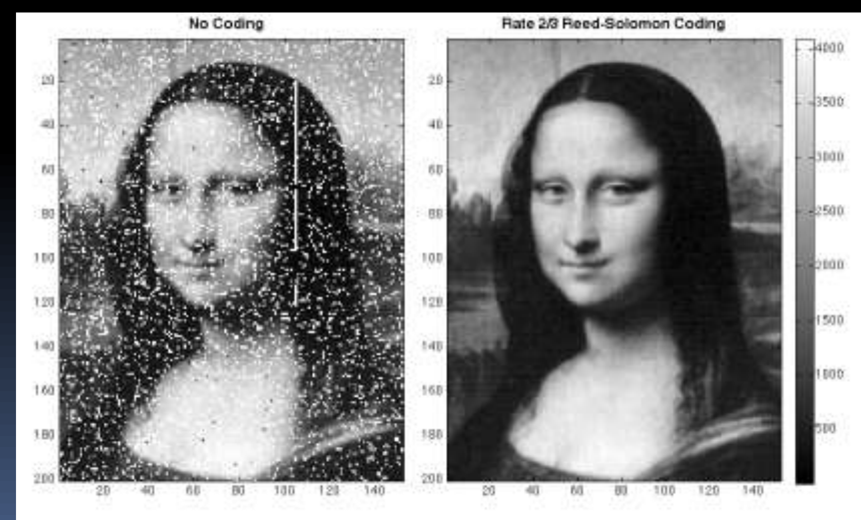
15-251: Great Theoretical Ideas in Computer Science

Fall 2016 Lecture 26

November 29, 2016

Error Correction



Recap: Polynomial Interpolation

Theorem:

Let arbitrary pairs (a_1,b_1), (a_2,b_2), …, (a_{d+1},b_{d+1}) from a field F be given (with all a_i's distinct). Then there always exists a unique polynomial $P(x)$ of degree $\leq d$ with $P(a_i) = b_i$ for all i.

- Uniqueness follows because a degree $\leq d$ polynomial has $\leq d$ distinct roots.
- Can construct a polynomial with $P(a_i) = b_i$ via Lagrange interpolation.

Lagrange Interpolation

a_1	b_1
a_2	b_2
a_3	b_3
…	…
a_d	b_d
a_{d+1}	b_{d+1}

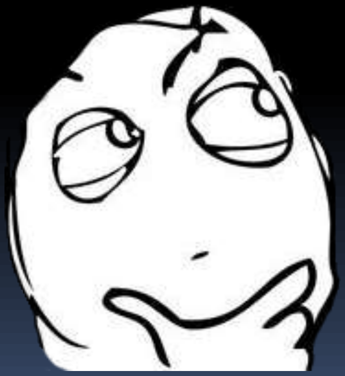
Want $P(x)$

(with degree $\leq$ d)

such that $P(a_i) = b_i \quad \forall i$.

Lagrange Interpolation

a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0



Can we do this special case?

Lagrange Interpolation

a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0

Numerator is a deg. d polynomial

Denominator is a nonzero field element

$$S_1(x) = \frac{(x-a_2)(x-a_3)\cdots(x-a_{d+1})}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_{d+1})}$$

Call this the selector polynomial for a_1.

a_1	0
a_2	1
a_3	0
…	…
a_d	0
a_{d+1}	0

What about above data?

$$S_2(x) = \frac{(x - a_1)(x - a_3)\cdots(x - a_{d+1})}{(a_2 - a_1)(a_2 - a_3)\cdots(a_2 - a_{d+1})}$$

a_1	0
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	1

And for this data,

$$S_{d+1}(x) = \frac{(x - a_1)(x - a_2)\cdots(x - a_d)}{(a_{d+1} - a_1)(a_{d+1} - a_2)\cdots(a_{d+1} - a_d)}$$

Polynomial Interpolation

a_1	b_1
a_2	b_2
a_3	b_3
$\ldots$	$\ldots$
a_d	b_d
a_{d+1}	b_{d+1}

$$P(x) = b_1 \cdot S_1(x) + b_2 \cdot S_2(x) + \cdots + b_{d+1} \cdot S_{d+1}(x)$$

Recall: Interpolation

Let pairs (a_1,b_1), (a_2,b_2), …, (a_{d+1},b_{d+1}) from a field F be given (with all a_i's distinct).

Theorem:

There is a unique degree d polynomial $P(x)$ satisfying $P(a_i) = b_i$ for all $i = 1 \ldots d+1$.

