15-251: Great Theoretical Ideas in Computer Science

Fall 2016 Lecture 27

December 1, 2016

Generating Functions

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} X^n = \frac{1}{(1-X)^k}$$

The Binomial Formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

The polynomial $(1+x)^n$ packages in convenient algebraic form information about the sequence

$\binom{n}{k}$ **k=0,1,…,n**

Generating functions are a formal algebraic view for (infinite) sequences

$(1+x)^n$ is the "generating function" for the sequence

$$\binom{n}{k} \quad k=0,1,\ldots,n$$

Generating functions are a formal algebraic representation for (infinite) sequences

Often, surprisingly powerful representation to understand the sequence!

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$

$$= \frac{1 - X^n}{1 - X}$$

Recall the Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots = \frac{1}{1 - X}$$

the Infinite Geometric Series

Holds when we plug X = a with |a| < 1

But also makes sense if we view the infinite sum on the left as a formal power series in variable X

$$P(X) = 1 + X^1 + X^2 + X^3 + \dots + X^n + \dots$$

$$-X * P(X) = - X^1 - X^2 - X^3 - \dots - X^n - X^{n+1} - \dots$$

$$(1 - X) P(X) = 1$$

$$\Rightarrow P(X) = \frac{1}{1 - X}$$

What is a Generating Function?

Just a particular representation of sequences…

$$\langle 1, 1, 1, \ldots \rangle$$

$$1 + 1x + 1x^2 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

In general, when a_n is a sequence…

$$\sum_{n=0}^{\infty} a_n x^n$$

Formal Power Series

$$P(X) = \sum_{n=0}^{\infty} a_n X^n$$

There are no worries about convergence issues.

This is a purely syntactic object.

Formal Power Series

$$P(X) = \sum_{i=0}^{\infty} a_i X^i$$

If you want, think of as the infinite vector
$V = < a_0, a_1, a_2, \ldots, a_n, \ldots >$

But, as you will see, thinking of as a "polynomial" is very natural and powerful.

…And why would I use one?

They're fun and powerful !

Solving (impossible looking) counting problems

Solving recurrences precisely

Proving identities

In Graham-Knuth-Patashnik's text "Concrete Mathematics: A Foundation for Computer Science", generating functions are described as "the most important idea in this whole book."

Generating functions transform problems about sequences into problems about functions, allowing us to put the piles of machinery available for manipulating functions to work for understanding sequences

Operations on Generating Functions

$$A(X) = a_0 + a_1 X + a_2 X^2 + \dots$$

$$B(X) = b_0 + b_1 X + b_2 X^2 + \dots$$

adding them together

$$(A+B)(X) = (a_0+b_0) + (a_1+b_1) X + (a_2+b_2) X^2 + \dots$$

like adding the vectors position-wise

<4,2,3,…> + <5,1,1,….> = <9,3,4,…>

Operations on Generating Functions

$$A(X) = a_0 X^0 + a_1 X^1 + a_2 X^2 + \dots$$

multiplying by X

$$X * A(X) = 0 X^0 + a_0 X^1 + a_1 X^2 + a_2 X^3 + \dots$$

like shifting the vector entries

SHIFT<4,2,3,…> = <0,4,2,3,…>

Example

Example:

V := <1,0,0,…>;

Loop n times
    V := V + SHIFT(V);

Store:

V = <1,0,0,0,…>
V = <1,1,0,0,…>
V = <1,2,1,0,…>
V = <1,3,3,1,…>

V = n'th row of Pascal's triangle
(binomial coefficients $\binom{n}{k}$)

Example

Example:

V := <1,0,0,…>; $P_V := 1;$

Loop n times
    V := V + SHIFT(V); $P_V := P_V * (1+X);$

V = n^{th} row of Pascal's triangle
(binomial coefficients $\binom{n}{k}$)

Example

Example:

```
V := <1,0,0,…>;

Loop n times
    V := V + SHIFT(V);
```

$$P_V = (1+X)^n$$

As expected, the coefficients of P_V give the binomial coefficients $\binom{n}{k}$

To repeat…

Given a sequence V = < a_0, a_1, a_2, ..., a_n, … >

associate a formal power series with it

$$P(X) = \sum_{i=0}^{\infty} a_i X^i$$

This is the "generating function" for V

Fibonacci Numbers

$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$

i.e., the sequence <0,1,1,2,3,5,8,13…>

is represented by the power series (generating function)

$0 + 1X^1 + 1X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + \ldots$

Two Representations

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$$

$$A(X) = 0 + 1X^1 + 1X^2 + 2\ X^3 + 3\ X^4 + 5\ X^5 + 8\ X^6 + \ldots$$

Can we write A(X) more succinctly?

$$A(X) = F_0 + F_1 X^1 + F2\, X^2 + F_3 X^3 + \ldots + F_n X^n + \ldots$$

$$= X^1 + (F_1 + F_0)X^2 + (F_2+F_1)\, X^3 + \ldots + (F_{n-1} + F_{n-2})\, X^n + \ldots$$

$$= X + \sum_{m=1}^{\infty} F_m X^{m+1} + \sum_{m=0}^{\infty} F_m X^{m+2}$$

$$= X + X(A(X) - F_0) + X^2 A(X)$$

$$= X + X\,A(X) + X^2 A(X)$$

$$A(X) = \frac{X}{(1 - X - X^2)}$$

G.F for Fibonaccis

$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$

has the generating function

$$A(X) = \frac{X}{(1 - X - X^2)}$$

i.e., the coefficient of X^n in $A(X)$ is F_n

$$X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6$$

$$1 - X - X^2 \;\overline{\big)\; X}$$

$$-(X - X^2 - X^3)$$

$$X^2 + X^3$$

$$-(X^2 - X^3 - X^4)$$

$$2X^3 + X^4$$

$$-(2X^3 - 2X^4 - 2X^5)$$

$$3X^4 + 2X^5$$

$$-(3X^4 - 3X^5 - 3X^6)$$

$$5X^5 + 3X^6$$

$$-(5X^5 - 5X^6 - 5X^7)$$

$$8X^6 + 5X^7$$

$$-(8X^6 - 8X^7 - 8X^8)$$

Closed form expression for F_n?

