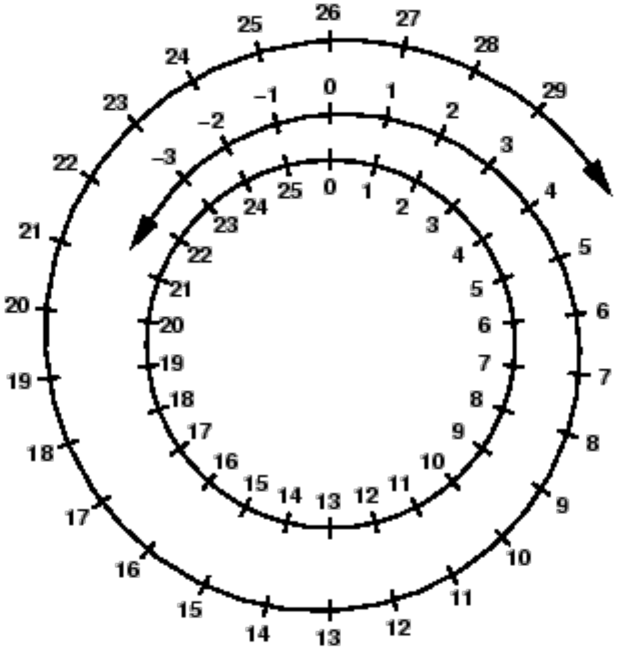
15-251
Great Theoretical Ideas in Computer Science

Lecture 21:
Modular Arithmetic

November 8th, 2016



Next 3 lectures

Modular arithmetic

+

Group theory
(a more abstract and general framework)

+

Cryptography
(in particular, "public-key" cryptography)

Main goal of this lecture

Goal:

Understanding modular arithmetic: **theory + algorithms**

Why:

1. When we do addition or multiplication, the universe is infinite (e.g. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$.)

   Sometimes we prefer to restrict ourselves to a finite universe (e.g. the modular universe).

2. Some hard-to-do arithmetic operations in $\mathbb{Z}$ or $\mathbb{Q}$ are easy in the modular universe.

3. Some easy-to-do arithmetic operations in $\mathbb{Z}$ or $\mathbb{Q}$ seem to be hard in the modular universe.

   And this is great for cryptography applications!

Main goal of this lecture

Modular Universe

- How to view the elements of the universe?
- How to do basic operations:

1. addition
2. subtraction
3. multiplication
4. division
5. exponentiation
6. taking roots
7. logarithm

theory
+
algorithms
(efficient (?))

The plan

Start with algorithms on good old integers.

Then move to the modular universe.

Integers

Algorithms on numbers involve **BIG** numbers.

3618502788666131106986593281521497110455743021169260358536775932020762686101
7237846234873269807102970128874356021481964232857782295671675021393065473695
3943653222082116941587830769649826310589717739181525033220266350650989268038
3194839273881505432422077179121838888281996148408052302196889866637200606252
6501310964926475205090003984176122058711164567946559044971683604424076996342
7183046544798021168297013490774140090476348290671822743961203698142307099664
3455133414637616824423860107889741058131271306226214208636008224651510961018
9789006815067664901594246966730927620844732714004599013904409378141724958467
7228950143608277369974692883195684314361862929679227167524851316077587207648
7845058367231603173079817471417519051357029671991152963580412838184841733782

Integers

$B = 5693030020523999993479642904621911725098567020556258102766251487234031094429$

$B \approx 5.7 \times 10^{75}$ (5.7 quattorvigintillion)

B is roughly the number of atoms in the universe

Definition: $\mathrm{len}(B) = \#\text{ bits to write } B$
$\approx \log_2 B$

For $B = 5693030020523999993479642904621911725098567020556258102766251487234031094429$

$\mathrm{len}(B) = 251$

(for crypto purposes, this is way too small)

Integers: Arithmetic

In general, arithmetic on numbers is not free!

Think of algorithms as performing string-manipulation.

The number of steps is measured with respect to the length of the input numbers.

