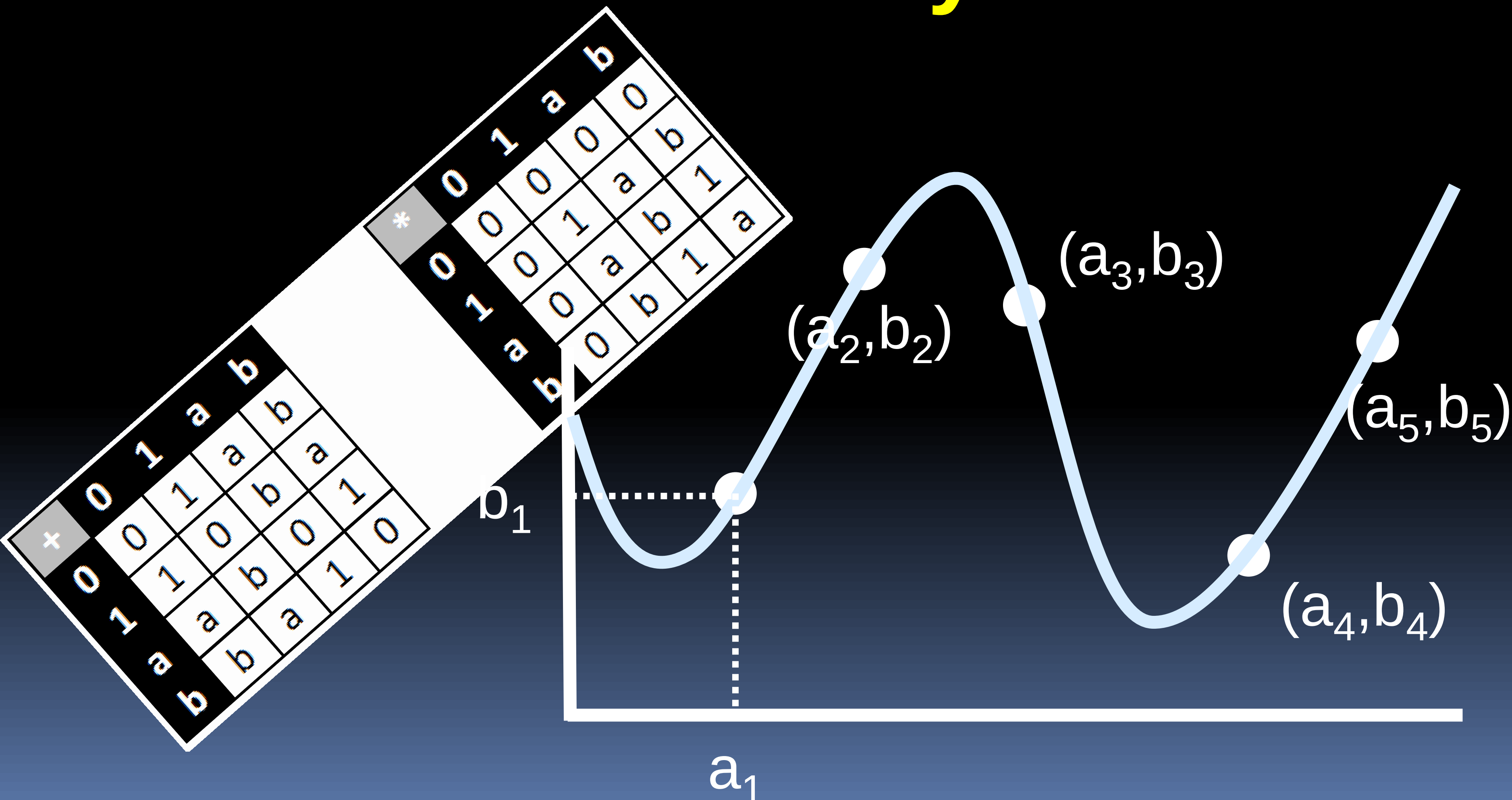
15-251: Great Theoretical Ideas in Computer Science

Fall 2016 Lecture 24

November 17, 2016

Fields and Polynomials



First, a little more Number Theory

Bezout's identity

Let a,b be arbitrary positive integers.
There exist integers r and s such that

$$r\,a + s\,b = \gcd(a,b)$$

Follows from Extended Euclid Algorithm

A non-algorithmic proof:

- Consider the set L of all *positive* integers that can be expressed as r a + s b for some integers r,s.
- L is non-empty (eg. $a \in S$)
- So L has a minimum element d

  (well-ordering principle $\Leftrightarrow$ principle of induction)

Claim: $d = \gcd(a,b)$

Claim: gcd(a,b) = d (the minimum positive integer expressible as ra+sb)

1. gcd(a,b) divides both a and b, and hence also divides d. So $d \geq \gcd(a,b)$

2. d divides both a and b, and hence $d \leq \gcd(a,b)$

Let's show $d \mid a$.
Write $a = q\, d + t$, with $0 \leq t < d$.