A linear algebra view

Let $p(x) = p_0 + p_1 x + p_2\, x^2 + \ldots + p_d\, x^d$
Need to find the coefficient vector $(p_0, p_1, \ldots, p_d)$

$$p(a) = p_0 + p_1\, a + \ldots + p_d\, a^d$$
$$= 1 \cdot p_0 + a \cdot p_1 + a^2 \cdot p_2 + \ldots + a^d \cdot p_d$$

Thus we need to solve:

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^d \\ 1 & a_2 & a_2^2 & \cdots & a_2^d \\ & & \vdots & & \\ 1 & a_{d+1} & a_{d+1}^2 & \cdots & a_{d+1}^d \end{pmatrix} \cdot \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_d \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{d+1} \end{pmatrix}$$

Lagrange interpolation

The $(d+1) \times (d+1)$ Vandermonde matrix

$$M = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^d \\ 1 & a_2 & a_2^2 & \cdots & a_2^d \\ & & \vdots & & \\ 1 & a_{d+1} & a_{d+1}^2 & \cdots & a_{d+1}^d \end{pmatrix}$$

is **invertible**.

- The determinant of M is nonzero when a_i's are distinct.

Thus can recover coefficient vector **as** $\vec{p} = M^{-1}\vec{b}$

The columns of M^{-1} are given by the coefficients of the various "selector" polynomials we constructed in Lagrange interpolation.

Representing Polynomials

Let $P(x) \in F[x]$ be a degree-d polynomial.

Representing $P(x)$ using $d+1$ field elements:

1. List the $d+1$ coefficients.
2. Give P's value at $d+1$ different elements.

Rep 1 to Rep 2: Evaluate at $d+1$ elements

Rep 2 to Rep 1: Lagrange Interpolation

Application of Fields/Polynomials

(and linear algebra):

Error-correcting codes

Sending messages on a noisy channel

Alice Bob

“ bit.ly/vrxUBN ”

The channel may corrupt up to k symbols.

How can Alice still get the message across?

Sending messages on a noisy channel

Let's say messages are sequences from $\mathbb{F}_{257}$

vrxUBN $\leftrightarrow$ 118 114 120 85 66 78

noisy channel

vrxUBN $\leftrightarrow$ 118 114 104 85 35 78

The channel may corrupt up to k symbols.

How can Alice still get the message across?

Sending messages on a noisy channel

Let's say messages are sequences from $\mathbb{F}_{257}$

vrxUBN $\leftrightarrow$ 118 114 120 85 66 78

noisy channel $\downarrow$

vrxUBN $\leftrightarrow$ 118 114 104 85 35 78

How to correct the errors?

How to even detect that there *are* errors?

Simpler case: "Erasures"

118 114 120 85 66 78

erasure channel

118 114 ?? 85 ?? 78

What can you do to handle up to k erasures?

Repetition code

Have Alice repeat each symbol k+1 times.

118 114 120 85 66 78

becomes

118 118 118 114 114 114 120 120 120 85 85 85 66 66 66 78 78 78

erasure channel

118 118 118 ?? ?? 114 120 120 120 85 85 85 66 66 66 78 78 78

If at most k erasures, Bob can figure out each symbol.

Repetition code – noisy channel

Have Alice repeat each symbol 2k+1 times.

118 114 120 85 66 78

becomes

118 118 118 114 114 114 120 120 120 85 85 85 66 66 66 78 78 78

noisy channel

118 118 118 114 223 114 120 120 120 85 85 85 66 66 66 78 78 78

At most k corruptions: Bob can take majority of each block.

This is pretty wasteful!

To send message of d+1 symbols and guard against k erasures, we had to send (d+1)(k+1) total symbols.

Can we do better?

This is pretty wasteful!

To send message of d+1 symbols and

guard against k erasures, we had

to send (d+1)(k+1) total symbols.

To send even 1 message symbol with
k erasures, ***need*** to send k+1 total symbols.

Maybe for d+1 message symbols with k erasures,
d+k+1 total symbols can suffice??

Enter polynomials

Say Alice's message is $d+1$ elements from $\mathbb{F}_{257}$

118 114 120 85 66 78

Alice thinks of it as the coefficients of a degree-d polynomial $P(x) \in \mathbb{F}_{257}[x]$

$$P(x) = 118x^5 + 114x^4 + 120x^3 + 85x^2 + 66x + 78$$

Now trying to send the degree-d polynomial P(x).