$F_0 = 0, F_1 = 1,$
$F_n = F_{n-1} + F_{n-2}$

$$A(X) = \frac{X}{(1 - X - X^2)}$$

let's factor $(1 - X - X^2)$

$$(1 - X - X^2) = (1 - \varphi_1 X)(1 - \varphi_2 X)$$

where $\varphi_1 = \frac{1 + \sqrt{5}}{2}$

$\varphi_2 = \frac{1 - \sqrt{5}}{2}$

Let's simplify

$F_0 = 0, F_1 = 1,$
$F_n = F_{n-1} + F_{n-2}$

$$A(X) = \frac{X}{(1 - \varphi_1 X)(1 - \varphi_2 X)}$$

some elementary algebra omitted…*

$$A(X) = \frac{1}{\sqrt{5}} \frac{1}{(1 - \varphi_1 X)} + \frac{-1}{\sqrt{5}} \frac{1}{(1 - \varphi_2 X)}$$

*you are not allowed to say this in <u>your</u> answers…

$$A(X) = \frac{1}{\sqrt{5}} \frac{1}{(1 - \phi_1 X)} + \frac{-1}{\sqrt{5}} \frac{1}{(1 - \phi_2 X)}$$

$$\frac{1}{(1 - \phi_1 X)} = 1 + \varphi_1 X + \varphi_1^{\,2} X^2 + \ldots + \varphi_1^{\,n} X^n + \ldots$$

$$\frac{1}{1 - Y} = 1 + Y^1 + Y^2 + Y^3 + \ldots + Y^n + \ldots$$

the <u>Infinite</u> Geometric Series

$$A(X) = \frac{1}{\sqrt{5}} \frac{1}{(1-\phi_1 X)} + \frac{-1}{\sqrt{5}} \frac{1}{(1-\phi_2 X)}$$

$$\frac{1}{(1-\phi_1 X)} = 1 + \varphi_1 X + \varphi_1^2 X^2 + \ldots + \varphi_1^n X^n + \ldots$$

$$\frac{1}{(1-\phi_2 X)} = 1 + \varphi_2 X + \ldots + \varphi_2^n X^n + \ldots$$

$\Rightarrow$ the coefficient of X^n in $A(X)$ is…

$$\frac{1}{\sqrt{5}} \varphi_1^n + \frac{-1}{\sqrt{5}} \varphi_2^n$$

Closed form for Fibonaccis

$$F_n = \frac{1}{\sqrt{5}} \varphi^n + \frac{-1}{\sqrt{5}} (-1/\varphi)^n$$

where $\phi = \dfrac{1 + \sqrt{5}}{2}$

"golden ratio"

Closed form for Fibonaccis

$$F_n = \frac{1}{\sqrt{5}} \varphi^n + \frac{-1}{\sqrt{5}} (-1/\varphi)^n$$

$F_n =$ closest integer to $\frac{1}{\sqrt{5}} \varphi^n$

To recap…

Given a sequence V = < a_0, a_1, a_2, ..., a_n, … >

associate a formal power series with it

$$P(X) = \sum_{i=0}^{\infty} a_i X^i$$

This is the “generating function” for V

We just used this for solving the Fibonacci recurrence…

Multiplication

$$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$$

$$B(X) = b_0 + b_1 X + b_2 X^2 + \ldots$$

multiply them together

$$(A*B)(X) = (a_{0*}b_0) + (a_0b_1 + a_1b_0) X$$
$$+ (a_0b_{2\,+}\, a_1b_1 + a_2b_0) X^2$$
$$+ (a_0b_{3\,+}\, a_1b_2 + a_2b_1 + a_3b_0) X^3$$
$$+ \ldots$$

seems a bit less natural in the vector representation

(it's called a "convolution" there)

Multiplication: special case

$$A(X) = a_0 + a_1 X + a_2 X^2 + \dots$$

Special case: $B(X) = 1 + X + X^2 + \dots \quad = \frac{1}{1-X}$

multiply them together

$$(A*B)(X) = a_0 + (a_0 + a_1) X + (a_0 + a_1 + a_2) X^2 + (a_0 + a_1 + a_2 + a_3) X^3 + \dots$$

It gives us partial sums!

Poll time

What's a closed form for the generating function of the sequence of natural numbers $\langle 0,1,2,3,4,\ldots\rangle$, i.e., the sequence $a_n = n$ for $n \geq 0$?

$$X + 2\,X^2 + 3\,X^3 + \cdots + n\,X^n + \cdots$$

equals

$$\frac{X}{(1-X)^2}$$

$$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$$

$$B(X) = 1 + X + X^2 + \ldots = \frac{1}{1-X}$$

$$(A*B)(X) = a_0 + (a_0 + a_1) X + (a_0 + a_1 + a_2) X^2 + (a_0 + a_1 + a_2 + a_3) X^3 + \ldots$$

It gives us partial sums.

Apply with A(X)=B(X)

$$1 + 2X + 3X^2 + 4X^3 + \cdots + n\, X^{n-1} + \cdots = \frac{1}{(1-X)^2}$$

To get generating function for naturals $\langle 0,1,2,3,\ldots\rangle$, which is a shift of $\langle 1,2,3,\ldots\rangle$, multiply the G.F by X

What happens if we again take prefix sums?