1. Addition in integers

$$
\begin{array}{rrl}
 & 36185027886661311069865932815214971104 & A \\
+ & 65743021169260358536775932020762686101 & B \\
\hline
 & 101928049055921669606641864835977657205 & C
\end{array}
$$

Grade school addition is linear time:

if $\operatorname{len}(A), \operatorname{len}(B) \leq n$

number of steps to produce C is $O(n)$

2. Subtraction in integers

$$
\begin{array}{rrl}
 & 101928049055921669606641864835977657205 & A \\
- & 36185027886661311069865932815214971104 & B \\
\hline
 & 65743021169260358536775932020762686101 & C
\end{array}
$$

Grade school subtraction is linear time:

if $\mathrm{len}(A), \mathrm{len}(B) \leq n$

number of steps to produce C is $O(n)$

3. Multiplication in integers

```
               36185027886661311069865932815214971104   A
x                                    5932020762686101   B
-------------------------------------------------------
               XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
              XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
             XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
            XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
           XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
          XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
         XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
        XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
       XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
      XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
     XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
    XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
   XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
  XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
 XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
-------------------------------------------------------
214650336722050463946651358202698404452609868137425504  C
```

 # steps: $O(\mathrm{len}(A) \cdot \mathrm{len}(B))$

$= O(n^2)$ **if** $\mathrm{len}(A), \mathrm{len}(B) \leq n$

4. Division in integers

```
                     6099949635084593037586   Q
B   5932020762686101 | 36185027886661311069865932815214971104   A
                       XXXXXXXXXXXXXXXXX
                        XXXXXXXXXXXXXXXXX
                         XXXXXXXXXXXXXXXXX
                          XXXXXXXXXXXXXXXXX
                           XXXXXXXXXXXXXXXXX
                            XXXXXXXXXXXXXXXXX
                             XXXXXXXXXXXXXXXXX
                              XXXXXXXXXXXXXXXXX
                               XXXXXXXXXXXXXXXXX
                                XXXXXXXXXXXXXXXXX
                                 XXXXXXXXXXXXXXXXX
                                  XXXXXXXXXXXXXXXXX
                                   XXXXXXXXXXXXXXXXX
                                    XXXXXXXXXXXXXXXXX
                                     XXXXXXXXXXXXXXXXX
                                      XXXXXXXXXXXXXXXXX
                                       XXXXXXXXXXXXXXXXX
                                        XXXXXXXXXXXXXXXXX
                                         XXXXXXXXXXXXXXXXX
                                        ---------------------
                                           3960087002178918   R
```

$$A = Q \cdot B + R$$

$$\boxed{R = A \bmod B}$$

 # steps: $O(\mathrm{len}(A) \cdot \mathrm{len}(B))$

5. Exponentiation in integers

Given as input B, compute 2^B.

If

$B =$ 5693030020523999993479642904621911725098567020556258102766251487234031094429

$\text{len}(B) = 251$

but $\text{len}(2^B)$ ~ 5.7 quattorvigintillion

(output length exceeds number of particles in the universe)

exponential in input length

6. Taking roots in integers

Given as input A, E , compute $A^{1/E}$.

From midterm 1: binary search.

7. Taking logarithms in integers

Given as input A, B, compute $\log_B A$.

i.e., find X such that $B^X = A$.

From Homework 4, Q3b:

Try X = 1, 2, 3, …

Stop when $B^X \geq A$.

Bonus problem 1: integer factorization

$A = 5693030020523999993479642904621911725098567020556258102766251487234031094429$

<u>Goal</u>: find one (non-trivial) factor of A

> **for** B = 2, 3, 4, 5, …
>   test **if** A mod B = 0.

It turns out:

$A = 68452332409801603635385895997250919383$ x
$83167801886452917478124266362673045163$

Each factor ≈ age of the universe in Planck time.

Worst case: $\sqrt{A}$ iterations.

$$\sqrt{A} = \sqrt{2^{\log_2 A}} = \sqrt{2^{\mathrm{len}(A)}} = 2^{\mathrm{len}(A)/2}$$

exponential in input length

Bonus problem 1: integer factorization

Fastest known algorithm is exponential time!