Send it in the Values Representation!

$P(x) = 118x^5 + 114x^4 + 120x^3 + 85x^2 + 66x + 78$

Alice sends P(x)'s values on d+k+1 inputs:

$P(1), P(2), P(3), \ldots, P(d+k+1)$

This is called the **Reed–Solomon encoding**.

Send it in the Values Representation!

$$P(x) = 118x^5 + 114x^4 + 120x^3 + 85x^2 + 66x + 78$$

Alice sends P(x)'s values on d+k+1 inputs:

P(1), P(2), P(3), …, P(d+k+1)

If there are at most k erasures, then
Bob still knows P's value on d+1 points.

Bob recovers P(x) using Lagrange Interpolation!

Example

How good is our encoding?

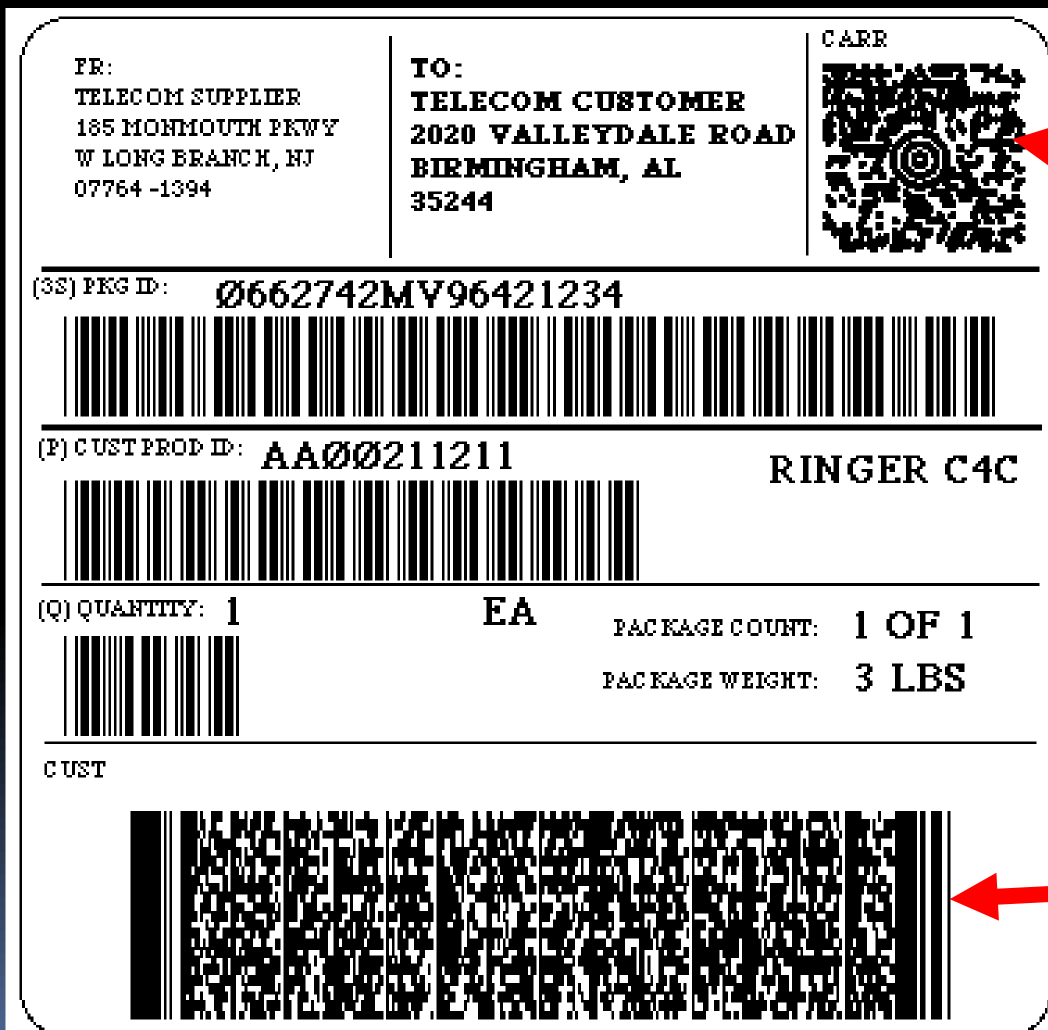
Naïve Repetition:
To send d+1 numbers with k erasure recovery
sent (d+1)(k+1) numbers

Polynomial Coding:
To send d+1 numbers with k erasures recovery,
sent (d+k+1) numbers

Reed−Solomon codes are used a lot in practice!