Take $1 + 2X + 3X^2 + 4X^3 + \ldots = \dfrac{1}{(1-X)^2}$

multiplying through by $1/(1-X)$

$$\Delta_1 + \Delta_2 X + \Delta_3 X^2 + \cdots = \frac{1}{(1-X)^3}$$

where $\Delta_n = \binom{n+1}{2}$ is the sequence of triangular numbers

What's the pattern?

$$\langle 1,1,1,1,\ldots \rangle = \frac{1}{1-X}$$

$$\langle 1,2,3,4,\ldots \rangle = \frac{1}{(1-X)^2}$$

$$\langle 1,3,6,10,\ldots \rangle = \frac{1}{(1-X)^3}$$

$$??? = \frac{1}{(1-X)^k}$$

What's the pattern?

$$\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \ldots = \frac{1}{1-X}$$

$$\langle 1,2,3,4,\ldots \rangle = \frac{1}{(1-X)^2}$$

$$\langle 1,3,6,10,\ldots \rangle = \frac{1}{(1-X)^3}$$

$$??? = \frac{1}{(1-X)^n}$$

What's the pattern?

$$\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \ldots = \frac{1}{1-X}$$

$$\binom{1}{1}, \binom{2}{1}, \binom{3}{1}, \binom{4}{1}, \ldots = \frac{1}{(1-X)^2}$$

$$\langle 1,3,6,10,\ldots \rangle = \frac{1}{(1-X)^3}$$

$$??? = \frac{1}{(1-X)^n}$$

What's the pattern?

$$\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \cdots = \frac{1}{1-X}$$

$$\binom{1}{1}, \binom{2}{1}, \binom{3}{1}, \binom{4}{1}, \cdots = \frac{1}{(1-X)^2}$$

$$\binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \binom{5}{2}, \cdots = \frac{1}{(1-X)^3}$$

??? $$= \frac{1}{(1-X)^k}$$

What's the pattern?

$$\binom{0}{0}, \binom{1}{0}, \binom{2}{0}, \binom{3}{0}, \ldots = \frac{1}{1-X}$$

$$\binom{1}{1}, \binom{2}{1}, \binom{3}{1}, \binom{4}{1}, \ldots = \frac{1}{(1-X)^2}$$

$$\binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \binom{5}{2}, \ldots = \frac{1}{(1-X)^3}$$

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} X^n = \frac{1}{(1-X)^k}$$

Another way to see it…

What is the coefficient of X^n in the expansion of:

$$(1 + X + X^2 + X^3 + X^4 + \ldots)^k \,?$$

To get X^n we need to pick X^{e_i} in $i'th$ factor, for $i = 1,2,\ldots,k$ with $e_1 + e_2 + \cdots + e_k = n$.

Each exponent can be any natural number.

∴ coefficient of X^n is the number of non-negative solutions to:

$$e_1 + e_2 + \ldots + e_k = n$$

which is $\binom{n+k-1}{k-1}$

The Convolution Rule

$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$ $B(X) = b_0 + b_1 X + b_2 X^2 + \ldots$

GF for selecting items from set ***A***

GF for selecting items from set ***B***

A and ***B*** disjoint

Suppose there is a bijection between n-element selections from ***A*** $\cup$***B*** and ordered pairs of selections from ***A*** and ***B*** containing total of n els.

Then, number of ways to select n items total from ***A*** $\cup$***B*** = $a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n b_0$

$\therefore$ GF for selecting items from disjoint union ***A*** $\cup$***B***

$= A(X)\, B(X)$

Now to a seemingly
over the top
counting problem…

Suppose that we want to fill a basket with fruit, but we impose on ourselves some very quirky constraints:

1. The number of apples must be a multiple of five (an apple a [week]day...)
2. The number of bananas must be even (eaten before 15-251 on Tues/Thurs...)
3. We can take at most four oranges (too acidic...).
4. There can be at most one pear (get mushy too fast...)

Let c_n = number of ways to pick exactly n fruits.

E.g., $c_5 = 6$

apples	0	0	0	0	0	5
bananas	4	4	2	2	0	0
oranges	1	0	2	3	4	0
pears	0	1	1	0	1	0

What is a closed form for c_n?

Recall Convolution Rule

If $A(x)$, $B(x)$, $O(x)$ and $P(x)$
are the generating functions for the number of ways to fill baskets using only one kind of fruit

Then the generating function for number of ways to fill basket using any of these fruits is given by
$C(x) = A(x)B(x)O(x)P(x)$

Suppose that we want to fill a basket with fruit, but we impose on ourselves some very quirky constraints:

1. The number of apples must be a multiple of five (an apple a [week]day...)
2. The number of bananas must be even (eaten before 15-251 on Tues/Thurs...)
3. We can take at most four oranges (too acidic...).
4. There can be at most one pear (get mushy too fast...)

Suppose we only pick bananas

b_n = number of ways to pick n fruits, only bananas.

$$\langle 1,0,1,0,1,0,\ldots \rangle$$

$$B(x) = 1 + x^2 + x^4 + x^6 + \ldots \quad = \frac{1}{1-x^2}$$

Suppose that we want to fill a basket with fruit, but we impose on ourselves some very quirky constraints:

1. The number of apples must be a multiple of five (an apple a [week]day...)
2. The number of bananas must be even (eaten before 15-251 on Tues/Thurs...)
3. We can take at most four oranges (too acidic...).
4. There can be at most one pear (get mushy too fast...)

Suppose we only pick apples

a_n = number of ways to pick n fruits, only apples.

$<1,0,0,0,0,1,\ldots>$

$$A(x) = 1 + x^5 + x^{10} + x^{15} + \ldots = \frac{1}{1-x^5}$$

Suppose that we want to fill a basket with fruit, but we impose on ourselves some very quirky constraints:

1. The number of apples must be a multiple of five (an apple a [week]day...)
2. The number of bananas must be even (eaten before 15-251 on Tues/Thurs...)
3. We can take at most four oranges (too acidic...).
4. There can be at most one pear (get mushy too fast...)