$t = a - q\, d$ is also expressible as a combination $r' a + s' b$.
Contradicts minimality of d.

Extended Euclid & Unique Factorization

Lemma: If gcd(a,b)=1 and a | bc, then a | c.

Proof: Let r,s be such that r a + s b =1

$r\,a\,c + s\,b\,c = c$

a | bc and a | r a c, so a | c. □

Corollary: If p is a prime and $p \mid q_1\, q_2 \ldots q_k$, then p must divide some q_i.

If the q_i's are also prime, then $p = q_i$ for some i.

Uniqueness of prime factorization follows from this!

Poll

Which of these numbers is congruent to 1 (mod 5), 6 (mod 7), and 8 (mod 9)?

- No such number exists
- 91
- 136
- 197
- 251
- 291
- None of the above
- Beats me

Chinese Remaindering

Chinese Remainder Theorem: Suppose positive integers $n_1, n_2, \ldots, n_k$ are ***pairwise coprime***. Then, for all integers $b_1, b_2, \ldots, b_k$, there exists an integer x solving the below system of simultaneous congruences

$$\begin{aligned} x &\equiv b_1 \pmod{n_1} \\ x &\equiv b_2 \pmod{n_2} \\ &\vdots \\ x &\equiv b_k \pmod{n_k}. \end{aligned}$$

Further, all solutions x are congruent to each other modulo $N = \prod_{i=1}^{k} n_i$.

Uniqueness of solutions modulo N

If x,y are two solutions, then n_i divides x-y, for i=1,2,…k

Since the n_i are pairwise coprime, this means the product $N = n_1\, n_2 \ldots n_k$ divides (x-y), thus $x \equiv y \pmod{N}$

Extended Euclid and Chinese Remaindering

Chinese Remainder Theorem: Suppose positive integers $n_1, n_2, \ldots, n_k$ are ***pairwise coprime***. Then, for all integers $b_1, b_2, \ldots, b_k$, there exists an integer x solving the below system of simultaneous congruences

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Further, all solutions x are congruent to each other modulo $N = \prod_{i=1}^{k} n_i$.

Proof for k=2:

$$\text{Take } x = b_1 \underbrace{(n_2^{-1} \bmod n_1)\, n_2}_{\substack{\text{Divisible by } n_2, \\ \text{Remainder 1 mod } n_1}} + b_2 \underbrace{(n_1^{-1} \bmod n_2)\, n_1}_{\substack{\text{Divisible by } n_1 \\ \text{Remainder 1 mod } n_2}}$$

Can compute x efficiently (by computing modular inverses)

Chinese Remainder Theorem: Suppose positive integers $n_1, n_2, \ldots, n_k$ are ***pairwise coprime***. Then, for all integers $b_1, b_2, \ldots, b_k$, there exists an integer x solving the below system of simultaneous congruences

$$\begin{aligned} x &\equiv b_1 \pmod{n_1} \\ x &\equiv b_2 \pmod{n_2} \\ &\vdots \\ x &\equiv b_k \pmod{n_k}. \end{aligned}$$

Further, all solutions x are congruent to each other modulo $N = \prod_{i=1}^{k} n_i$.

For arbitrary k: Let $m_i = N/n_i$

Note $\gcd(m_i, n_i) = 1$

$n_i \mid m_j$ for $j \neq i$

Take $x = b_1 (m_1^{-1} \bmod n_1)\, m_1 + b_2 (m_2^{-1} \bmod n_2)\, m_2 + \ldots + b_k (m_k^{-1} \bmod n_k)\, m_k$

First term contributes the remainder mod n_1 (rest are divisible by n_1), …. ,

k'th term contributes the remainder mod n_k

Quick Recap: Groups

Recap: Definition of a **group**

G is a "**group** under operation $\bullet$" if:

0. [Closure] G is closed under $\bullet$

   i.e., $a \bullet b \in G \qquad \forall\, a,b \in G$

1. [Associativity] Operation $\bullet$ is associative:

   i.e., $a \bullet (b \bullet c) = (a \bullet b) \bullet c \qquad \forall\, a,b,c \in G$

2. [Identity] There exists an element $e \in G$

   (called the "identity element") such that

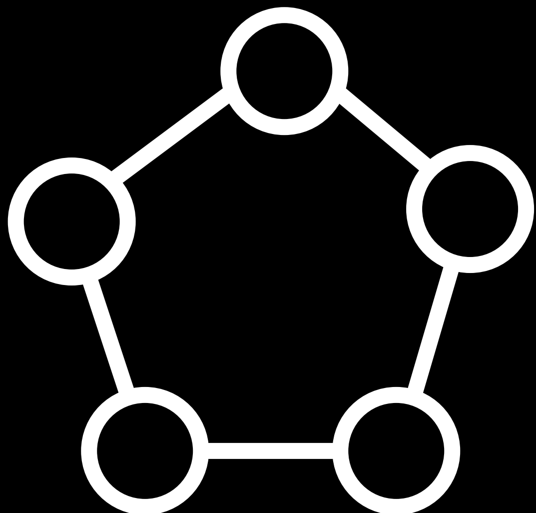
   $a \bullet e = a, \; e \bullet a = a \qquad \forall\, a \in G$

