15-251: Great Theoretical Ideas In Computer Science

Recitation 6 : Circuits and Matching

Recap

- A matching for a bipartite graph is called **stable** if it doesn't contain any *rogue couples*, i.e. a man and a woman who would prefer each other over their matched partners.
- Given a list of men and women along with their preference lists, the **Traditional Matching Algorithm** (TMA) produces a stable matching in polynomial time. The algorithm is optimal for the gender that proposes (if men are proposing, each man will get his best possible partner).
- A **circuit family** is an infinite sequence of circuits $\langle C_i \rangle$, where C_i is a circuit with i input bits. We say that a circuit family $\langle C_i \rangle$ *decides* a language $L \subseteq \Sigma^*$ if the set of strings accepted by C_i is exactly $\{s : s \in L \text{ and } |s| = i\}$

Gates has 3 floors

Show that any boolean function $f : \{0,1\}^n \to \{0,1\}$ (i.e. a function that takes in n input bits and outputs 1 bit) can be computed by a circuit of depth at most 3 (This means that the longest path from any input bit to the output should have at most 3 gates). Your gates may have any number of inputs.
What is the size (in big-O) of such a circuit in the worst case?

Bounds on Circuit Size

Let $x_1, x_2, \ldots, x_n$ be input bits ($n \geq 2$). We use the convention that truth assignments are either 0 or 1. We are interested in computing the following Boolean function:

$$H(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if at least } 2 \text{ of the } x_i\text{'s are assigned } 1, \\ 0 & \text{if fewer than } 2 \text{ of the } x_i\text{'s are assigned } 1. \end{cases}$$

Prove there is a circuit computing H that uses at most $C \cdot n$ gates. Here C should be some fixed positive number, like 3 or 4 or 10. (Your C should work for every choice of n.) Your circuit can use any type of gate with fan-in at most 2 (though perhaps you will only need AND and OR gates?). If it helps you, you may assume that n is a power of 2.

[(**Bonus.** Lower bounds are hard. Prove that any circuit computing H must have at least $2n - 3$ gates]

It's All the Same to Me

(a) Suppose that, in a stable marriage scenario, a man was the last choice for every women, and a woman was the last choice for every man. Is it necessary that in a stable matching, the man and women are paired together?

(b) Give a polynomial time algorithm to decide if a given instance of the stable matching problem has a unique solution.

I Couldn't Find a Suitable Marriage Pun

Hall's Marriage Theorem states that a bipartite graph $G(X, Y, E)$ has a perfect matching iff $|X| = |Y|$ and for every $S \subseteq X$, $|N(S)| \geq |S|$. Prove this properly.