Suppose we only pick oranges

o_n = number of ways to pick n fruits, only oranges.

$\langle 1,1,1,1,1,0,0,0,\ldots \rangle$

$$O(x) = 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}$$

Suppose that we want to fill a basket with fruit, but we impose on ourselves some very quirky constraints:

1. The number of apples must be a multiple of five (an apple a [week]day...)
2. The number of bananas must be even (eaten before 15-251 on Tues/Thurs...)
3. We can take at most four oranges (too acidic...).
4. There can be at most one pear (get mushy too fast...)

Suppose we only pick pears

p_n = number of ways to pick n fruits, only pears.

$$\langle 1,1,0,0,0,0,0,\ldots \rangle$$

$$P(x) = 1 + x \quad = \frac{1-x^2}{1-x}$$

Suppose that we want to fill a basket with fruit, but we impose on ourselves some very quirky constraints:

1. The number of apples must be a multiple of five (an apple a [week]day...)
2. The number of bananas must be even (eaten before 15-251 on Tues/Thurs...)
3. We can take at most four oranges (too acidic...).
4. There can be at most one pear (get mushy too fast...)

Let c_n = number of ways to pick exactly n fruits of any type

$$\sum c_n x^n = A(x)\,B(x)\,O(x)\,P(x)$$

$$= \frac{1}{1-x^5}\;\frac{1}{1-x^2}\;\frac{1-x^5}{1-x}\;\frac{1-x^2}{1-x} = \frac{1}{(1-x)^2}$$

Suppose that we want to fill a basket with fruit, but we impose on ourselves some very quirky constraints:

1. The number of apples must be a multiple of five (an apple a [week]day...)
2. The number of bananas must be even (eaten before 15-251 on Tues/Thurs...)
3. We can take at most four oranges (too acidic...).
4. There can be at most one pear (get mushy too fast...)

Let c_n = number of ways to pick exactly n fruits of any type

c_n is coefficient of X^n in $\dfrac{1}{(1-X)^2}$

<1,2,3,4,…>

$\therefore\ c_n = n+1.$

Another useful operation: Differentiation

$$A(X) = a_0 + a_1 X + a_2 X^2 + \dots$$

differentiate it…

$$A'(X) = a_1 + 2a_2 X + 3a_3 X^2 \dots$$

$$A'(X) = \sum_{i=0}^{\infty} (i+1)a_{i+1} X^i$$

$$X\, A'(X) = \sum_{i=0}^{\infty} i a_i X^i$$

Example of differentiation in action

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} X^n = \frac{1}{(1-X)^k}$$

$$\begin{aligned}
\frac{1}{(1-X)^k} &= \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \left(\frac{1}{1-X} \right) \\
&= \frac{1}{(k-1)!} \sum_{\ell=(k-1)}^{\infty} \ell(\ell-1)\cdots(\ell-(k-2))\, X^{\ell-(k-1)} \\
&= \sum_{n=0}^{\infty} \frac{(n+k-1)(n+k-2)\cdots(n+1)}{(k-1)!}\, X^n \\
&= \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} X^n .
\end{aligned}$$

Differentiation in action

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} X^n = \frac{1}{(1-X)^k}$$

Fact: For a generating function $A(X) = \sum_{n=0}^{\infty} a_n X^n$

$$a_n = \frac{A^{(n)}(0)}{n!}$$

where $A^{(n)}(X)$ is the n'th order derivative of $A(X)$

For $A(X) = \frac{1}{(1-X)^k}$, we have $A^{(n)}(X) = \frac{k(k+1)\cdots(k+n-1)}{(1-X)^{k+n}}$

Differentiation in use

Exercise: Prove that the generating function for squares, i.e.,
the sequence $a_n = n^2$, $n=0,1,2\ldots$ equals

$$\frac{X(1+X)}{(1-X)^3}$$

One approach: Use differentiation + shifting twice

Integration

$A(X) = a_0 + a_1 X + a_2 X^2 + \ldots$

Integrating both sides ….

$$\int_0^X A(t)dt = a_0 X + a_1 \frac{X^2}{2} + a_2 \frac{X^3}{3} + \cdots$$

$$\frac{1}{X}\int_0^X A(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} X^n$$