3. [Inverse] For each $a \in G$ there is an element $a^{-1} \in G$

   (called the "inverse of a") such that

   $a \bullet a^{-1} = e, \; a^{-1} \bullet a = e$

Symmetries of undirected cycle: dihedral group



$G =$

$\{ Id, r_1, r_2, r_3, r_4, f_1, f_2, f_3, f_4, f_5 \}$

•	Id	r_1	r_2	r_3	r_4	f_1	f_2	f_3	f_4	f_5
Id	Id	r_1	r_2	r_3	r_4	f_1	f_2	f_3	f_4	f_5
r_1	r_1	r_2	r_3	r_4	Id	f_4	f_5	f_1	f_2	f_3
r_2	r_2	r_3	r_4	Id	r_1	f_2	f_3	f_4	f_5	f_1
r_3	r_3	r_4	Id	r_1	r_2	f_5	f_1	f_2	f_3	f_4
r_4	r_4	Id	r_1	r_2	r_3	f_3	f_4	f_5	f_1	f_2
f_1	f_1	f_3	f_5	f_2	f_4	Id	r_3	r_1	r_4	r_2
f_2	f_2	f_4	f_1	f_3	f_5	r_2	Id	r_3	r_1	r_4
f_3	f_3	f_5	f_2	f_4	f_1	r_4	r_2	Id	r_3	r_1
f_4	f_4	f_1	f_3	f_5	f_2	r_1	r_4	r_2	Id	r_3
f_5	f_5	f_2	f_4	f_1	f_3	r_3	r_1	r_4	r_2	Id

Abelian groups

In a group we do NOT NECESSARILY have

$$a \bullet b = b \bullet a$$

Definition:

"$a,b \in G$ commute" means $ab = ba$.

Definition:

A group is said to be abelian if all pairs $a,b \in G$ commute.

Order of a group element

Let G be a ***finite*** group. Let $a \in G$.

Definition: The order of x, denoted ord(a), is the

smallest $m \geq 1$ such that $a^m = 1$.

Note that $a, a^2, a^3, \ldots, a^{m-1}, a^m=1$ all distinct.

Order Theorem: For every $a \in G$,

ord(a) divides $|G|$.

Corollary: $a^{|G|}=1$ for all $a \in G$.

Corollary (Euler's Theorem): For $a \in Z_n^*$, $a^{\phi(n)} = 1$

That is, if $\gcd(a,n)=1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$

Corollary (Fermat's little theorem):

For prime p, if $\gcd(a,p)=1$, then

$a^{p-1} \equiv 1 \pmod{p}$

Cyclic groups

A finite group G of order n is cyclic if
G= $\{e,b,b^2,\ldots,b^{n-1}\}$ for some group element b

In such a case, we say b ***"generates"*** G,
or b is a *"generator"* of G.

Examples:

- $(Z_n, +)$ (1 is a generator)
- C_4 (Rot_{90} is a generator)

Non-examples: Mattress group; dihedral group;
any non-abelian group.

Lagrange's Theorem: If G is a finite group, and H is a subgroup then |H| divides |G|.

A useful corollary: If G is a finite group and H is a proper subgroup of G, then $|H| \leq |G|/2$

Feature Presentation:
Field Theory

Find out about the wonderful world of $\mathbb{F}_{2^k}$
where two equals zero, plus is minus,
and squaring is a linear operator!

– Richard Schroeppel

A group is a set with a single binary operation.

Number-theoretic sets often have more than one operation defined on them.

For example, in $\mathbb{Z}$, we can do both addition and multiplication.

Same in Z_n (we can add and multiply modulo n)

For reals $\mathbb{R}$ or rationals $\mathbb{Q}$, we can also divide (inverse operation for multiplication).

Fields

Informally, it's a place where you can
add, subtract, multiply, and divide.

Examples:

Real numbers	$\mathbb{R}$	
Rational numbers	$\mathbb{Q}$	
Complex numbers	$\mathbb{C}$	
Integers mod ***prime***	Z_p	(Why?)

NON-examples:

Integers $\mathbb{Z}$	division??
Non-negative reals $\mathbb{R}^+$	subtraction??

Field – formal definition

A ***field*** is a set F with *two* binary operations, called + and •.

(F,+) an abelian group, with identity element called 0

(F \ {0},•) an abelian group, identity element called 1

Distributive Law holds:

a•(b+c) = a•b + a•c

Example:

$\mathbb{F}_3 = Z_3^*$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

•	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Fields: familiar examples

Real numbers	$\mathbb{R}$
Rational numbers	$\mathbb{Q}$
Complex numbers	$\mathbb{C}$
Integers mod ***prime***	Z_p

The last one is a finite field

Example

Quadratic "number field"

$$\mathbb{Q}(\sqrt{2}) = \{ a + b\sqrt{2} : a,b \in \mathbb{Q} \}$$

Addition: $(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a+c) + (b+d)\sqrt{2}$

Multiplication:

$$(a + b\sqrt{2}) \bullet (c + d\sqrt{2}) = (ac+2bd) + (ad+bc)\sqrt{2}$$

Exercise: Prove above defines a field.