That turns out to be a good thing:

If there is an efficient algorithm to solve
the factoring problem

can break most cryptographic systems
used on the internet

Bonus problem 2: isPrime

Your favorite function from 15-112

```
def isPrime(n):
    if (n < 2):
        return False
    for factor in range(2,n):
        if (n % factor == 0):
            return False
    return True
```

iterations: $\approx$ n

$$n = 2^{\log_2 n} = 2^{\mathrm{len}(n)}$$

exponential in input length

Bonus problem 2: isPrime

```python
def fasterIsPrime(n):
    if (n < 2):
        return False
    if (n == 2):
        return True
    if (n % 2 == 0):
        return False
    maxFactor = round(n**0.5)
    for factor in range(3,maxFactor+1,2):
        if (n % factor == 0):
            return False
    return True
```

<u>Exercise</u>: Show that this is still exponential time.

Bonus problem 2: isPrime

Amazing result from 2002:

There is a poly-time algorithm for isPrime.

Agrawal, Kayal, Saxena

undergraduate students at the time

However, best known implementation is ~ $O(n^6)$ time.

Not feasible when $n = 2048$.

(n = len(input))

Bonus problem 2: isPrime

So that's not what we use in practice.

Everyone uses the Miller-Rabin algorithm (1975).

CMU
Professor

The running time is ~ $O(n^2)$.

It is a Monte Carlo algorithm with tiny error probability

(say $1/2^{300}$)

Bonus problem 3: generating a prime number

Task: Given n, generate n-bit prime number (in poly(n) time)

> **repeat:**
> let A be a random n-bit number
> test if A is prime

Prime Number Theorem (informal):

About 1/n fraction of n-bit numbers are prime.

$\Longrightarrow$ expected run-time of the above algorithm ~ $O(n^3)$.

No poly-time deterministic algorithm is known!!

The plan

Start with algorithms on good old integers.

Then move to the modular universe.

Main goal of this lecture

Modular Universe

- How to view the elements of the universe?

- How to do basic operations:

1. addition
2. subtraction
3. multiplication
4. division
5. exponentiation
6. taking roots
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theory
+
algorithms
(efficient (?))

Modular Operations:
Basic Definitions and Properties

Modular universe: How to view the elements

Hopefully everyone already knows:

Any integer can be reduced mod *N*.

$\boxed{A \bmod N \text{ = remainder when you divide } A \text{ by } N}$

Example

$$N = 5$$

$$\begin{array}{ccccc|ccccc|cccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \bmod 5 \\
0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & \cdots
\end{array}$$

Modular universe: How to view the elements

We write $A \equiv B \bmod N$ or $A \equiv_N B$

when $A \bmod N = B \bmod N$.

(In this case, we say A is congruent to B modulo N.)