Example

Evaluate the sum $$\sum_{i=0}^{n} \binom{n}{i} \frac{1}{(i+1)}.$$

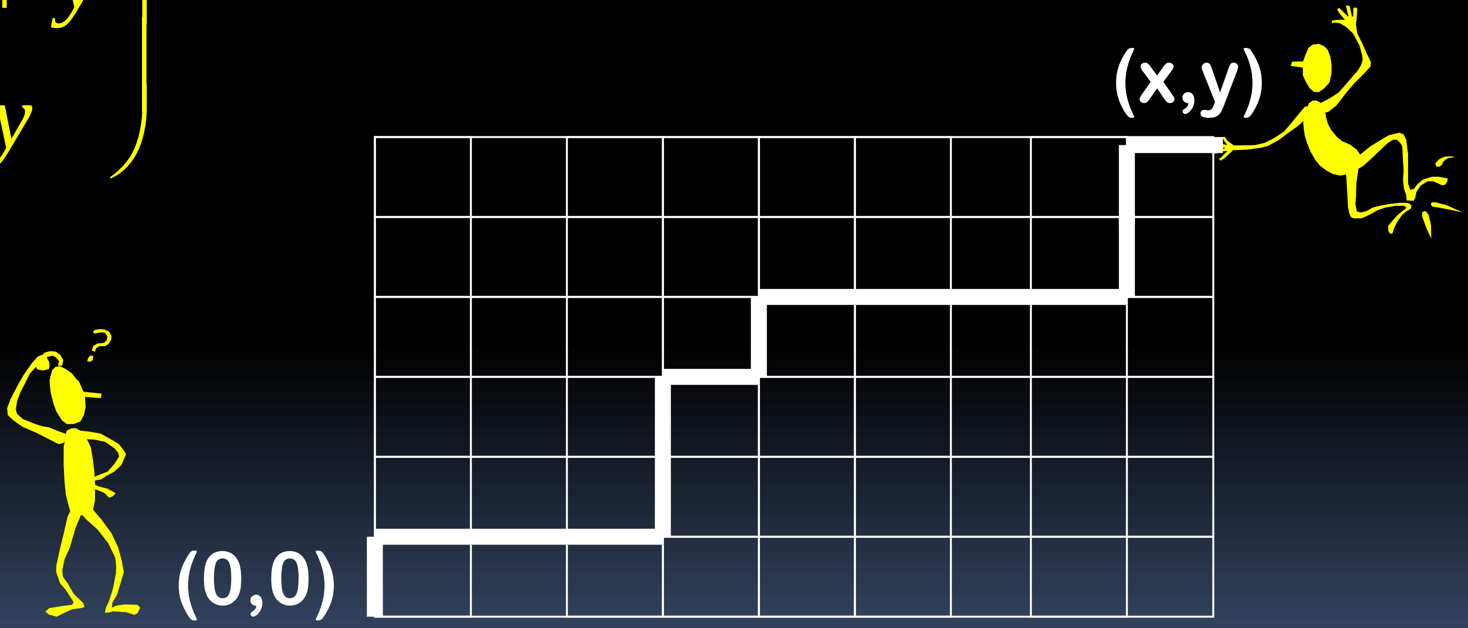
$$\sum_{i=0}^{n} \frac{\binom{n}{i}}{i+1} X^i = \frac{1}{X} \int_0^X (1+t)^n \, dt = \frac{(1+X)^{n+1} - 1}{X(n+1)}$$

Substituting X=1, answer = $$\frac{2^{n+1} - 1}{n+1}$$

Manhattan walk

All the avenues numbered 0 through *x*, run north-south, and all streets, numbered 0 through y, run east-west. The number of [sensible] ways to walk from the corner of (0,0) to (x,y) (total x+y steps) equals:

