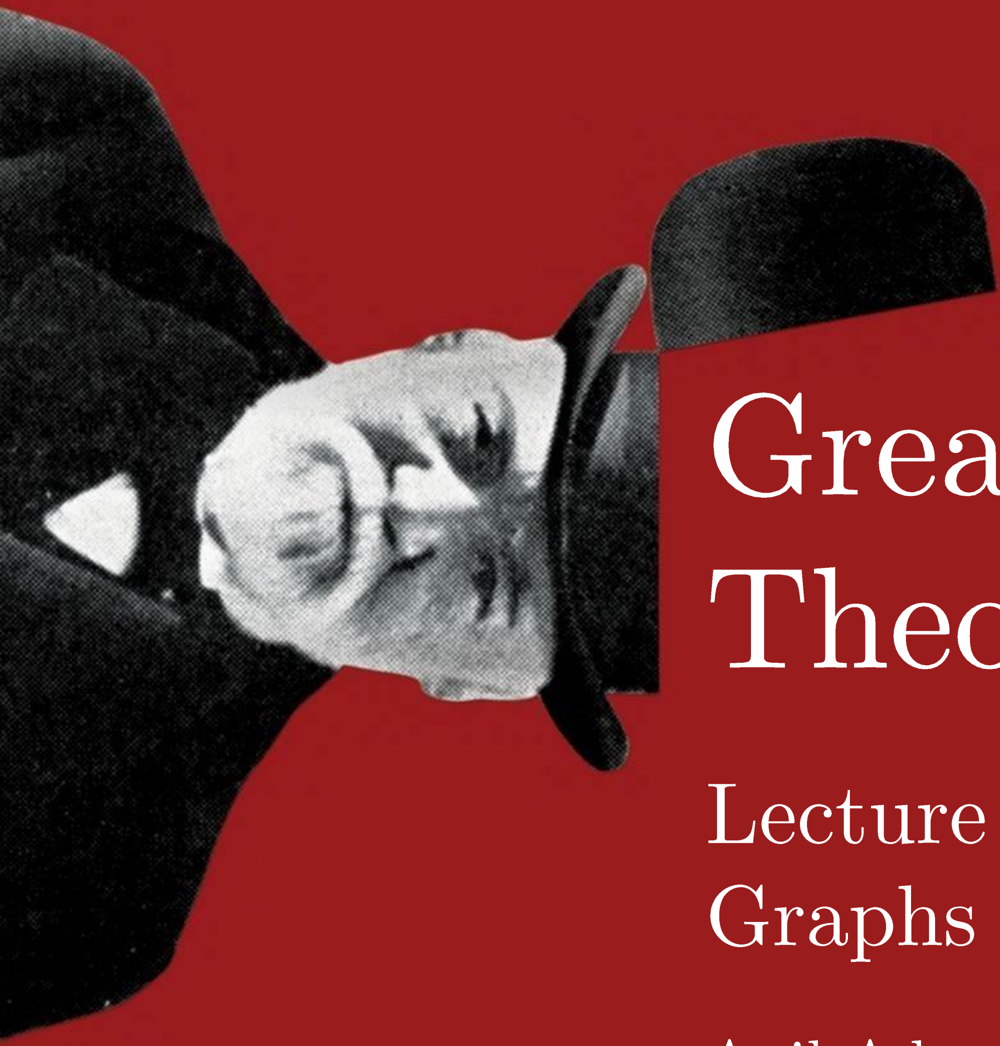
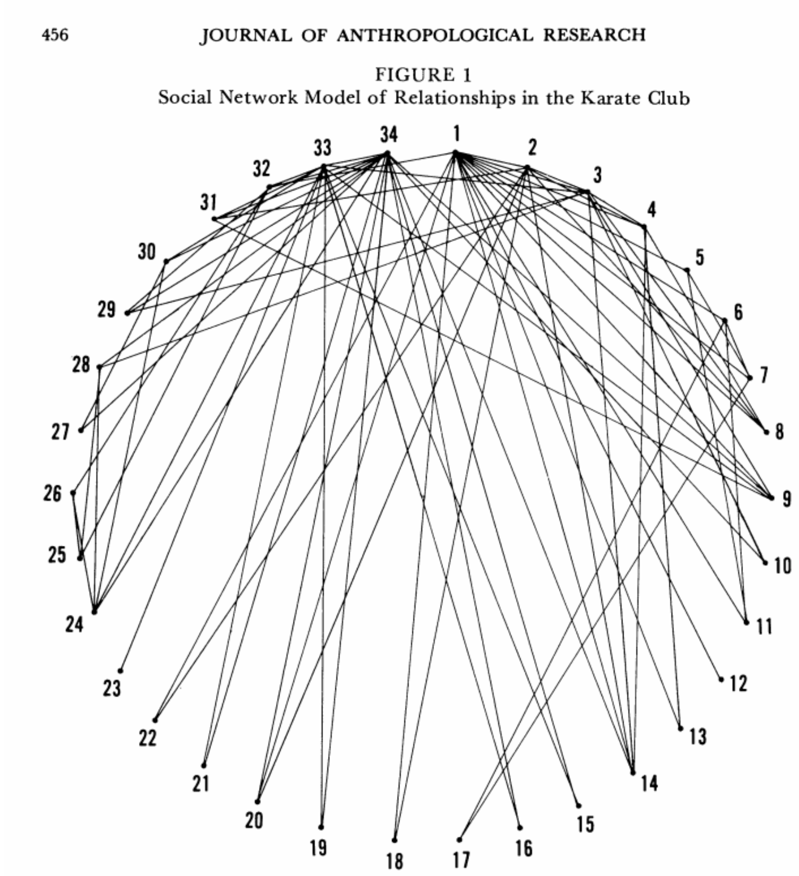