Another everyday use:

CD/DVDs, hard discs, satellite communication, …



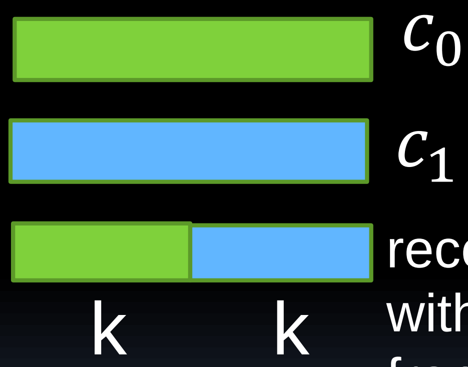
Maxicodes
= “UPS codes”
= another 2-d
Reed-Solomon codes

PDF417 codes
= 2-d Reed-Solomon
codes

What about corruptions/errors

To send message of d+1 symbols
and enable correction from up to k errors,
repetition code has to send (d+1)(2k+1) total symbols.

To communicate even 1 symbol while enabling recovery from k errors, ***need*** to send at least 2k+1 total codeword symbols.



received word within k errors from both c_0 and c_1

Maybe for d+1 message symbols with k errors,
d+2k+1 total symbols can suffice??

Want to send a polynomial of degree-d subject to at most k corruptions.

First simpler problem: **Error detection**

Suppose we try the same idea

- Evaluate P(X) at d+1+k points
- Send P(0), P(1), P(2), …, P(d+k)

At least d+1 of these values will be unchanged (because we assume at most k errors)

Example

$P(X) = 2X^2 + 1$, and $k = 1$.

So I sent $P(0)=1$, $P(1)=3$, $P(2)=9$, $P(3)=19$

Corrupted email says (1, 4, 9, 19)

Choosing (1, 4, 9) will give us $Q(X) = X^2 + 2X + 1$

We can now *detect* (up to k) errors

Evaluate P(X) at d+1+k points

Send P(0), P(1), P(2), …, P(d+k)

Received as P(0), P(1)* , P(2), P(3), P(4)*, …, P(d+k)

- At least d+1 of these values assumed correct

- Using these d+1 correct values for interpolation will give P(X)
- Using any of the incorrect values for interpolation will give *some other polynomial*

Quick way of detecting errors

- Interpolate first d+1 points to get Q(X)
- Check that all other received values are consistent with this polynomial Q(X)
- If all values consistent, no errors.

In this case, we know Q(X) = P(X)

else there were errors…

How good is our encoding?

Naïve Repetition:
To send d+1 numbers with error detection,
sent $(d+1)(k+1)$ numbers

Polynomial Coding:
To send d+1 numbers with error detection,
sent $(d+k+1)$ numbers

How about error ***correction***?

Requires more redundancy

To send d+1 numbers in such a way
that we can correct up to k errors,
need to send d+1+2k numbers.

Similar encoding scheme

Evaluate degree-d P(x) at d+1+2k points

Send P(0), P(1), P(2), …, P(d+2k)

Receive P(0), P(1)* , P(2), P(3), P(4)*, …, P(d+2k)

At least d+1+k of these values will be correct (since we assume at most k corruptions)

Trouble: We do NOT know which ones are correct

Correct polynomial determined from noisy data

Suppose $\langle P(0), P(1), \dots, P(d+2k)\rangle$ is transmitted and $\langle r_0, r_1, \dots, r_{d+k}\rangle$ is received, with $\leq k$ errors

Theorem: $P(X)$ is the unique degree-d polynomial that differs from the received data on $\leq k$ points.

Proof: Clearly, the original polynomial P(X) obeys $P(i) \neq r_i$ for $\leq k$ values of i

Suppose a different degree-d polynomial $Q(X)$ did so as well. Then $P(i) \neq Q(i)$ for $\leq 2k$ values of i. (Why?)