$$\binom{x+y}{y}$$

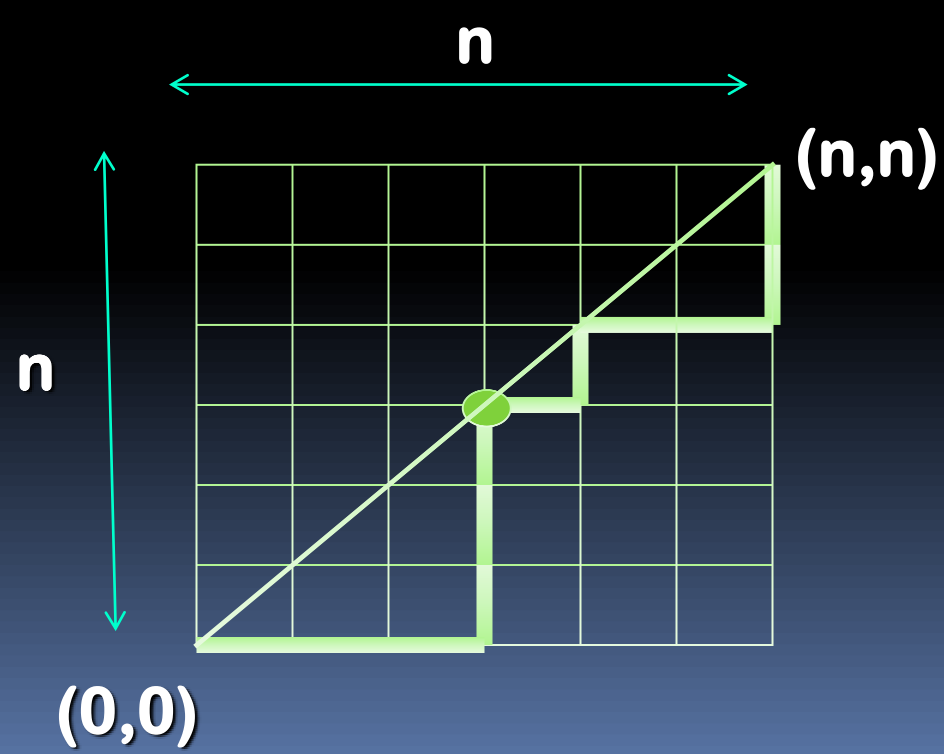


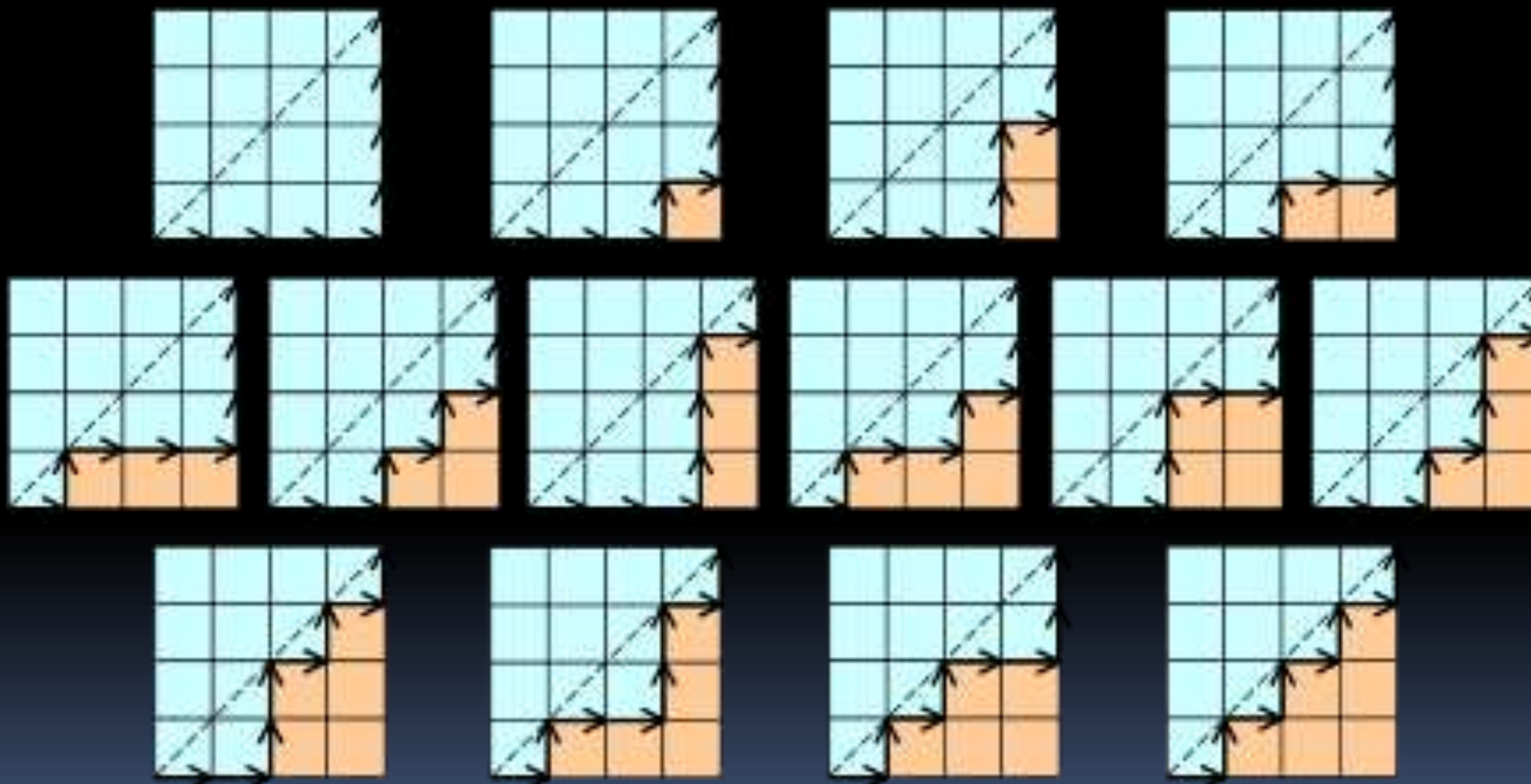
Noncrossing Manhattan walk

What if we require the Manhattan walk to **never cross the diagonal**?

How many ways can we walk from (0,0) to (n,n) along the grid subject to this rule?

This number, say c_n, is called the n'th Catalan number





A recurrence

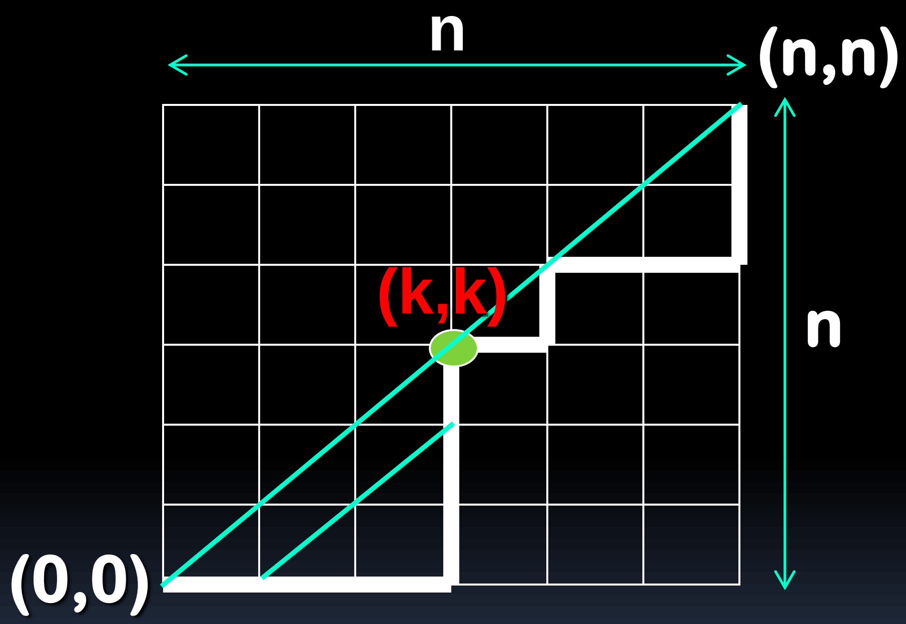
c_n = # Manhattan walks from (0,0) to (n,n) that never cross the diagonal (define $c_0=1$).

The walk must hit the diagonal at least once (perhaps only at the end).

walks that hit the diagonal at (k,k) for the ***first*** time?

$(1 \le k \le n)$

<u>Answer:</u> $c_{k-1}\ c_{n-k}$



$$c_n = \sum_{k=1}^{n} c_{k-1} c_{n-k} = \sum_{i=0}^{n-1} c_i c_{n-1-i} \quad \text{for } n \ge 1$$

Generating Function

- Define $C(x) = \sum_{n=0}^{\infty} c_n x^n$

c_n = coefficient of x^{n-1} in $C(x)^2$

$$c_n = \sum_{k=1}^{n} c_{k-1} c_{n-k} = \sum_{i=0}^{n-1} c_i c_{n-1-i} \quad \text{for } n \geq 1$$

Together with $c_0=1$ we get

$$C(x) = 1 + x\,C(x)^2$$

Catalan generating function

$$C(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$x\,C(x)^2 - C(x) + 1 = 0$$

Solving the quadratic: $C(x) = \dfrac{1 - \sqrt{1-4x}}{2x}$.

