cantor's in developing his notions about infinity was to find an easier proof that trancendental numbers exist. Armed with his ideas, you too can prove the existance of trancendental numbers! Do so.

Finite is like, totally undecidable!

Prove that TOTAL $\leq_T$ FINITE.

Bonus: Leftover From Lecture

Give an explicit bijection between $\{0,1\}^\infty$ and $[0,1]$.