Great Ideas in Theoretical CS

Lecture 11: Graphs I: Basics

Anil Ada
Ariel Procaccia (this time)

Zachary Karate Club

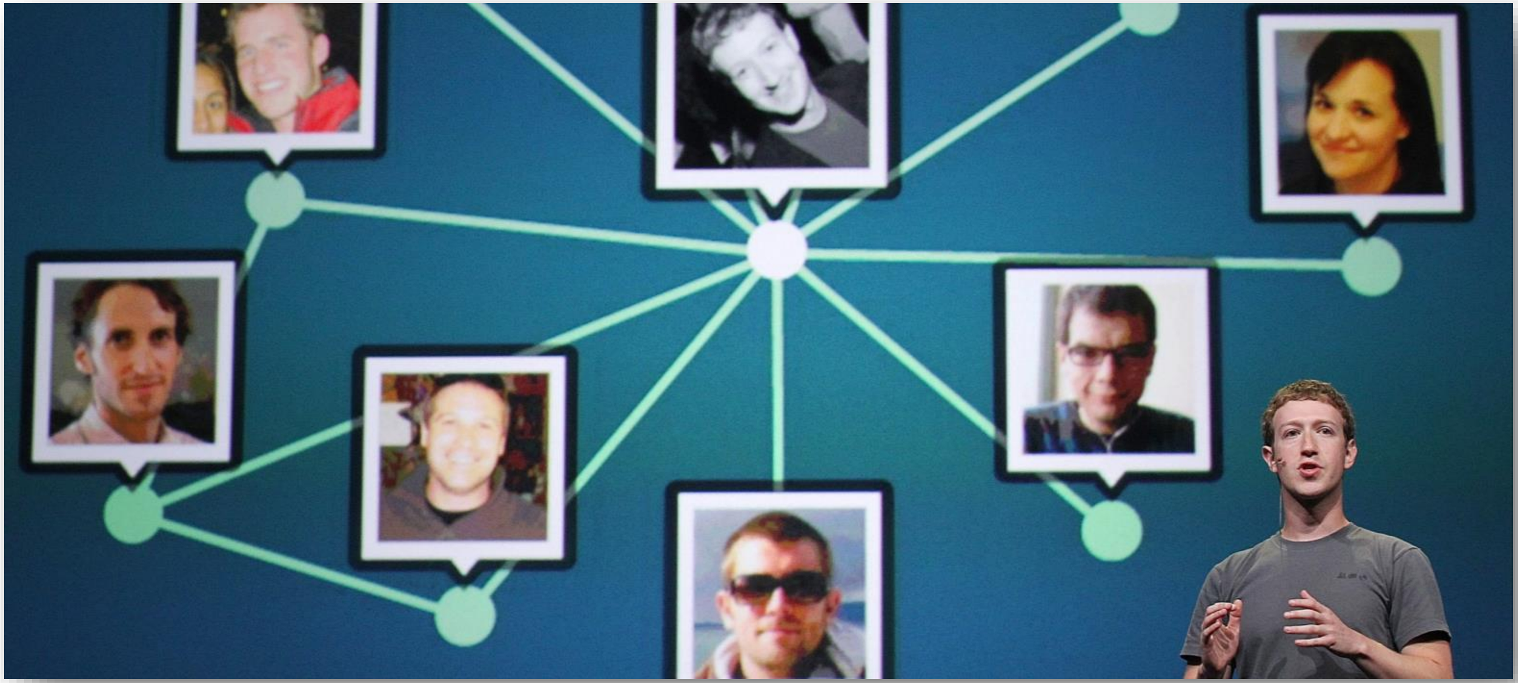


34 vertices (karatekas), 78 edges (friendships)