Using this, one can calculate $c_n = \dfrac{1}{n+1}\dbinom{2n}{n}$

Define $D(x) = 2x\,C(x) = 1 - \sqrt{1-4x} = \sum_{n=0}^{\infty} d_n x^n$

$$d_n = \frac{D^{(n)}(0)}{n!} = \frac{2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-5) \cdot (2n-3)}{n!}$$

$$c_n = \frac{d_{n+1}}{2}$$

Another take on Catalan Generating Fn.

Let E(X) be the GF for *super non-crossing* Manhattan walks on n x n grids that *never touch the diagonal* (except at endpoints)

Fact 1: E(X) = X C(X)

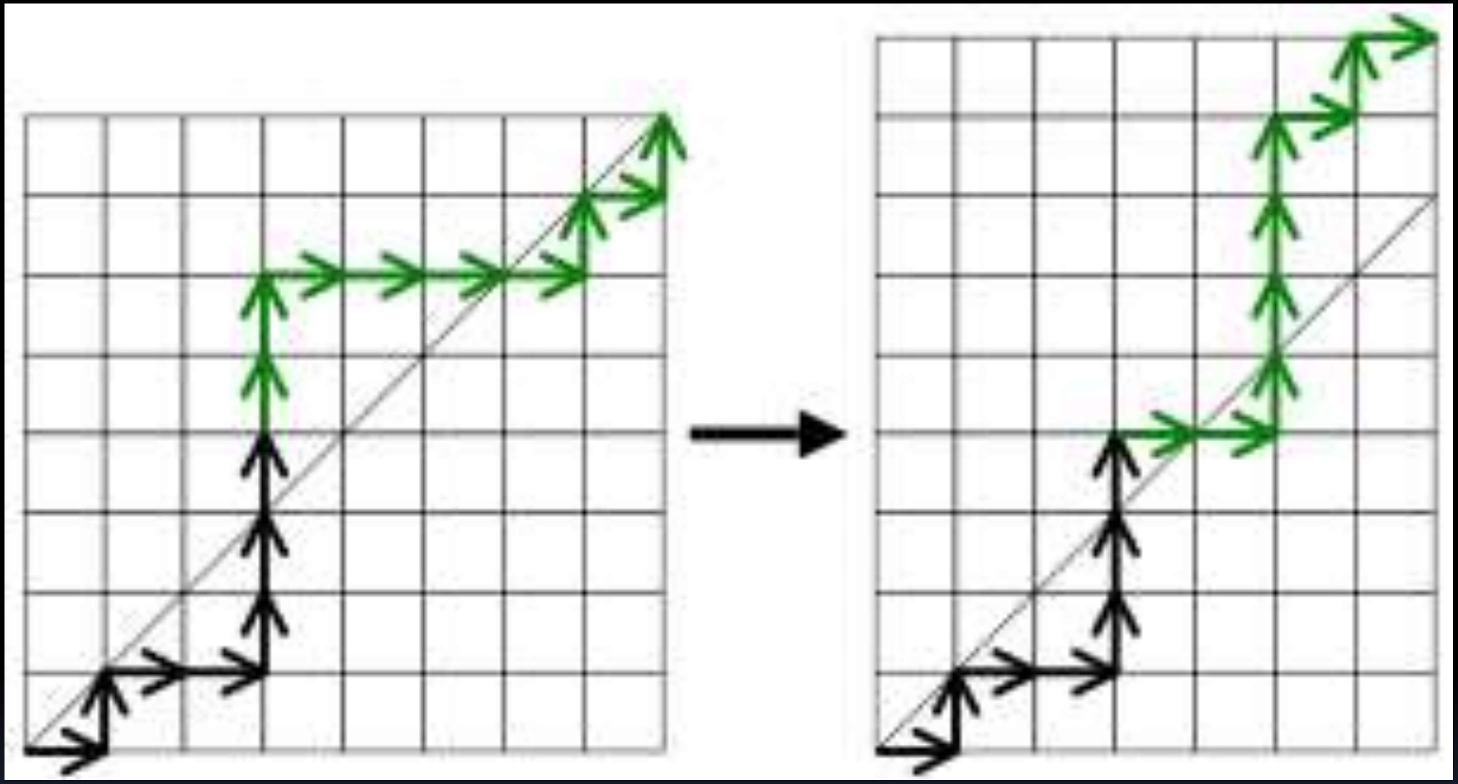
Fact 2: C(X) = 1 + E(X) + $(E(X))^2$ + $(E(X))^3$ + ….

Together these imply

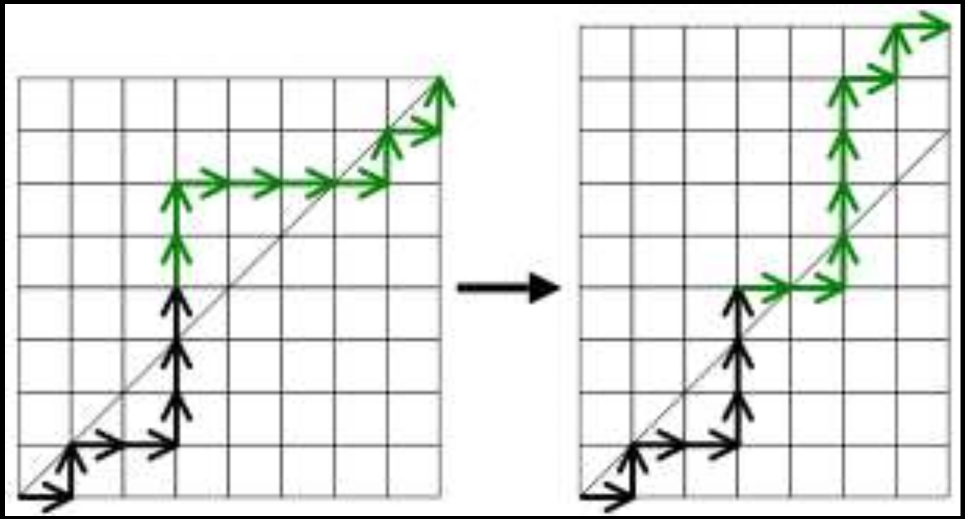
$$C(X) = \frac{1}{1 - XC(X)}$$

Here's yet another take,
this time without
Generating Functions (Yay!)