Examples

$$5 \equiv_5 100$$

$$13 \equiv_7 27$$

Exercise

$$A \equiv_N B \iff N \text{ divides } A - B$$

Modular universe: How to view the elements

2 Points of View

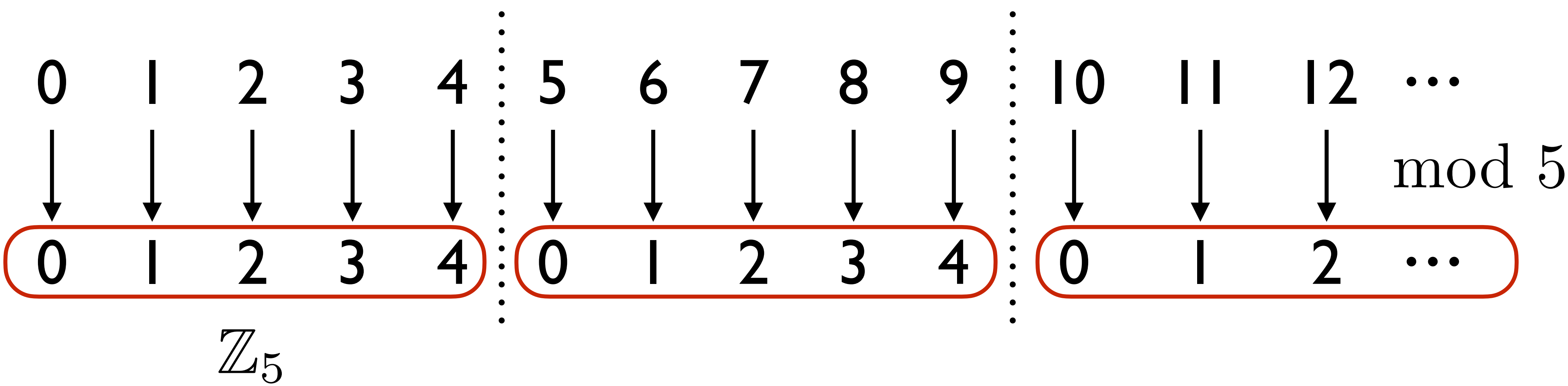
View 1

The universe is $\mathbb{Z}$.

Every element has a "mod *N*" representation.

View 2

The universe is the finite set $\mathbb{Z}_N = \{0, 1, 2, \ldots, N-1\}$.



Modular universe: Addition

Can define a "plus" operation in $\mathbb{Z}_N$:

$$\underbrace{A +_N B}_{\boxed{\text{"plus" in } \mathbb{Z}_N}} \quad = \quad (A \underbrace{+}_{\boxed{\text{"plus" in } \mathbb{Z}}} B) \bmod N$$

Modular universe: Addition

Addition table for $\mathbb{Z}_5$

$+_N$	**0**	**1**	**2**	**3**	**4**
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

0 is called the (additive) identity: $\mathbf{0} +_N A = A +_N \mathbf{0} = A$

for any A

Modular universe: Addition

In $\mathbb{Z}$		In $\mathbb{Z}_5$
3019573	$\longrightarrow$	3
912382236	$\longrightarrow$	1
3019573 + 912382236	$\xrightarrow{?}$	4

YES!

Modular universe: Addition

In $\mathbb{Z}$		In $\mathbb{Z}_5$
A	$\longrightarrow$	3
B	$\longrightarrow$	1
$A + B$	$\xrightarrow{?}$	4

YES!

Modular universe: Addition

In $\mathbb{Z}$		In $\mathbb{Z}_N$
A	$\longrightarrow$	$A \bmod N$
B	$\longrightarrow$	$B \bmod N$
$A+B$	$\xrightarrow{?}$	$(A \bmod N) +_N (B \bmod N)$

Is $(A+B) \bmod N \;=\; (A \bmod N) +_N (B \bmod N)$?

YES!

Modular universe: Subtraction

How about subtraction in $\mathbb{Z}_N$?

What does $A - B$ mean?

It is actually addition in disguise: $A + (-B)$

Then what does $-B$ mean in $\mathbb{Z}_N$?

Definition:

Given $B \in \mathbb{Z}_N$, its *additive inverse*, denoted by $-B$, is the element in $\mathbb{Z}_N$ such that $B +_N -B = 0$.

$$A -_N B = A +_N -B$$

Modular universe: Subtraction

Addition table for $\mathbb{Z}_5$

$+_N$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$-0 = 0$

$-1 = 4$

$-2 = 3$

$-3 = 2$

$-4 = 1$

Modular universe: Subtraction

Addition table for $\mathbb{Z}_5$

$+_N$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Note:

For every $A \in \mathbb{Z}_N$, $-A$ exists.

Why? $-A = N - A$

This implies:

A row contains distinct elements.

i.e. every row is a permutation of $\mathbb{Z}_N$.