$\Longrightarrow P(i) = Q(i)$ for $\geq (d+2k+1) - 2k = d+1$ values of i

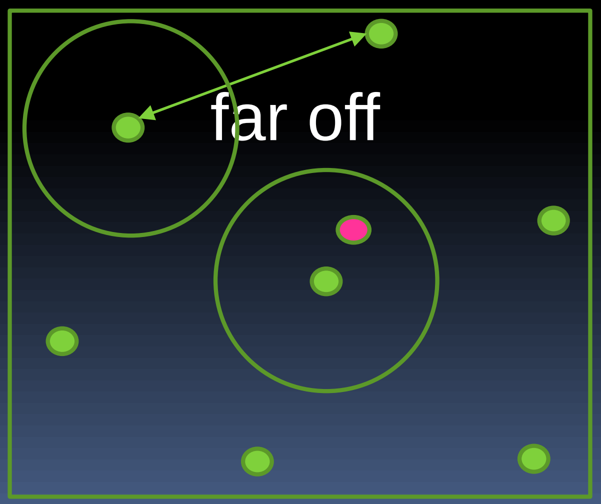
Thus $P(X), Q(X)$ agree with each other on $d+1$ points. So being degree-d polynomials, they must be equal.

A geometric view

The evaluation encodings of two different degree-d polynomials $P(X)$ and $R(X)$ differ on at least $2k+1$ of the $d+2k+1$ points.

Viewed as points in F_{257}^{d+2k+1} their “Hamming distance” (number of positions where they differ) is $\geq 2k+1$

So if $\leq k$ corruptions occur, the original polynomial is the unique closest one (in Hamming distance) to the noisy received word.



Theorem: The transmitted polynomial P(X) is the unique degree-d polynomial that agrees with the received data on at least d+1+k points

Brute-force Algorithm to find P(X):

Interpolate each subset of (d+1) points

Check if the resulting polynomial agrees with received data on d+1+k pts

Takes too much time…

A fast (cubic runtime) algorithm to do error correction and find $P(X)$ was given by [Peterson, 1960]

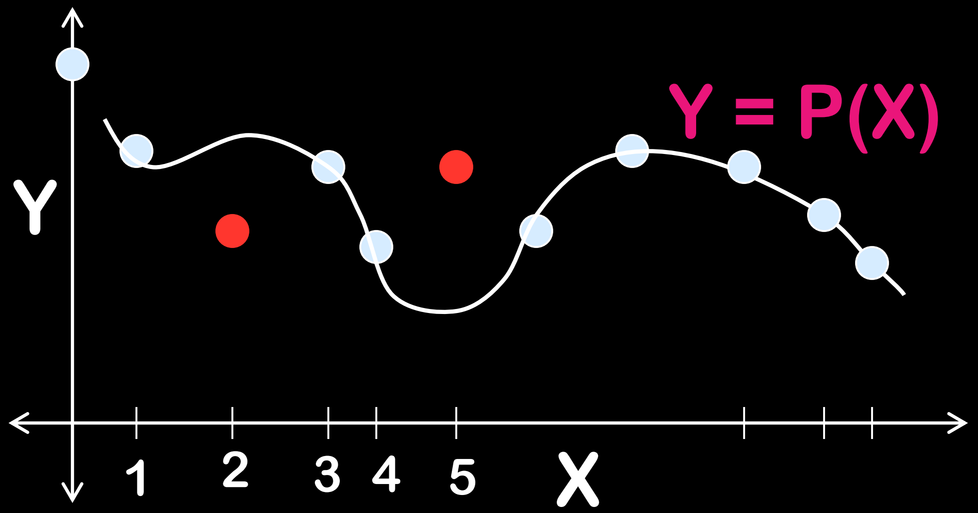
Later improvements by Berlekamp and Massey gave practical algorithms.

We will now sketch an elegant approach (buried in a patent by Welch-Berlekamp) to efficiently recover the original polynomial when there are k corruptions

Locating the errors

Noisy points mess up the interpolation

Let Err := $\{i : P(i) \neq r_i\}$ be the set of error locations



If only we knew the error locations, we'd be done

Define the *error locator polynomial* with roots at error locations:

$$E(X) := \prod_{i \in Err} (X - i)$$

Of course we don't know $E(X)$

A valid equation for all points

Err := $\{i : P(i) \neq r_i\}$
Error locator polynomial: $E(X) \coloneqq \prod_{i \in Err}(X - i)$

Key equation: For ***all*** evaluation points i,

$$E(i) r_i = E(i) P(i)$$

Proof:

- If $i \in \mathrm{Err}$, $E(i) = 0$ so both sides are 0.
- If $i \notin$ Err, $r_i = P(i)$ so both sides are equal.