Let’s count # violating paths, that ***do*** cross the diagonal
Will do so by a bijection.



Find first step above the diagonal.
“Flip” the portion of the path ***after*** that step.



Flip the portion of the path ***after*** the first edge above the diagonal.

Observe: New path goes to (n-1,n+1)

Claim: The above is a *bijection* from *crossing* Manhattan walks in n x n grid to *unconstrained* Manhattan walks in (n-1,n+1) grid

Thus, number of noncrossing Manhattan walks on n x n grid =

$$\binom{2n}{n} - \binom{2n}{n-1}$$

How many sequences of balanced paranthesis with n ('s and n 1)'s are there?

Answer: The n'th Catalan number

$$c_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

Some Common GFs

Sequence	Generating Function
$\langle 1, 1, 1, \ldots \rangle$	$\frac{1}{1-x}$
$\langle 1, 2, 4, \ldots \rangle$	$\frac{1}{1-2x}$
$\langle 1, 2, 3, \ldots \rangle$	$\frac{1}{(1-x)^2}$
$\langle 0, 1, 1, 2, 3, \ldots \rangle$	$\frac{x}{1-x-x^2}$

Supplementary material: Another recurrence example

$$d_n = 2d_{n-1} + 3d_{n-2} \qquad d_0 = 0 \quad d_1 = 1$$

Goal: derive a closed form using generating functions.

Let $$D(x) = \sum_{n=0}^{\infty} d_n x^n$$

Proceeding as in Fibonacci example...

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = x + \sum_{n=2}^{\infty} (2d_{n-1} + 3d_{n-2}) x^n$

$$= x + \sum_{n=2}^{\infty} 2d_{n-1} x^n + \sum_{n=2}^{\infty} 3d_{n-2} x^n$$

$$= x + 2x \sum_{n=2}^{\infty} d_{n-1} x^{n-1} + 3x^2 \sum_{n=2}^{\infty} d_{n-2} x^{n-2}$$

$$= x + 2x \sum_{n=1}^{\infty} d_n x^n + 3x^2 \sum_{n=0}^{\infty} d_n x^n$$

$$= x + 2x(D(x) - d_0) + 3x^2 D(x)$$

A closed form

$$D(x) = x + 2xD(x) + 3x^2 D(x)$$

$$(1 - 2x - 3x^2)D(x) = x$$

$$D(x) = \frac{x}{1 - 2x - 3x^2}$$

Simplifying to retrieve d_n

$$D(x) = \sum_{n=0}^{\infty} d_n x^n = \frac{x}{1-2x-3x^2} = \frac{-1}{4(1+x)} + \frac{1}{4(1-3x)}$$

Factorize denominator to break it into smaller pieces!

$$\frac{x}{1-2x-3x^2} = \frac{x}{(1+x)(1-3x)} = \frac{A}{1+x} + \frac{B}{1-3x}$$

$$x = (1-3x)A + (1+x)B$$

$$1 = -3A + B$$

$$0 = A + B$$

$$A = \frac{-1}{4}$$

$$B = \frac{1}{4}$$

Retrieving d_n

$$D(x) = \sum_{n=0}^{\infty} d_n x^n = \frac{x}{1 - 2x - 3x^2} = \frac{-1}{4(1+x)} + \frac{1}{4(1-3x)}$$

$$= \frac{-1}{4} \sum_{n=0}^{\infty} (-x)^n + \frac{1}{4} \sum_{n=0}^{\infty} (3x)^n$$

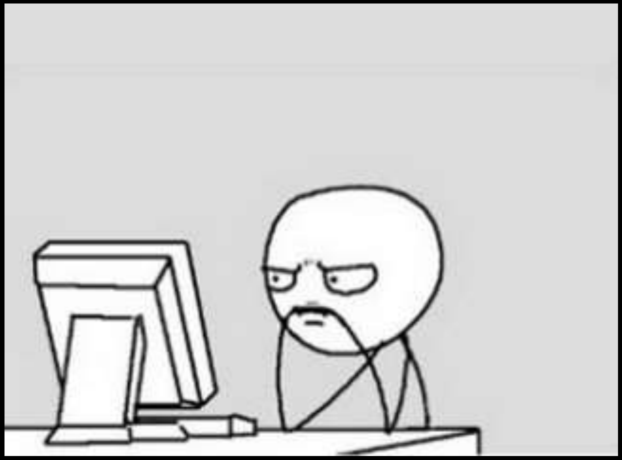
$$= \sum_{n=0}^{\infty} \frac{1}{4} \left((-1)^{n+1} + 3^n \right) x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

$$\frac{1}{1-(3x)} = \sum_{n=0}^{\infty} (3x)^n$$

$$d_n = \frac{1}{4} \left((-1)^{n+1} + 3^n \right)$$



Study Guide

Formal Power Series

Basic operations on Formal Power Series

Solving recurrences using generating functions
(handle base cases carefully!)

Solving G.F. to get closed form

G.F.s for common sequences

Prefix sums using G.F.s

Using G.F.s to solve counting problems