Fix row A: $\underbrace{A}_{\text{row}} +_N \underbrace{B}_{\text{col}} = \underbrace{A}_{\text{row}} +_N \underbrace{B'}_{\text{col}} \implies \underbrace{B = B'}_{\text{same col}}$

Modular universe: Multiplication

Can define a "multiplication" operation in $\mathbb{Z}_N$:

$$\underbrace{A \cdot_N B}_{\text{"multiplication" in } \mathbb{Z}_N} \quad = \quad (\underbrace{A \cdot B}_{\text{"multiplication" in } \mathbb{Z}}) \bmod N$$

Modular universe: Multiplication

Multiplication table for $\mathbb{Z}_5$

$\bullet_N$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

1 is called the (multiplicative) identity: $1 \bullet_N A = A \bullet_N 1 = A$

for any A

Modular universe: Multiplication

In $\mathbb{Z}$		In $\mathbb{Z}_N$
A	$\cdots\cdots\cdots\blacktriangleright$	$A \bmod N$
B	$\cdots\cdots\cdots\blacktriangleright$	$B \bmod N$
$A \cdot B$	$\overset{?}{\cdots\cdots\cdots\blacktriangleright}$	$(A \bmod N) \cdot_N (B \bmod N)$

Is $(A \cdot B) \bmod N = (A \bmod N) \cdot_N (B \bmod N)$?

YES!

Modular universe: Division

How about division in $\mathbb{Z}_N$?

What does A/B mean?

It is actually multiplication in disguise: $A \cdot \dfrac{1}{B} = A \cdot B^{-1}$

Then what does B^{-1} mean in $\mathbb{Z}_N$?

Definition:

Given $B \in \mathbb{Z}_N$, its *multiplicative inverse*, denoted by B^{-1}, is the element in $\mathbb{Z}_N$ such that $B \cdot_N B^{-1} = 1.$

$$A /_N B = A \cdot_N B^{-1}$$

Modular universe: Division

Multiplication table for $\mathbb{Z}_5$

$\bullet_N$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$$0^{-1} = \text{undefined}$$
$$1^{-1} = 1$$
$$2^{-1} = 3$$
$$3^{-1} = 2$$
$$4^{-1} = 4$$

Modular universe: Division

Multiplication table for $\mathbb{Z}_6$

$\bullet_N$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

$$0^{-1} = \text{undefined}$$
$$1^{-1} = 1$$
$$2^{-1} = \text{undefined}$$
$$3^{-1} = \text{undefined}$$
$$4^{-1} = \text{undefined}$$
$$5^{-1} = 5$$

WTF?

Modular universe: Division

Multiplication table for $\mathbb{Z}_7$

$\bullet_N$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Every number except 0 has a multiplicative inverse.

Modular universe: Division

Multiplication table for $\mathbb{Z}_8$

$\bullet_N$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

{1, 3, 5, 7} have inverses. Others don't.

Modular universe: Division

Fact: $A^{-1} \in \mathbb{Z}_N$ exists <u>if and only if</u> $\gcd(A, N) = 1.$

$\gcd(a, b)$ = greatest common divisor of a and b.

<u>Examples</u>:

$$\gcd(12, 18) = 6$$

$$\gcd(13, 9) = 1$$

$$\gcd(1, a) = 1 \quad \forall a$$

$$\gcd(0, a) = a \quad \forall a$$

If $\gcd(a, b) = 1$, we say a and b are **relatively prime**.

Modular universe: Division

Fact: $A^{-1} \in \mathbb{Z}_N$ exists if and only if $\gcd(A, N) = 1.$

Definition: $\mathbb{Z}_N^* = \{A \in \mathbb{Z}_N : \gcd(A, N) = 1\}.$

Definition: $\varphi(N) = |\mathbb{Z}_N^*|$

Note that $\mathbb{Z}_N^*$ is "closed" under multiplication,

i.e., $A, B \in \mathbb{Z}_N^* \implies A \cdot_N B \in \mathbb{Z}_N^*$

(Why?)

