15-251: Great Theoretical Ideas In Computer Science

Recitation 9 : Probability

Review of Definitions and Theorems

Let A and B be events.

- If $\Pr[A] \neq 0$, the conditional probability of B given A is defined to be
$$\Pr[B \mid A] = \frac{\Pr[B \cap A]}{\Pr[A]}.$$
- The law of total probability states that
$$\Pr[B] = \Pr[B \mid A] \cdot \Pr[A] + \Pr[B \mid \overline{A}] \cdot \Pr[\overline{A}].$$
- We say that A and B are independent if
$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B].$$

Let X, Y be random variables with sample space Ω.

- The expectation of X is defined to be
$$\mathbb{E}[X] = \sum_{\ell \in \Omega} \Pr[\ell] \cdot X(l).$$
- Linearity of expectation says that (even when X and Y are not independent)
$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Professors' Probabilistic Pancakes

Professor Ada is making pancakes for students at his office hours. Unfortunately, his not-so-trusty griddle is not feeling very well today and randomly burns pancakes with probability $\frac{1}{3}$. In addition, with Professor Ada's impressive pancake making skills, with a functioning griddle he only burns a pancake with probability $\frac{1}{4}$.

You may assume that Professor Ada does not get demoralized if he burns a pancake (i.e. he does not burn other pancakes with higher or lower probability as a result).

(a) What is the probability that a pancake is burnt?

(b) Suppose Professor Ada made three pancakes. Given that at least one pancake burns, what is the probability that they all burn?

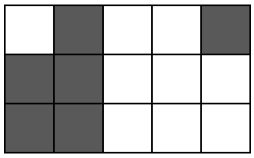
(c) Professor Guruswami joins Professor Ada at his office hours with a new griddle. Professor Guruswami's pancake making skills are even more impressive and only burns a pancake with probability $\frac{1}{5}$, and makes pancakes twice as fast! They make pancakes on their individual griddles and collate the pancakes in a single pile. Josh comes along and grabs a burnt pancake off the pile. What is the probability that Professor Guruswami burnt the pancake?

Expected Expectation

Suppose $X \sim$ Geometric(p) for some $0 < p \leq 1$. Use the Law of Total Expectation to show that $\mathbb{E}[X] = 1/p$.

Colored Components by Chance

We have an $m \times n$ grid of squares ($m, n \geq 2$). Each square is randomly colored black or white with probability $\frac{1}{2}$ each, independently for all squares. (An example outcome with $m = 3, n = 5$ is shown below.)



Define a *pane* to be a 2×2 subgrid. Let X be the number of panes whose four squares all have the same color. (These panes may overlap. E.g., above $X = 4$ because there are 3 all-white panes and 1 all-black pane.) Determine (with proof) a formula for $\mathbb{E}[X]$ in terms of m and n.

Markov Marked Off

Recall that Markov's inequality states that for any nonnegative random variable X,

$$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a} \qquad \text{for every } a > 0.$$

(a) Suppose we flip n fair coins. Using Markov's inequality, find an upper bound on the probability that at least $3/4$ of the coins land heads.

(b) Suppose $n = 7$. Compute exactly the probability that at least $3/4$ of the coins land heads. How loose is the bound given by Markov's inequality?

(c) Can you come up with a (non-constant) distribution for X and some a such that Markov's inequality is tight?