Finite fields

Some familiar *infinite* fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ (now $\mathbb{Q}(\sqrt{2})$)

Finite fields we know: Z_p aka $\mathbb{F}_p$ for p a prime

Is there a field with 2 elements? Yes

Is there a field with 3 elements? Yes

Is there a field with 4 elements? Yes

$\mathbb{F}_4$

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

•	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Evariste Galois (1811−1832) introduced the concept of a finite field (also known as a Galois Field in his honor)

Finite fields

Is there a field with 2 elements?	Yes
Is there a field with 3 elements?	Yes
Is there a field with 4 elements?	Yes
Is there a field with 5 elements?	Yes
Is there a field with 6 elements?	No
Is there a field with 7 elements?	Yes
Is there a field with 8 elements?	Yes
Is there a field with 9 elements?	Yes
Is there a field with 10 elements?	No

Finite fields

Theorem (which we won’t prove):

There is a field with q elements
if and only if q is a power of a prime.

Up to *isomorphism*, it is unique.

That is, all fields with q elements have the
same addition and multiplication tables,
after renaming elements.

This field is denoted $\mathbb{F}_q$ (also $GF(q)$)

Finite fields

Question:

If q is a prime power but not just a prime, what ***are*** the addition and multiplication tables of $\mathbb{F}_q$?

Answer:

It's a bit hard to describe.

We'll tell you later, but for 251's purposes, you mainly only need to know about prime q.

Polynomials

Polynomials

Informally, a polynomial is an expression that looks like this:

$$6x^3 - 2.3x^2 + 5x + 4.1$$

x is a symbol, called the *variable* *(or indeterminate)*

the 'numbers' standing next to powers of x are called the *coefficients*

Polynomials

Informally, a polynomial is an expression that looks like this:

$\in \mathbb{R}[x]$

$$6x^3 - 2.3x^2 + 5x + 4.1$$

Actually, coefficients can come from any *field*.

Can allow multiple variables, but we won't.

Set of polynomials with variable x and coefficients from field F is denoted **F[x]**.

Polynomials – formal definition

Let F be a field and let x be a variable symbol.

$F[x]$ is the set of polynomials over F,

defined to be expressions of the form

$$c_d\, x^d + c_{d-1}\, x^{d-1} + \cdots + c_2\, x^2 + c_1\, x + c_0$$

where each c_i is in F, and $c_d \neq 0$.

We call d the degree of the polynomial.

Also, the expression 0 is a polynomial.

(By convention, we call its degree $-\infty$.)

Adding and multiplying polynomials

You can add and multiply polynomials.

Example. Here are two polynomials in $\mathbb{F}_{11}[x]$

$$P(x) = x^2 + 5x - 1$$
$$Q(x) = 3x^3 + 10x$$

$$P(x) + Q(x) = 3x^3 + x^2 + 15x - 1$$
$$= 3x^3 + x^2 + 4x - 1$$
$$= 3x^3 + x^2 + 4x + 10$$

Adding and multiplying polynomials

You can add and multiply polynomials (they are a "ring" but we'll skip a formal treatment of rings)

Example. Here are two polynomials in $\mathbb{F}_{11}[x]$

$$P(x) = x^2 + 5x - 1$$
$$Q(x) = 3x^3 + 10x$$

$$\begin{aligned} P(x) \bullet Q(x) &= (x^2 + 5x - 1)(3x^3 + 10x) \\ &= 3x^5 + 15x^4 + 7x^3 + 50x^2 - 10x \\ &= 3x^5 + 4x^4 + 7x^3 + 6x^2 + x \end{aligned}$$

Adding and multiplying polynomials

Polynomial addition is associative and commutative.

$0 + P(x) = P(x) + 0 = P(x)$.

$P(x) + (-P(x)) = 0$.

So $(F[x], +)$ is an abelian group!

Polynomial multiplication is associative and commutative.

$1 \cdot P(x) = P(x) \cdot 1 = P(x)$.

Multiplication distributes over addition:

$$P(x) \cdot (Q(x) + R(x)) = P(x) \cdot Q(x) + P(x) \cdot R(x)$$

If $P(x) / Q(x)$ were always a polynomial,

then $F[x]$ would be a field! Alas…

Dividing polynomials?

P(x) / Q(x) is not necessarily a polynomial.

So F[x] is not quite a field.

(It's a "ring")

Same with $\mathbb{Z}$, the integers:

it has everything except division.

Actually, there are many analogies between F[x] and $\mathbb{Z}$.

- starting point for rich interplay between algebra, arithmetic, and geometry in mathematics

Dividing polynomials?