Modular universe: Division

$$\mathbb{Z}_5^*$$

$\bullet_N$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$$\varphi(5) = 4$$

Modular universe: Division

$$\mathbb{Z}_5^*$$

$\bullet_N$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$$\varphi(5) = 4$$

Modular universe: Division

$$\mathbb{Z}_5^*$$

$\bullet_N$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

For P prime, $\varphi(P) = P - 1.$

Modular universe: Division

$$\mathbb{Z}_8^*$$

$\bullet_N$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

$$\varphi(8) = 4$$

Modular universe: Division

$$\mathbb{Z}_8^*$$

$\bullet_N$	**1**	**3**	**5**	**7**
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

$$\varphi(8) = 4$$

Modular universe: Division

$$\mathbb{Z}_{15}^*$$

$\cdot_N$	**1**	**2**	**4**	**7**	**8**	**11**	**13**	**14**
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1

$$\varphi(15) = 8$$

Modular universe: Division

$$\mathbb{Z}_{15}^*$$

$\bullet_N$	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1

Exercise: For P, Q distinct primes, $\varphi(PQ) = (P-1)(Q-1)$.

Modular universe: Division

$\mathbb{Z}_8^*$

$\cdot_N$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

$\varphi(8) = 4$

For every $A \in \mathbb{Z}_N^*$, A^{-1} exists.

This implies:

A row contains distinct elements.

i.e. every row is a permutation of $\mathbb{Z}_N^*$.

$$A \cdot_N B = A \cdot_N B' \implies B = B'$$

Summary so far

$+_N$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\mathbb{Z}_N$

behaves nicely
with respect to
addition / subtraction

$\bullet_N$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

$\mathbb{Z}_N^*$

behaves nicely
with respect to
multiplication / division

Modular universe: Exponentiation

Exponentiation in $\mathbb{Z}_N$

Notation:

For $A \in \mathbb{Z}_N$, $E \in \mathbb{N}$,

$$A^E = \underbrace{A \cdot_N A \cdot_N \cdots \cdot_N A}_{E \text{ times}}$$

Modular universe: Exponentiation

Exponentiation in $\mathbb{Z}_N^*$

(Same as before)

Notation:

For $A \in \mathbb{Z}_N^*$, $E \in \mathbb{N}$,

$$A^E = \underbrace{A \cdot_N A \cdot_N \cdots \cdot_N A}_{E \text{ times}}$$

There is more though…

Modular universe: Exponentiation

Exponentiation in $\mathbb{Z}_N^*$

$\mathbb{Z}_5^*$

$\cdot_N$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$$\varphi(8) = 4$$

1	1^2	1^3	1^4	1^5	1^6	1^7	1^8
1	1	1	1	1	1	1	1
2	2^2	2^3	2^4	2^5	2^6	2^7	2^8
2	4	3	1	2	4	3	1
3	3^2	3^3	3^4	3^5	3^6	3^7	3^8
3	4	2	1	3	4	2	1
4	4^2	4^3	4^4	4^5	4^6	4^7	4^8
4	1	4	1	4	1	4	1

2 and 3 are called **generators**.

Modular universe: Exponentiation

Exponentiation in $\mathbb{Z}_N^*$

$\mathbb{Z}_8^*$

$\cdot_N$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

$$\varphi(8) = 4$$

1	1^2	1^3	1^4	1^5	1^6	1^7	1^8
1	1	1	1	1	1	1	1
3	3^2	3^3	3^4	3^5	3^6	3^7	3^8
3	1	3	1	3	1	3	1
5	5^2	5^3	5^4	5^5	5^6	5^7	5^8
5	1	5	1	5	1	5	1
7	7^2	7^3	7^4	7^5	7^6	7^7	7^8
7	1	7	1	7	1	7	1

Modular universe: Exponentiation

Euler's Theorem:

For any $A \in \mathbb{Z}_N^*$, $\quad A^{\varphi(N)} = 1$.