Define $N(X) \coloneqq E(X)P(X)$; the degree of $N(X)$ is $\leq d + k$

There is a rational function $\mathrm{R(X)} = \frac{N(X)}{E(X)}$
with $\deg(N) \leq d + k$ and $\deg(E) \leq k$
such that $R(i) = r_i$ for $i = 0,1, \ldots, d + 2k$

(Let's say 0/0 is equal to any desired value)

Error-correction algorithm

1. Interpolate a rational function $\tilde{R}(\mathrm{X}) = \frac{\tilde{N}(X)}{\tilde{E}(X)}$ with $\deg(\tilde{E}) \leq k$ and $\deg(\tilde{N}) \leq d + k$ such that $\tilde{R}(i) = r_i$ for $i = 0,1, \ldots, d + 2k$

2. If $\tilde{R}(X)$ is a polynomial of degree $\leq d$, output it; otherwise declare more than k errors occurred.

Efficiency?

Similar to polynomial interpolation, Step 1 can be implemented by solving a system of linear equations (a solution exists by previous slide)

Correctness

Interpolate a rational function $\tilde{R}(\mathrm{X}) = \frac{\tilde{N}(X)}{\tilde{E}(X)}$ with $\deg(\tilde{E}) \leq k$ and $\deg(\tilde{N}) \leq d + k$ such that $\tilde{R}(i) = r_i$ for $i = 0,1, \dots, d + 2k$

Claim: There is a unique such rational function.

Proof: We proved existence of a solution $\mathrm{R}(\mathrm{X}) = \frac{N(X)}{E(X)}$ via error locator polynomial $E(X)$.

If we had another solution $\tilde{R}(\mathrm{X}) = \frac{\tilde{N}(X)}{\tilde{E}(X)}$, then

$$\tilde{N}(i)E(i) = N(i)\tilde{E}(i) \text{ for } i = 0,1, \dots, d + 2k.$$

This implies $\tilde{N}(X)E(X) = N(X)\tilde{E}(X)$ as polynomials (Why?)

So $\tilde{R}(X) = R(X) = P(X)$.

How good is Reed-Solomon encoding?

Naïve Repetition:
To send d+1 numbers with
error correction of up to k corruptions,
sent (d+1)(2k+1) numbers

Polynomial (Reed-Solomon) Coding:
To send d+1 numbers with
error correction of up to k corruptions,
sent (d+2k+1) numbers (optimal!)

Sending messages on a noisy channel

Alice

Um, what if
$d+2k+1 > 257$?

Message: $d+1$ symbols from $\mathbb{F}_{257}$

Reed–Solomon: To guard against k corruptions, treat message as coeffs of poly P, send P(1), P(2), …, P(d+2k+1)

Sending messages on a noisy channel

Alice

Um, what if
d+2k+1 > 257?

What if the noisy
channel corrupts **bits**,
not bytes?
(Can we have fewer
redundant **bits**?)

$\mathbb{F}_{257}$

ainst k corruptions,
ssage as coeffs of poly P,
send P(1), P(2), …, P(d+2k+1)

Sending messages on a noisy channel

Alice wants to send an n-bit message to Bob.

The channel may flip up to k bits.

How can Alice get the message across?

Sending messages on a noisy channel

Alice wants to send an $(n-1)$-bit message to Bob.

The channel may flip up to 1 bit.

How can Alice get the message across?

Q1: How can Bob detect if there's been a bit-flip?

Parity-check solution

Alice tacks on a bit, equal to
the parity of the message's $n-1$ bits.

Alice's n-bit 'encoding' always has
an even number of 1's.

Bob can detect if the channel flips a bit:
if he receives a string with an odd # of 1's.

1-bit error-detection for 2^{n-1} messages
by sending n bits: optimal! (exercise)

Linear Algebra perspective

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

G: an n×(n−1) 'generator' matrix

Alice's message $x \in \mathbb{F}_2^{n-1}$

Bob receives

Linear Algebra perspective

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} \stackrel{?}{=} 0$$

H: a 1×n ‘parity check’ matrix

Bob receives

Bob checks this to detect if no errors

Solves 1-bit error detection, but not correction

If Bob sees z = (1, 0, 0, 0, 0, 0, 0),

did Alice send y = (0, 0, 0, 0, 0, 0, 0),

or y = (1, 1, 0, 0, 0, 0, 0),

or y = (1, 0, 1, 0, 0, 0, 0),

or… ?