$\mathbb{Z}$ has the concept of "division with remainder":

Given $a,b \in \mathbb{Z}$, $b \neq 0$, can write

$$a = q \cdot b + r,$$

where r is "smaller than" b.

$F[x]$ has the same concept:

Given $A(x), B(x) \in F[x]$, $B(x) \neq 0$, can write

$$A(x) = Q(x) \cdot B(x) + R(x),$$

where $\deg(R(x)) < \deg(B(x))$.

"Division with remainder" for polynomials

Example: Divide $6x^4+8x+1$ by $2x^2+4$ in $\mathbb{F}_{11}[x]$

$$
\begin{array}{r|l}
 & 3x^2 \;+5 \\
\hline
2x^2+4 & 6x^4+8x+1 \\
- & 6x^4+x^2 \\
\hline
 & \quad -x^2+8x+1 \\
 - & \quad -x^2+9 \\
\hline
 & \qquad 8x+3
\end{array}
$$

Check:

$6x^4+8x+1$

$= (3x^2+5)(2x^2+4)+(8x+3)$

(in $\mathbb{F}_{11}[x]$)

Integers $\mathbb{Z}$

"size" = absolute value

"division":

$a = qb + r, \quad |r| < |b|$

can use Euclid's Algorithm to find GCDs

p is "prime": no nontrivial divisors

$\mathbb{Z}$ mod p: a field iff p is prime

Polynomials F[x]

"size" = degree

"division":

$A(x) = Q(x)B(x)+R(x),$
$\deg(R) < \deg(B)$

can use Euclid's Algorithm to find GCDs

P(x) is "irreducible": no nontrivial divisors

F[x] mod P(x): a field iff P(x) is irreducible (with $|F|^{\deg(P)}$ elements)

The field with 4 elements

Degree < 2 polynomials $\{0,1,x,1+x\} \subseteq \mathbb{F}_2[x]$

Addition and multiplication modulo $1+x+x^2$

$\mathbb{F}_4$

a=x
b=1+x

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

•	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

The field with p^d elements

Degree < d polynomials $\subseteq \mathbb{F}_p[x]$

Addition and multiplication modulo h(x), which is any degree d ***irreducible*** polynomial in $\mathbb{F}_p[x]$

- Fact: Irreducibles of every degree exist in $\mathbb{F}_p[x]$

Field with 9 elements:

$\mathbb{F}_3[x]$ mod (x^2+1)

Field with 8 elements:

$\mathbb{F}_2[x]$ mod (x^3+x+1)

Enough algebraic theory.

Let’s play with polynomials!

Evaluating polynomials

Given a polynomial $P(x) \in F[x]$,

$P(a)$ means its evaluation at element a.

E.g., if $P(x) = x^2+3x+5$ in $\mathbb{F}_{11}[x]$

$P(6) = 6^2+3\cdot 6+5 = 36+18+5 = 59 = 4$

$P(4) = 4^2+3\cdot 4+5 = 16+12+5 = 33 = 0$

Definition: α is a **root** of $P(x)$ if $P(\alpha) = 0$.

Polynomial roots

Theorem:

Let $P(x) \in F[x]$ have degree 1.

Then $P(x)$ has exactly 1 root.

Proof:

Write $P(x) = cx + d$ (where $c \neq 0$).

$$
\begin{aligned}
\text{Then } P(r) = 0 \quad &\Leftrightarrow \quad cr + d = 0 \\
&\Leftrightarrow \quad cr = -d \\
&\Leftrightarrow \quad r = -d/c.
\end{aligned}
$$

Polynomial roots

Theorem:

Let $P(x) \in F[x]$ have degree 2.

Then $P(x)$ has… how many roots??

E.g.: x^2+1…

# of roots over $\mathbb{F}_2[x]$: 1 (namely, 1)

# of roots over $\mathbb{F}_3[x]$: 0

# of roots over $\mathbb{F}_5[x]$: 2 (namely, 2 and 3)

# of roots over $\mathbb{R}[x]$: 0

# of roots over $\mathbb{C}[x]$: 2 (namely, i and −i)

The single most important theorem about polynomials over fields:

A nonzero degree-d polynomial has at most d roots.

Theorem: Over a field, for all $d \geq 0$, a nonzero degree-d polynomial P has at most d roots.

Proof by induction on $d \in \mathbb{N}$:

Recall our convention: $\deg(0) = -\infty$

Base case: If $P(x)$ is degree-0 then $P(x) = a$ for some $a \neq 0$. This has 0 roots.

Induction:

Assume true for $d \geq 0$. Let $P(x)$ have degree $d+1$.

If $P(x)$ has 0 roots: we're done! Else let b be a root.

Divide with remainder: $P(x) = Q(x)(x-b) + R(x)$. $(*)$

$\deg(R) < \deg(x-b) = 1$, so $R(x)$ is a constant. Say $R(x)=r$.

Plug $x = b$ into $(*)$: $0 = P(b) = Q(b)(b-b)+r = 0+r = r$.

So $P(x) = Q(x)(x-b)$. Now, $\deg(Q) = d$. $\therefore$ Q has $\leq d$ roots.