Equivalently, for $A \in \mathbb{Z}, N \in \mathbb{N}$ with $\gcd(A, N) = 1$,

$$A^{\varphi(N)} \equiv 1 \bmod N$$

When *N* is a prime, this is known as:

Fermat's Little Theorem:

Let P be a prime. For any $A \in \mathbb{Z}_P^*$, $\quad A^{P-1} = 1.$

Equivalently, for any A not divisible by P,

$$A^{P-1} \equiv 1 \bmod P$$

Poll

What is $213^{248} \bmod 7$?

- 0
- 1
- 2
- 3
- 4
- 5
- 6
- Beats me.

Poll Answer

Euler's Theorem:

For any $A \in \mathbb{Z}_N^*$, $\quad A^{\varphi(N)} = 1\,.$

$$
\begin{array}{cccc|ccc|cc}
A^0 & A^1 & A^2 & \cdots & A^{\varphi(N)} & A^{\varphi(N)+1} & \cdots & A^{2\varphi(N)} & A^{2\varphi(N)+1} \\
\| & & & & \| & \| & & \| & \| \\
1 & & & & A^0 & A^1 & \cdots & A^0 & A^1
\end{array}
$$

In other words, the exponent can be reduced $\bmod \varphi(N)$.

$$213^{248} \equiv_7 3^{248}$$

$$3^{248} \equiv_7 3^2 \qquad = 2$$

Poll Answer

IMPORTANT!!!

When exponentiating elements $\boxed{A \in \mathbb{Z}_N^*}$

can think of the exponent living in the universe $\mathbb{Z}_{\varphi(N)}$.

Modular Operations: Computational Complexity

Complexity of Addition

Input: $A, B \in \mathbb{Z}_N$

Output: $A +_N B$

Compute $(A + B) \bmod N$.

Poly-time 

Complexity of Subtraction

Input: $A, B \in \mathbb{Z}_N$

Output: $A -_N B$

Compute $(A + (N - B)) \bmod N.$

Poly-time ✓

Complexity of Multiplication

Input: $A, B \in \mathbb{Z}_N$

Output: $A \cdot_N B$

Compute $(A \cdot B) \bmod N.$

Poly-time 

Complexity of Division

Input: $A, B \in \mathbb{Z}_N$

Output: $A /_N B$ (if the answer exists)

Now things get interesting.

$$A /_N B = A \cdot_N B^{-1}$$

Questions:

1. Does B^{-1} exist?
2. If it does, how do you compute it?

Complexity of Division

Recall: B^{-1} exists **iff** $\gcd(B, N) = 1$.

So to determine if B has an inverse, we need to compute $\gcd(B, N)$.

Euclid's Algorithm finds gcd in polynomial time.

Arguably the first ever algorithm. ~ 300 BC

Complexity of Division

Euclid's Algorithm

```
gcd(A, B):
    if B == 0, return A
    return gcd(B,  A mod B)
```

Recitation or Homework or Practice

Why does it work?

Why is it polynomial time?

Major open problem in Computer Science

Is gcd computation efficiently parallelizable?

i.e., is there a circuit family of
- poly(n) size
- polylog(n) depth

that computes gcd?

Complexity of Division

Ok, Euclid's Algorithm tells us whether an element has an inverse. How do you find it if it exists?

Claim: An extension of Euclid's Algorithm gives us the inverse.

First, a definition:

Definition: We say that C is a miix of A and B if

$$C = k \cdot A + \ell \cdot B$$

not a real term 😋

for some $k, \ell \in \mathbb{Z}$.

Examples:

2 is a miix of 14 and 10: 2 = (-2)·14 + 3·10

7 is not a miix of 55 and 40. (why?)

Complexity of Division

Fact: C is a miix of A and B <u>if and only if</u> C is a multiple of $\gcd(A,B)$.

So $\gcd(A,B) = k \cdot A + \ell \cdot B$

<u>Exercise</u>: The coefficients k and ℓ can be found by slightly modifying Euclid's Algorithm (in poly-time).

<u>Finding B^{-1}:</u>

If $\gcd(B,N) = 1$, we can find $k, \ell \in \mathbb{Z}$ such that

$$1 = \underbrace{k}_{=B^{-1}} \cdot B + \ell \cdot N$$

Therefore found B^{-1}

Complexity of Division

Summary for the complexity of division

To compute $A/_N B = A \cdot_N B^{-1}$, we need to compute B^{-1} (if it exists).

