Great Ideas in Theoretical CS

Lecture 10: Cake Cutting

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How to **fairly** divide a heterogeneous divisible good between players with different preferences?
THE PROBLEM

- Cake is interval $[0,1]$
- Set of players $N = \{1, \ldots, n\}$
- Piece of cake $X \subseteq [0,1]$: finite union of disjoint intervals
THE PROBLEM

- Each player $i \in N$ has a non-negative valuation $V_i$ over pieces of cake
- Additive: for $X \cap Y = \emptyset$, $V_i(X) + V_i(Y) = V_i(X \cup Y)$
- Normalized: For all $i \in N$, $V_i([0,1]) = 1$
- Divisible: $\forall \lambda \in [0,1]$ can cut $I' \subseteq I$ s.t. $V_i(I') = \lambda V_i(I)$
Fairness Properties

• Our goal is to find an allocation $A_1, \ldots, A_n$

• Proportionality:
  \[ \forall i \in N, V_i(A_i) \geq \frac{1}{n} \]

• Envy-Freeness (EF):
  \[ \forall i, j \in N, V_i(A_i) \geq V_i(A_j) \]

• Poll 1: For $n = 2$ which is stronger?
  1. Proportionality
  2. EF
  3. They are equivalent
  4. They are incomparable
FAIRNESS PROPERTIES

• Our goal is to find an allocation \( A_1, \ldots, A_n \)

• Proportionality:
  \[ \forall i \in N, V_i(A_i) \geq \frac{1}{n} \]

• Envy-Freeness (EF):
  \[ \forall i, j \in N, V_i(A_i) \geq V_i(A_j) \]

• Poll 2: For \( n \geq 3 \) which is stronger?
  1. Proportionality
  2. EF
  3. They are equivalent
  4. They are incomparable
**Cut-and-Choose**

- Algorithm for $n = 2$ [Procaccia and Procaccia, circa 1985]
- Player 1 divides into two pieces $X, Y$ s.t.
  \[ V_1(X) = \frac{1}{2}, V_1(Y) = \frac{1}{2} \]
- Player 2 chooses preferred piece
- This is EF (hence proportional)
TIME COMPLEXITY

• Player 1 divides into two pieces $X, Y$ s.t.
  $V_1(X) = 1/2, V_1(Y) = 1/2$

• Player 2 chooses preferred piece

What is the running time of Cut-and-Choose? What is the input size?
The Robertson-Webb Model

- Input size is $n$
- Two types of operations
  - $Eval_i(x, y)$ returns $V_i([x, y])$
  - $Cut_i(x, \alpha)$ returns $y$ such that $V_i([x, y]) = \alpha$

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\[ \begin{array}{c}
\text{eval output} \rightarrow \alpha \\
\end{array} \]

$\begin{array}{c}
x \\
y \leftarrow \text{cut output}
\end{array}$
THE ROBERTSON-WEBB MODEL

• Two types of operations
  - \( \text{Eval}_i(x, y) = V_i([x, y]) \)
  - \( \text{Cut}_i(x, \alpha) = y \text{ s.t. } V_i([x, y]) = \alpha \)

• Poll 3: \#operations needed to find an EF allocation when \( n = 2 \)?
  1. 1
  2. 2
  3. 3
  4. 4

This concrete complexity model is a great idea!
Dubins-Spanier

• Referee continuously moves knife
• Repeat: when piece left of knife is worth $1/n$ to player, player shouts “stop” and gets piece
• That player is removed
• Last player gets remaining piece
Dubins-Spanier Protocol
**Dubins-Spanier**

- **Claim:** The Dubins-Spanier protocol produces a proportional allocation

- **Proof:**
  - At stage 0, each of the $n$ players values the whole cake at 1
  - At each stage, the allocated piece of cake is worth at most $1/n$ to the remaining players
  - Hence, if at stage $k$ each of the remaining $n - k$ players has value at least $1 - k/n$ for the remaining cake, then at stage $k + 1$ each of the remaining $n - (k + 1)$ players has value at least $1 - \frac{k+1}{n}$ for the remaining cake
**Dubins-Spanier**

What is the complexity of Dubins-Spanier in the RW model?

- Moving knife is not really needed
- Repeat: each player makes a mark at his $1/n$ point, leftmost player gets piece up to its mark
DUBINS-SHANIER
Dubins-Spanier

\[ \frac{1}{3} \]
DUBINS-SPANIER
Dubins-Spanier
Dubins-Spanier

- **Poll 4:** So what is the complexity of Dubins-Spanier in the RW model?

  1. $\Theta(\sqrt{n})$
  2. $\Theta(n)$
  3. $\Theta(n \log n)$
  4. $\Theta(n^2)$

Can we do better?
**EVEN-PAZ**

- Given \([x, y]\), assume \(n = 2^k\)
- If \(n = 1\), give \([x, y]\) to the single player
- Otherwise, each player \(i\) makes a mark \(z\) s.t.

\[
V_i([x, z]) = \frac{1}{2} V_i([x, y])
\]

- Let \(z^*\) be the \(n/2\) mark from the left
- Recurse on \([x, z^*]\) with the left \(n/2\) players, and on \([z^*, y]\) with the right \(n/2\) players
EVEN-PAZ
**Even-Paz**

- **Claim:** The Even-Paz protocol produces a proportional allocation
- **Proof:**
  - At stage 0, each of the $n$ players values the whole cake at 1
  - At each stage the players who share a piece of cake value it at least at $V_i([x,y])/2$
  - Hence, if at stage $k$ each player has value at least $1/2^k$ for the piece he’s sharing, then at stage $k+1$ each player has value at least $\frac{1}{2^{k+1}}$
  - The number of stages is $\log n$ ■
\[
T(1) = 0, T(n) = 2n + 2T\left(\frac{n}{2}\right)
\]

Overall: \(2n \log n\)
Complexity of proportionality

- Theorem [Edmonds and Pruhs, 2006]: Any proportional protocol needs $\Omega(n \log n)$ operations in the RW model.
- The Even-Paz protocol is provably optimal!
- Envy-freeness is a much more complicated story.
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SUMMARY

• Terminology:
  o Proportionality / envy-freeness
  o The Robertson-Webb model
  o The Dubins-Spanier protocol
  o The Even-Paz protocol

• Principles:
  o Concrete complexity models for reasoning about time complexity