$\therefore$ $P(x)$ has $\leq d+1$ roots, completing the induction.

A useful corollary

Theorem: Over a field F, for all $d \geq 0$,
degree-d polynomials have at most d roots.

Corollary: Suppose a polynomial $R(x) \in F[x]$ is such that

(i) R has degree $\leq d$ and
(ii) R has $> d$ roots

Then R must be the 0 polynomial

I've used the above corollary *several times* in my research.

Theorem:

Over a field, degree-d polynomials have at most d roots.

Reminder:

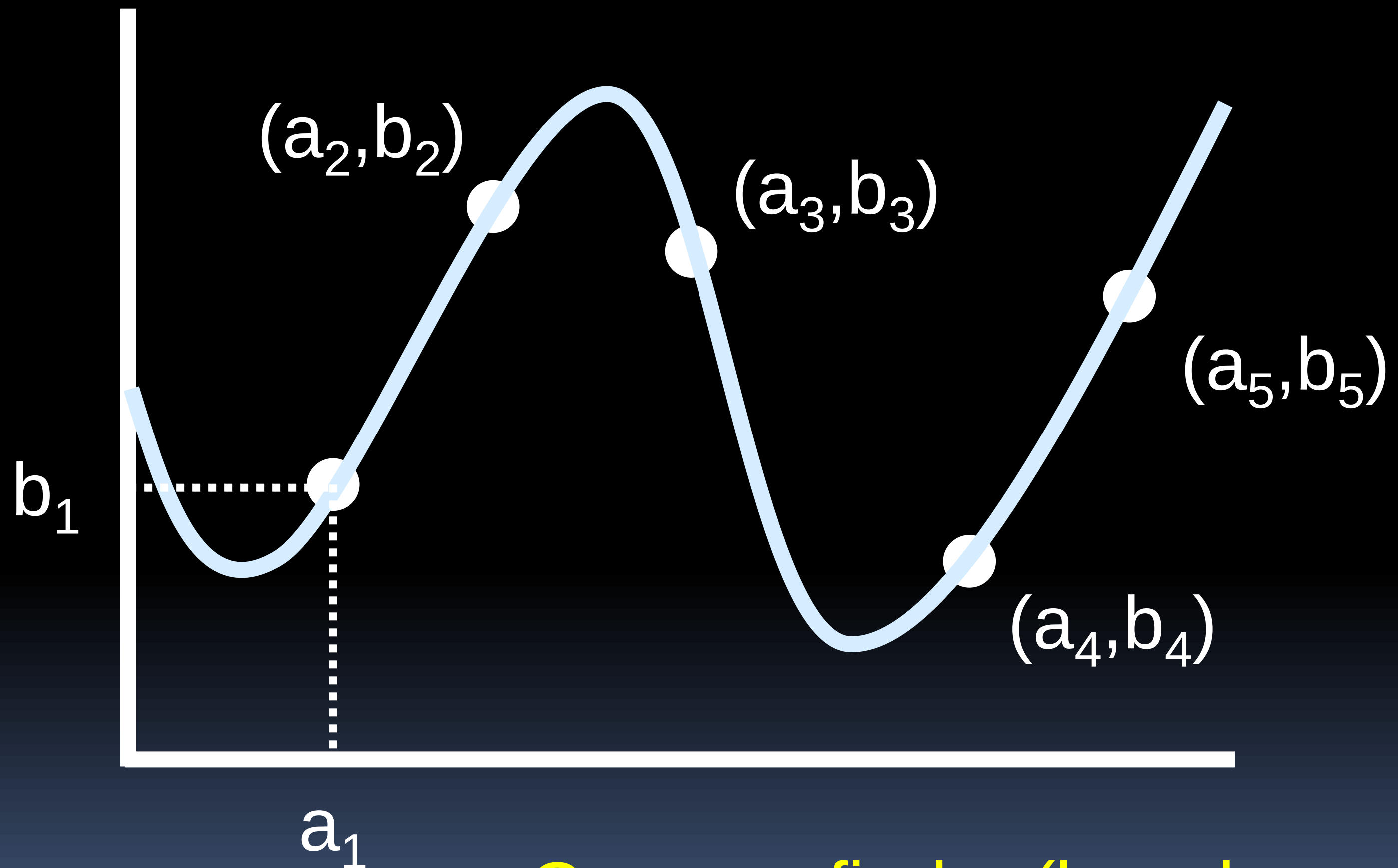
This is only true over a field.

E.g., consider $P(x) = 3x$ over Z_6.

It has degree 1, but 3 roots: 0, 2, and 4.

Interpolation

Say you're given a bunch of "data points"



Can you find a (low-degree) polynomial which "fits the data"?

Interpolation

Let pairs (a_1,b_1), (a_2,b_2), …, (a_{d+1},b_{d+1}) from a field F be given (with all a_i's distinct).

Theorem:

There is exactly one polynomial $P(x)$ of degree at most d such that $P(a_i) = b_i$ for all $i = 1 \ldots d+1$.