Zachary Karate Club CLUB

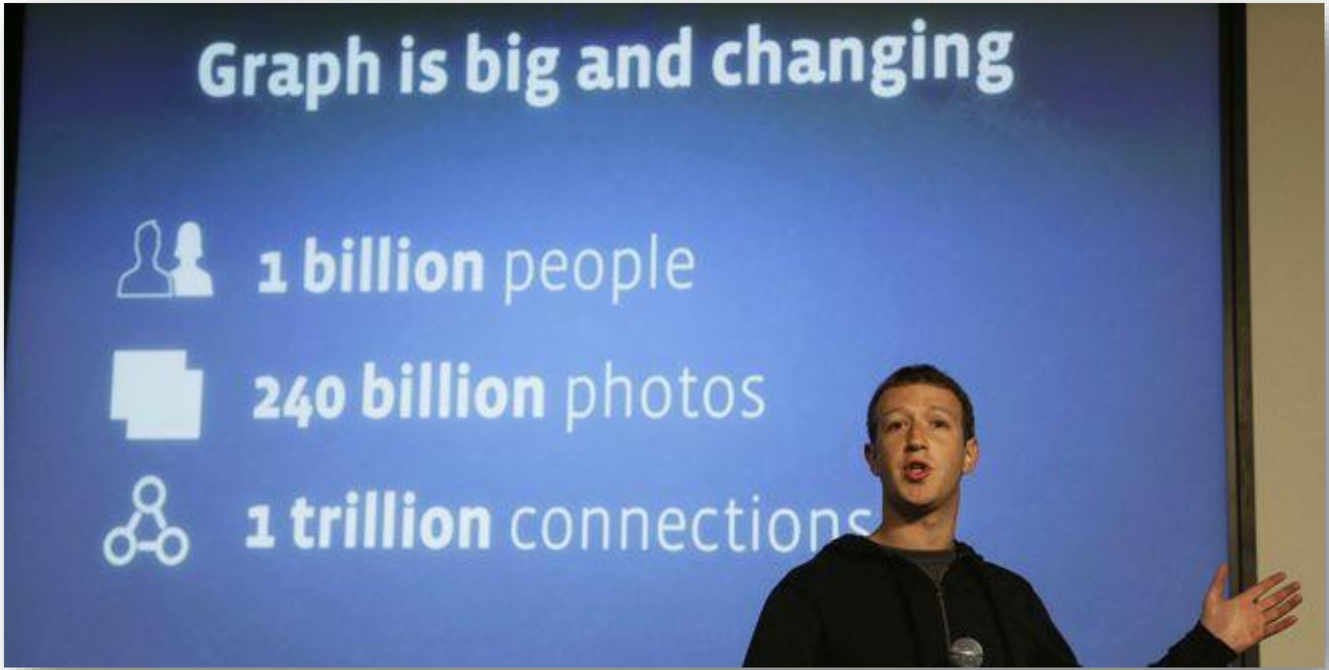
networkkarate.tumblr.com

Facebook



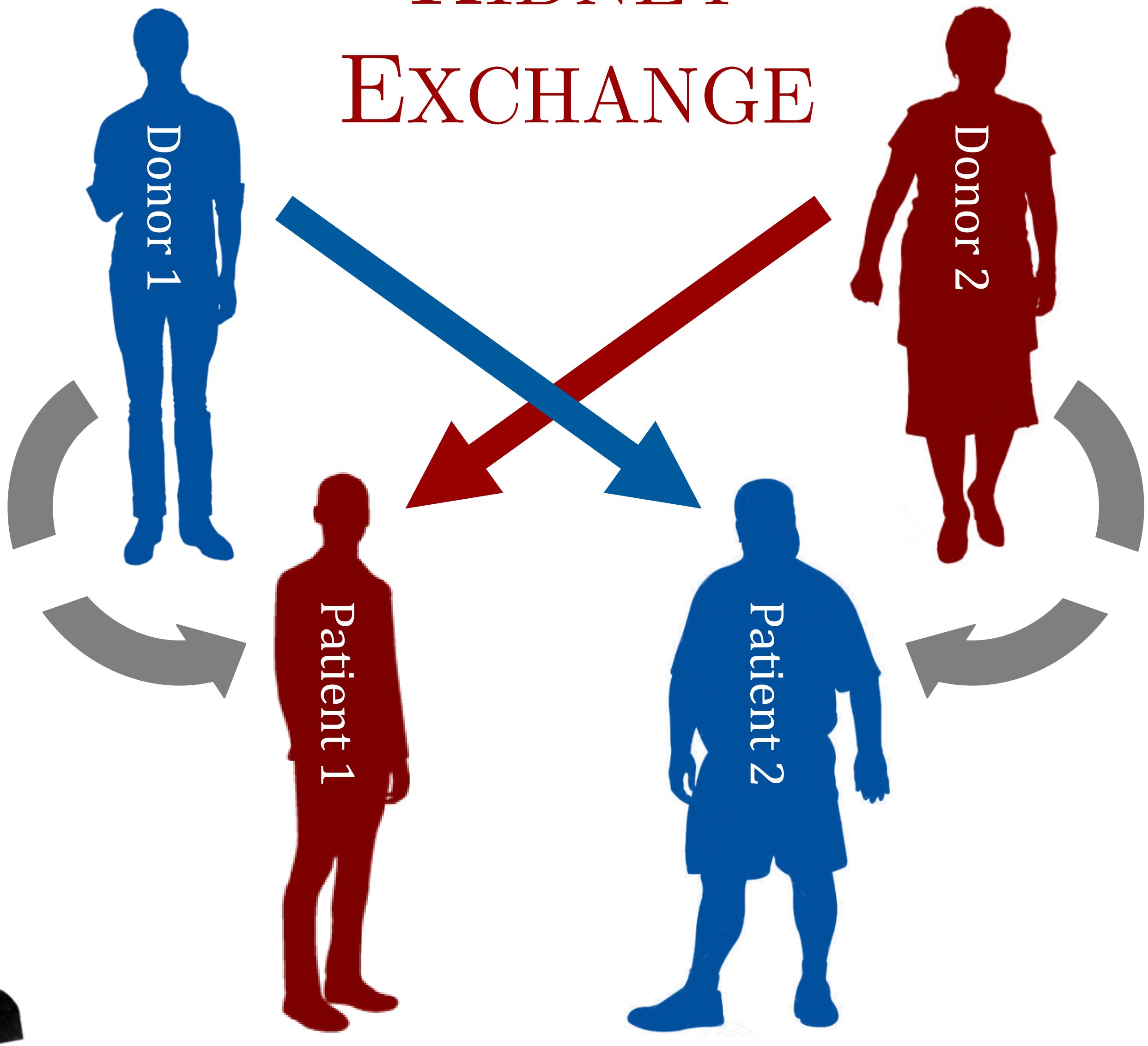