B^{-1} exists **iff** $\gcd(B, N) = 1$ (can be computed with Euclid).

Extension of Euclid gives us (in poly-time) $k, \ell \in \mathbb{Z}$ such that

$$\gcd(B, N) = 1 = k \cdot B + \ell \cdot N$$

$$B^{-1} = k \bmod N$$

Complexity of Exponentiation

Input: $A, E, N \in \mathbb{N}$

Output: $A^E \bmod N$

In the modular universe, length of output not an issue.

Can we compute this efficiently?

Complexity of Exponentiation

Example

Compute $2337^{32} \bmod 100$.

Naïve strategy:

2337 x 2337 = 5461569

2337 x 5461569 = 12763686753

2337 x 12763686753 = …

⋮ *(30 more multiplications later)*

62672756515215551165318888666866858831347582423666560739675500890577014623663553722821669603097061282892281**81**

Complexity of Exponentiation

Example

Compute $2337^{32} \bmod 100$.

2 improvements:

- Do mod 100 after every step.

- Don't multiply 32 times. Square 5 times.

$$2337 \longrightarrow 2337^2 \longrightarrow 2337^4 \longrightarrow 2337^8 \longrightarrow 2337^{16} \longrightarrow 2337^{32}$$

(what if the exponent is 53?)

Complexity of Exponentiation

Example

Compute $2337^{53} \bmod 100$.

(what if the exponent is 53?)

Multiply powers 32, 16, 4, 1. (53 = 32 + 16 + 4 + 1)

$$2337^{53} = 2337^{32} \cdot 2337^{16} \cdot 2337^{4} \cdot 2337^{1}$$

53 in binary = 110101

Complexity of Exponentiation

Input: $A, E, N \in \mathbb{N}$ (each at most n bits)

Output: $A^E \bmod N$

Algorithm:

- Repeatedly square A, always mod N. Do this n times.

- Multiply together the powers of A corresponding to the binary digits of E (again, always mod N).

Running time: a bit more than $O(n^2 \log n)$.

Complexity of Log

Input: A, B, P such that

- P is prime
- $A \in \mathbb{Z}_P^*$
- $B \in \mathbb{Z}_P^*$ is a generator.

Output: X such that $B^X \equiv_P A$.

Note: $\{B^0,\ B^1,\ B^2,\ B^3,\ \cdots,\ B^{P-2}\} = \mathbb{Z}_P^*$

Which one corresponds to A ?

It is like we want to compute $\log_B A$ in $\mathbb{Z}_P^*$.

Complexity of Log

Find X such that $B^X \equiv_P A$.

What do you think of this algorithm:

```
DiscreteLog(A, B, P):
    for X = 0, 1, 2, …, P-2:
        compute B^X    (use fast modular exponentiation)
        check whether P divides B^X - A
```

- simple and efficient. love it.
- simple but not efficient.
- I don't understand why we are checking if P divides B^X - A.
- I don't understand what is going on right now.

Complexity of Log

Input: A, B, P such that

- P is prime
- $A \in \mathbb{Z}_P^*$
- $B \in \mathbb{Z}_P^*$ is a generator.

Output: X such that $B^X \equiv_P A$.

We don't know how to compute this efficiently!

Complexity of Taking Roots

Input: A, E, N such that $A \in \mathbb{Z}_N^*$

Output: B such that $B^E \equiv_N A$

So we want to compute $A^{1/E}$ in $\mathbb{Z}_N^*$.

We don't know how to compute this efficiently!

Main goal of this lecture

Modular Universe

- How to view the elements of the universe?

- How to do basic operations:

1. addition
2. subtraction
3. multiplication
4. division
5. exponentiation
6. taking roots
7. logarithm

theory
+
algorithms
(efficient (?))

Thursday

Group Theory

Next Tuesday

Cryptography