E.g., through 2 points there is a unique linear polynomial.

Interpolation

There are two things to prove.

1. There is at *least* one polynomial of degree $\leq d$ passing through all d+1 data points.

2. There is at *most* one polynomial of degree $\leq d$ passing through all d+1 data points.

Let's prove #2 first.

Interpolation

<u>Theorem</u>: Let pairs (a_1,b_1), (a_2,b_2), …, (a_{d+1},b_{d+1}) from a field F be given (with all a_i's distinct). Then there is at most one polynomial $P(x)$ of degree at most d with $P(a_i) = b_i$ for all i.

<u>Proof</u>: Suppose $P(x)$ and $Q(x)$ both do the job.

Let $R(x) = P(x)-Q(x)$.

Since $\deg(P), \deg(Q) \leq d$ we must have $\deg(R) \leq d$.

But $R(a_i) = b_i-b_i = 0$ for all $i = 1\ldots d+1$.

Thus $R(x)$ has more roots than its degree.

$\therefore$ $R(x)$ must be the 0 polynomial, i.e., $P(x)=Q(x)$.

Interpolation

Now let's prove the other part,
that there is at least one polynomial.

Theorem:

Let pairs (a_1,b_1), (a_2,b_2), …, (a_{d+1},b_{d+1}) from a field F be given (with all a_i's distinct).
Then there exists a polynomial $P(x)$ of degree at most d with $P(a_i) = b_i$ for all i.

Interpolation

The method for constructing the polynomial is called Lagrange Interpolation.

Discovered in 1779 by Edward Waring.

Rediscovered in 1795 by J.-L. Lagrange.

Lagrange Interpolation

a_1	b_1
a_2	b_2
a_3	b_3
…	…
a_d	b_d
a_{d+1}	b_{d+1}

Want $P(x)$

(with degree $\leq d$)

such that $P(a_i) = b_i \ \forall i$.

Lagrange Interpolation

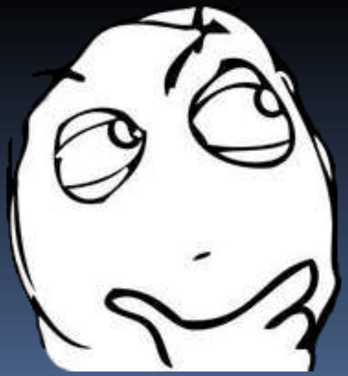
a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0

Can we do this special case?

Promise: once we solve this special case, the general case is very easy.

Lagrange Interpolation

a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0



Lagrange Interpolation

a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0

Just divide P(x) by this number.

Idea #1: $P(x) = (x-a_2)(x-a_3)\cdots(x-a_{d+1})$

Degree is d. ✓

$P(a_2) = P(a_3) = \cdot\cdot\cdot = P(a_{d+1}) = 0.$ ✓

$P(a_1) = (a_1-a_2)(a_1-a_3)\cdots(a_1-a_{d+1}).$??

Lagrange Interpolation

Numerator is a deg. d polynomial

a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0

Denominator is a nonzero field element

Idea #2:

$$S_1(x) = \frac{(x-a_2)(x-a_3)\cdots(x-a_{d+1})}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_{d+1})}$$

Call this the selector polynomial for a_1.

Lagrange Interpolation

a_1	0
a_2	1
a_3	0
…	…
a_d	0
a_{d+1}	0

Great! But what about this data?

$$S_2(x) = \frac{(x-a_1)(x-a_3)\cdots(x-a_{d+1})}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_{d+1})}$$

Lagrange Interpolation

a_1	0
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	1

Great! But what about this data?

$$S_{d+1}(x) = \frac{(x - a_1)(x - a_2)\cdots(x - a_d)}{(a_{d+1} - a_1)(a_{d+1} - a_2)\cdots(a_{d+1} - a_d)}$$

Lagrange Interpolation

a_1	b_1
a_2	b_2
a_3	b_3
…	…
a_d	b_d
a_{d+1}	b_{d+1}

Great! Finally, what about this data?

$$P(x) = b_1 \cdot S_1(x) + b_2 \cdot S_2(x) + \cdots + b_{d+1} \cdot S_{d+1}(x)$$

Lagrange Interpolation – example

Over Z_{11}, find a polynomial P of degree ≤ 2 such that $P(5) = 1$, $P(6) = 2$, $P(7) = 9$.

$$S_5(x) = 6\,(x-6)(x-7)$$

$$\frac{1}{(5-6)(5-7)}$$

$$S_6(x) = -1\,(x-5)(x-7)$$

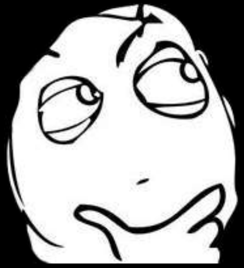
$$S_7(x) = 6\,(x-5)(x-6)$$

$$P(x) = 1\,S_5(x) + 2\,S_6(x) + 9\,S_7(x)$$

$$P(x) = 6(x^2-13x+42) - 2(x^2-12x+35) + 54(x^2-11x+30)$$

$$P(x) = 3x^2+x+9$$

The Chinese Remainder Theorem had a very similar proof



Not a coincidence:

algebraically, integers & polynomials share many common properties

Lagrange interpolation is the *exact analog* of Chinese Remainder Theorem for polynomials.