Vertices = people, edges = Friendships

Facebook



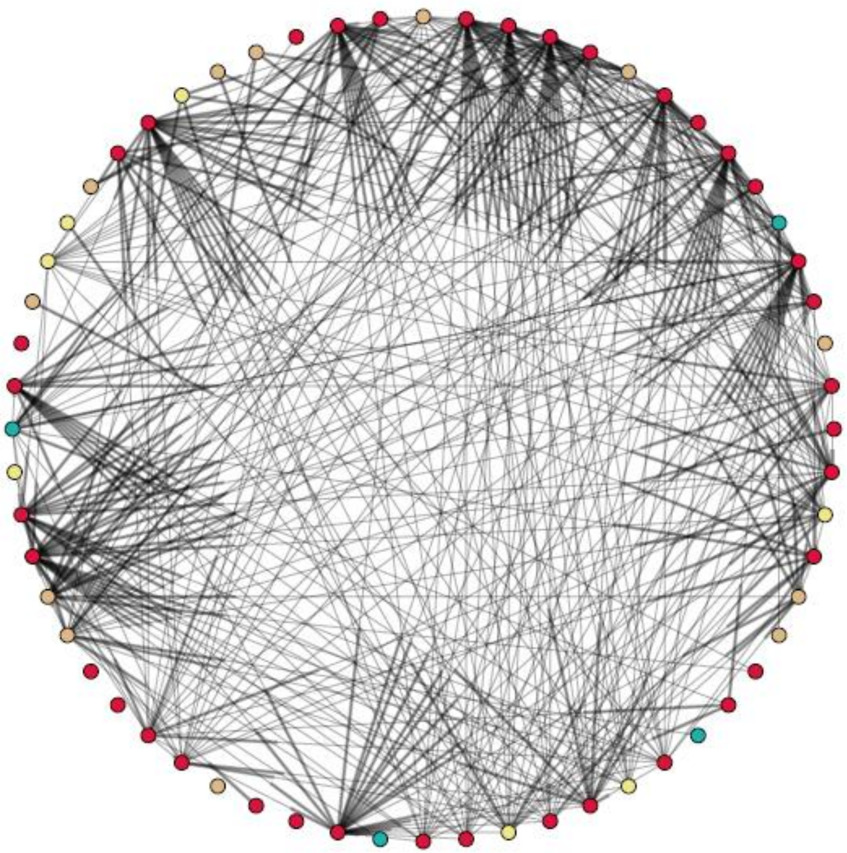
$\#\text{vertices } n = 10^9, \ \#\text{edges } m = 10^{12}$

