1 Pancake Sorting Problem

Definition 1.1 (Pancake numbers)**.**
We are given a stack of n pancakes, each of different size. Our goal is to sort this stack from smallest to largest (largest being on the bottom of the stack). The only thing we are allowed to do is to insert the spatula in between two pancakes (or between the bottom pancake and the plate), and flip over all the pancakes that are on top of the spatula.

We are interested in the maximum number of flips (in terms of n) we would need to sort a stack of n pancakes, where the maximum is over all stacks with n pancakes. In other words, we are interested in

$$P_n = \max_S \min_A \quad \text{number of flips that method } A \text{ takes to sort stack } S.$$

Here, the maximum is over all pancake stacks of size n, and the minimum is over all methods/algorithms for sorting a given stack of pancakes. ∎

Notation 1.2. We represent a stack of n pancakes with a permutation of $\{1, 2, \ldots, n\}$. Here, the numbers correspond to how large the pancake is, so 1 represents the smallest pancake and n represents the largest pancake. For example, (5 2 3 4 1) corresponds to a stack of 5 pancakes, where the largest pancake 5 is at the top of the stack, and the smallest pancake 1 is at the bottom.

Proposition 1.3 (Number of flips required for (5 2 3 4 1))**.** *Let X denote the minimum number of flips needed to sort the stack (5 2 3 4 1). Then $X = 4$.*

Proof. To prove $X \leq 4$, we show how to sort (5 2 3 4 1) in 4 flips:

$$(5\ 2\ 3\ 4\ 1) \to (1\ 4\ 3\ 2\ 5) \to (2\ 3\ 4\ 1\ 5) \to (4\ 3\ 2\ 1\ 5) \to (1\ 2\ 3\ 4\ 5).$$

We now prove $X \geq 4$. The proof is by contradiction, so assume that there is a way to sort (5 2 3 4 1) in 3 or less flips.
Observation. Right before a pancake is placed at the bottom of the stack, it must be placed at the top of the stack.
Claim. The first flip must put 5 on the bottom of the stack.
Proof of Claim. Suppose the first flip does not put 5 on the bottom of the stack, so it puts it somewhere in the middle. Then we can show that (5 2 3 4 1) cannot be sorted in 3 or less flips. We know that after 3 flips, 5 must be placed at the bottom of the

stack. The observation above implies that the second flip must send 5 to the top. So in the first two flips, 5 first gets sent from the top to somewhere in the middle, and then it gets flipped back up to the top. In other words, after 2 flips we end up with the original stack (5 2 3 4 1). There is no way to sort (5 2 3 4 1) with the remaining flip, which proves the claim.

So we know that the first flip must be $(5\ 2\ 3\ 4\ 1) \to (1\ 4\ 3\ 2\ 5)$. In the remaining two flips, 4 must be placed next to 5. It is clear that 5 should not be touched (i.e., we should not be flipping the whole stack). So we can ignore 5 and and just consider the stack of 4 pancakes (1 4 3 2). We need to put 4 at the bottom of this stack in 2 flips. Again, using the observation stated above, we know that 4 must be first placed at the top of the stack. So the 2 flips must be $(1\ 4\ 3\ 2) \to (4\ 1\ 3\ 2) \to (2\ 3\ 1\ 4)$. The resulting stack is not sorted, which is the desired contradiction. $\square$

Theorem 1.4. *For $n \geq 4$, we have*

$$n \leq P_n \leq 2n - 3.$$

The proof of the theorem follows from the following two lemmas.

Lemma 1.5. *For $n \geq 2$, we have $P_n \leq 2n - 3$.*

Proof. Consider the following algorithm for sorting an arbitrary stack of n pancakes.

- If $n = 1$: do nothing.

- If $n = 2$: sort the pancakes in one flip if they are not already sorted.

- Else (if $n \geq 3$):

    - Bring the largest pancake to the bottom of the stack in 2 flips.

    - Recursively sort the remaining $n - 1$ pancakes.

Clearly,[1] the algorithm correctly sorts a given stack of pancakes. Let $T(n)$ be the number of flips that this algorithm uses to sort a stack of n pancakes. By the definition of P_n, $P_n \leq T(n)$. So we are done once we show $T(n) \leq 2n - 3$ for $n \geq 2$. The recursive relation that $T(n)$ satisfies is

$$\begin{aligned}
T(1) &= 0, \\
T(2) &\leq 1, \\
T(n) &\leq 2 + T(n - 1) \quad \text{for } n \geq 3.
\end{aligned}$$

This implies that $T(n) \leq 2n - 3$ for $n \geq 2$, which completes the proof.[2] $\square$

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[1]You should be careful using the word "clearly". In this case, it is justified.

[2]To be more complete, you can prove $T(n) \leq 2n - 3$ for $n \geq 2$ with a quick induction. This part is omitted.

Exercise 1.6. Show by induction that the recurrence relation in the above proof solves to $T(n) \leq 2n - 3$ for $n \geq 2$.

Lemma 1.7. *For $n \geq 4$, we have $P_n \geq n$.*

Proof. Given $i, j \in \{1, 2, \ldots, n\}$ and a pancake stack, we say that (i, j) form a *bad* pair with respect to that stack if they are adjacent in the stack, and $|i - j| > 1$ (i.e., they are not supposed to be adjacent once the stack is sorted). Observe that if two pancakes are adjacent in a stack, they will remain adjacent if the spatula is never inserted in between them. This means that if (i, j) form a bad pair, then any sorting method that sorts the stack *must* insert the spatula in between i and j at some point. Note that we can also consider the bottom pancake and the plate as a bad pair too. If we never insert the spatula at the bottom of the stack, then the bottom pancake and the plate will remain adjacent. So we extend our definition of a bad pair to include the plate too.

Now we can conclude that a stack with b bad pairs needs at least b flips to be sorted. We finish the proof by showing that for $n \geq 4$, there is a stack of n pancakes containing n bad pairs. We do this by considering two cases: when n is even and when n is odd. When n is even, the following stack has n bad pairs:

$$(2\ 4\ 6\ \cdots\ n-2\ n\ 1\ 3\ 5\ \cdots\ n-1).$$

When n is odd, the following stack has n bad pairs:

$$(1\ 3\ 5\ \cdots\ n-2\ n\ 2\ 4\ 6\ \cdots\ n-1).$$

(Note that the assumption $n \geq 4$ is required so that the pancakes right in the middle of the stacks form a bad pair.) $\qquad\square$

Exercise 1.8. Suppose we are allowed to take any contiguous set of pancakes and flip them in place (they need not be on the top of the stack). Let Q_n be the maximum over stacks of size n of the minimum number of flips required to sort that stack, using this new flipping operation. Show that $n/2 \leq Q_n \leq n - 1$ for all $n \geq 2$.