Chinese Remainder Theorem: Suppose $n_1, n_2, \ldots, n_k$ are pairwise coprime. Then, for all integers $a_1, a_2, \ldots, a_k$, there exists an integer x solving the below system of simultaneous congruences

$$x \equiv a_1 \pmod{n_1}$$

$$x \equiv a_2 \pmod{n_2}$$

$$\vdots$$

$$x \equiv a_k \pmod{n_k} .$$

Further, all solutions x are congruent modulo $N = \prod_{i=1}^{k} n_i$.

Let $m_i = N/n_i$

i'th "selector" number: $T_i = (m_i^{-1} \bmod n_i)\, m_i$

$$x = a_1 T_1 + a_2 T_2 + \ldots + a_k T_k$$

Recall: Interpolation

Let pairs (a_1,b_1), (a_2,b_2), …, (a_{d+1},b_{d+1}) from a field F be given (with all a_i's distinct).

Theorem:

There is a unique degree d polynomial $P(x)$ satisfying $P(a_i) = b_i$ for all $i = 1 \ldots d+1$.

A linear algebra view

Let $p(x) = p_0 + p_1 x + p_2\, x^2 + \ldots + p_d\, x^d$
Need to find the coefficient vector $(p_0, p_1, \ldots, p_d)$

$$p(a) = p_0 + p_1\, a + \ldots + p_d\, a^d$$
$$= 1 \cdot p_0 + a \cdot p_1 + a^2 \cdot p_2 + \ldots + a^d \cdot p_d$$

Thus we need to solve:

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^d \\ 1 & a_2 & a_2^2 & \cdots & a_2^d \\ & & \vdots & & \\ 1 & a_{d+1} & a_{d+1}^2 & \cdots & a_{d+1}^d \end{pmatrix} \cdot \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_d \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{d+1} \end{pmatrix}$$

Lagrange interpolation

The $(d+1) \times (d+1)$ Vandermonde matrix

$$M = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^d \\ 1 & a_2 & a_2^2 & \cdots & a_2^d \\ & & \vdots & & \\ 1 & a_{d+1} & a_{d+1}^2 & \cdots & a_{d+1}^d \end{pmatrix}$$

is **invertible**.

- The determinant of M is nonzero when a_i's are distinct.

Thus can recover coefficient vector **as** $\vec{p} = M^{-1}\vec{b}$

The columns of M^{-1} are given by the coefficients of the various "selector" polynomials we constructed in Lagrange interpolation.

Representing Polynomials

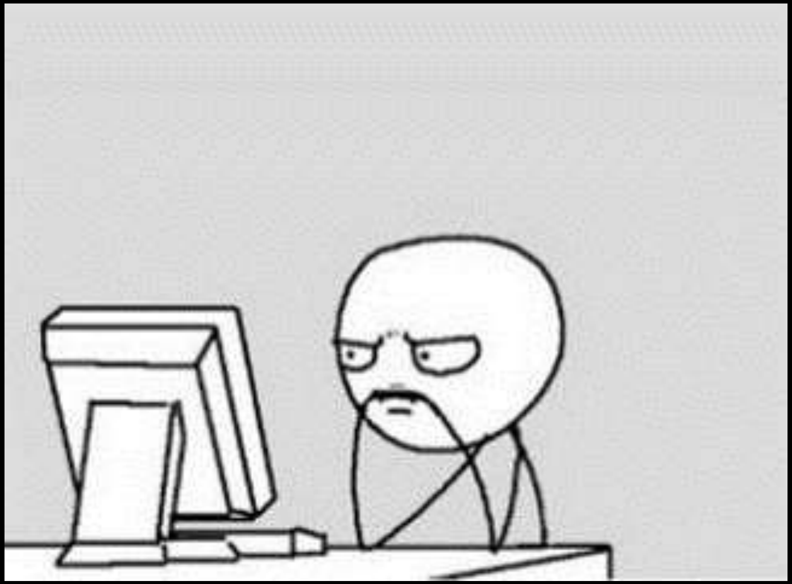
Let $P(x) \in F[x]$ be a degree-d polynomial.

Representing $P(x)$ using $d+1$ field elements:

1. List the $d+1$ coefficients.
2. Give P's value at $d+1$ different elements.

Rep 1 to Rep 2: Evaluate at $d+1$ elements

Rep 2 to Rep 1: Lagrange Interpolation



Study Guide

Number Theory:

Unique factorization

Chinese Remainder theorem

Fields:

Definitions

Examples

Finite fields of prime order

Polynomials:

Degree-d polys have ≤ d roots.

Polynomial division with remainder

Lagrange Interpolation