Kidney Exchange



Kidney Exchange

Vertices = patient-donor pairs, edges = compatibility



UNOS pool, Dec 2010 [Courtesy John Dickerson, CMU]

World Wide Web

2.2 Link Structure of the Web

While estimates vary, the current graph of the crawlable Web has roughly 150 million nodes (pages) and 1.7 billion edges (links). Every page has some number of forward links (outedges) and backlinks (inedges) (see Figure 1). We can never know whether we have found all the backlinks of a particular page but if we have downloaded it, we know all of its forward links at that time.

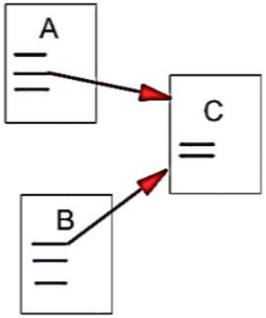


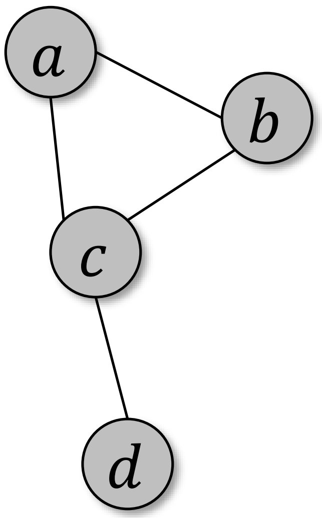
Figure 1: A and B are Backlinks of C

Web pages vary greatly in terms of the number of backlinks they have. For example, the Netscape home page has 62,804 backlinks in our current database compared to most pages which have just a few backlinks. Generally, highly linked pages are more "important" than pages with few links. Simple citation counting has been used to speculate on the future winners of the Nobel Prize [San95]. PageRank provides a more sophisticated method for doing citation counting.

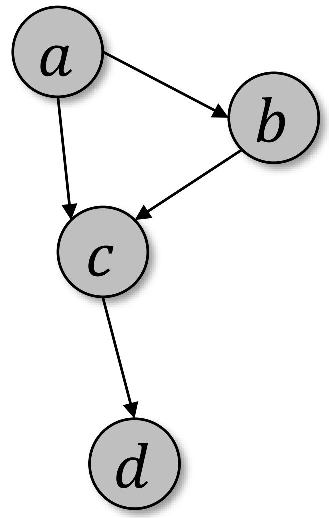
Vertices = pages, edges = hyperlinks

If your problem has a graph, great. If not, try to make it have a graph!