The Hamming(7,4) Code

Alice communicates 4-bit messages (16 possible messages)

by transmitting 7 bits.

$$G' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Alice encodes $x \in \mathbb{F}_2^4$ by G' x,

which looks like x followed by 3 extra bits.

The Hamming(7,4) Code

Alice sends 4-bit messages ($x_1x_2x_3x_4$) using 7 'codeword' bits.

Codeword y = Gx satisfies:

$y_3=x_1 \quad y_5=x_2 \quad y_6=x_3 \quad y_7=x_4$

$$y_1 = x_1 + x_2 + x_4$$

$$y_2 = x_1 + x_3 + x_4$$

$$y_4 = x_2 + x_3 + x_7$$

Let's permute the output 7 bits (rows of G')

G =

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Hamming(7,4) Code

Alice sends 4-bit messages using 7 codeword bits.

Any 'codeword' $y = Gx$ satisfies some 'parity checks':

$$y_1 = y_3 + y_5 + y_7$$
$$y_2 = y_3 + y_6 + y_7$$
$$y_4 = y_5 + y_6 + y_7$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

I.e., $Hy = 0$

Let's permute the output 7 bits (rows of G')

$$G = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Hamming(7,4) Code

Alice communicates 4-bit messages using 7 bits.

Columns are 1…7 in binary!

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$Hy = 0$, because $HG = 0$.

The Hamming(7,4) Code

On receiving $z \in \mathbb{F}_2^7$, Bob computes Hz.

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

If no errors, $z = Gx$, so $Hz = HGx = 0$.

If jth coordinate corrupted, $z = Gx+e_j$.

Then $Hz = H(Gx+e_j) = HGx + He_j$

$= He_j =$ (j'th column of H) = binary rep. of j

Bob knows where the error is, can recover msg!

Sending longer messages: General Hamming Code

By sending $n = 7$ bits, Alice can communicate one of 16 messages, guarding against 1 bit flip.

This scheme generalizes: Let $n = 2^r - 1$,
take H to be the $r \times (2^r - 1)$ matrix whose columns are the numbers $1 \ldots 2^r-1$ in binary.

There are $2^{n-r} = 2^n/(n+1)$ solutions $z \in \{0,1\}^n$ to the check equations $Hz = 0$.

- These are *codewords* of the Hamming code of length n

Summary: Parity Check & Hamming code

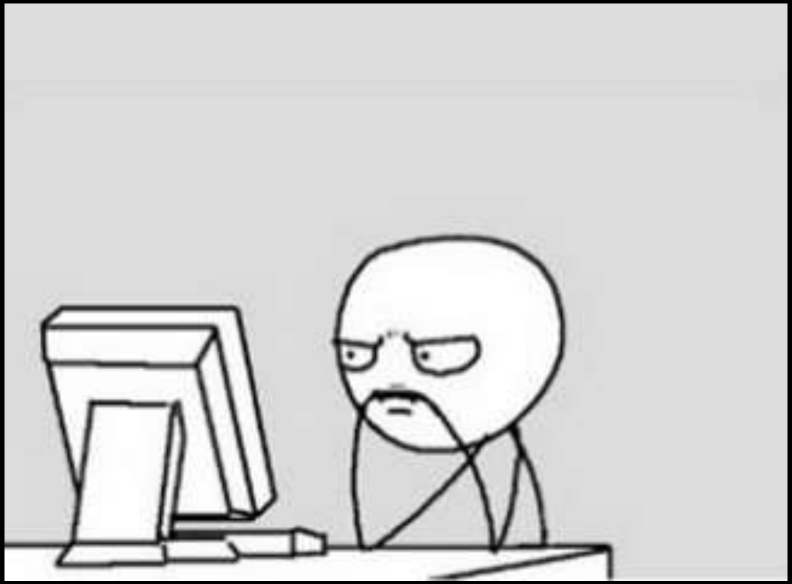
To ***detect*** 1 bit error in n transmitted bits:

- one parity check bit suffices,
- can communicate 2^{n-1} messages by sending n bits.

To ***correct*** 1 bit error in n transmitted bits:

- for $n = 2^r - 1$, r check bits suffice
- can communicate $2^n/(n+1)$ messages by sending n bits

Fact (left as exercise): Both are optimal (in terms of number of messages communicated through n codeword bits)



Study Guide

Polynomials:

Lagrange Interpolation

Reed-Solomon codes:

Erasure correction via interpolation

Error correction

Hamming codes:

Correcting 1 bit error