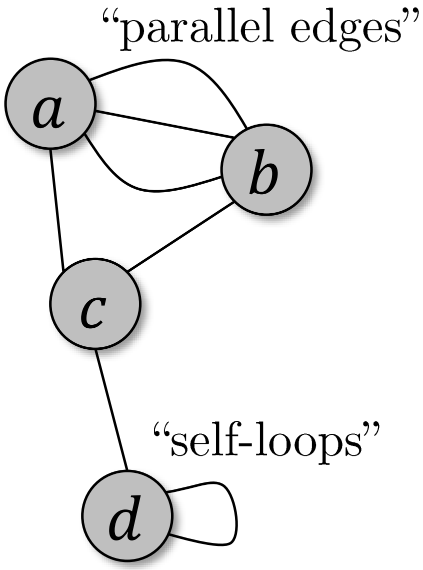
Types of graphs



Simple Undirected Graphs



Directed Graphs



General Graphs

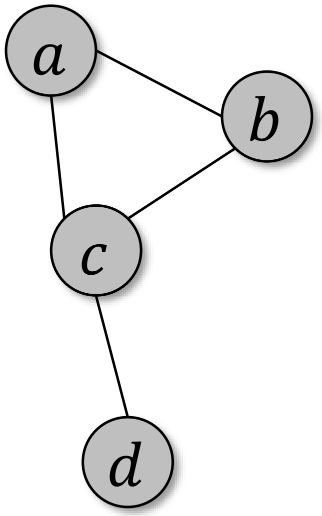
Retronym

Acoustic
Guitar

Electric
Guitar

Basic Definitions

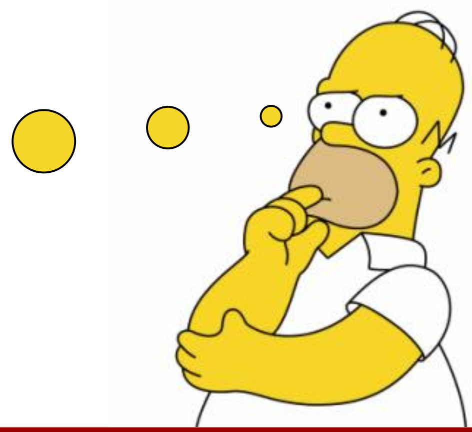
- A graph G is a pair:
  - V is the set of vertices/nodes; $|V| = n$
  - E is the set of edges; $|E| = m$
- Each edge is a pair $\{u, v\}$, where $u \neq v$
- Example:
  - $V = \{a, b, c, d\}$
  - $E = \{\, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}\, \}$



Edge Cases

- A graph with no edges is called an empty graph
- Example:
  - $V = \{1,2,3,4\}$
  - $E = \emptyset$

Graph with no vertices?



The Null Graph

IS THE NULL-GRAPH A POINTLESS CONCEPT?

Frank Harary
University of Michigan
and Oxford University

Ronald C. Read
University of Waterloo

ABSTRACT

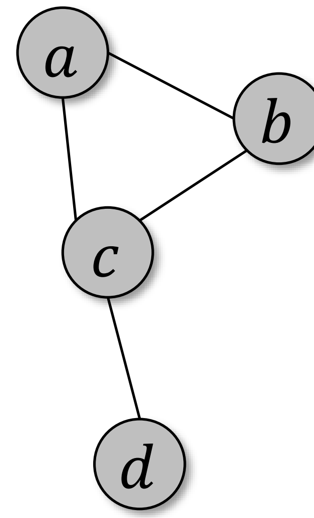
The graph with no points and no lines is discussed critically. Arguments for and against its official admittance as a graph are presented. This is accompanied by an extensive survey of the literature. Paradoxical properties of the null-graph are noted. No conclusion is reached.

The Null Graph

Figure 1. The Null Graph

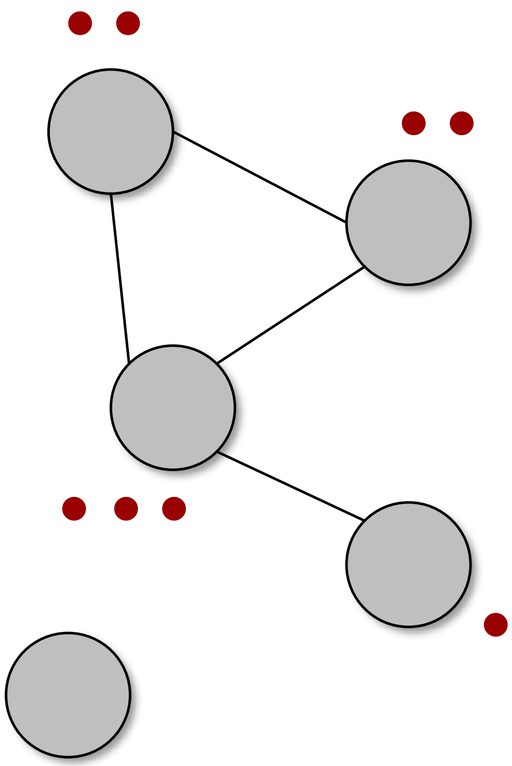
Mr. Vertex's Neighborhood

- If $\{u, v\} \in E$, u is a neighbor of v
- The neighborhood $N(u)$ of u is $\{v \in V \mid \{u, v\} \in E\}$
- The degree $\deg(u)$ of u is $|N(u)|$



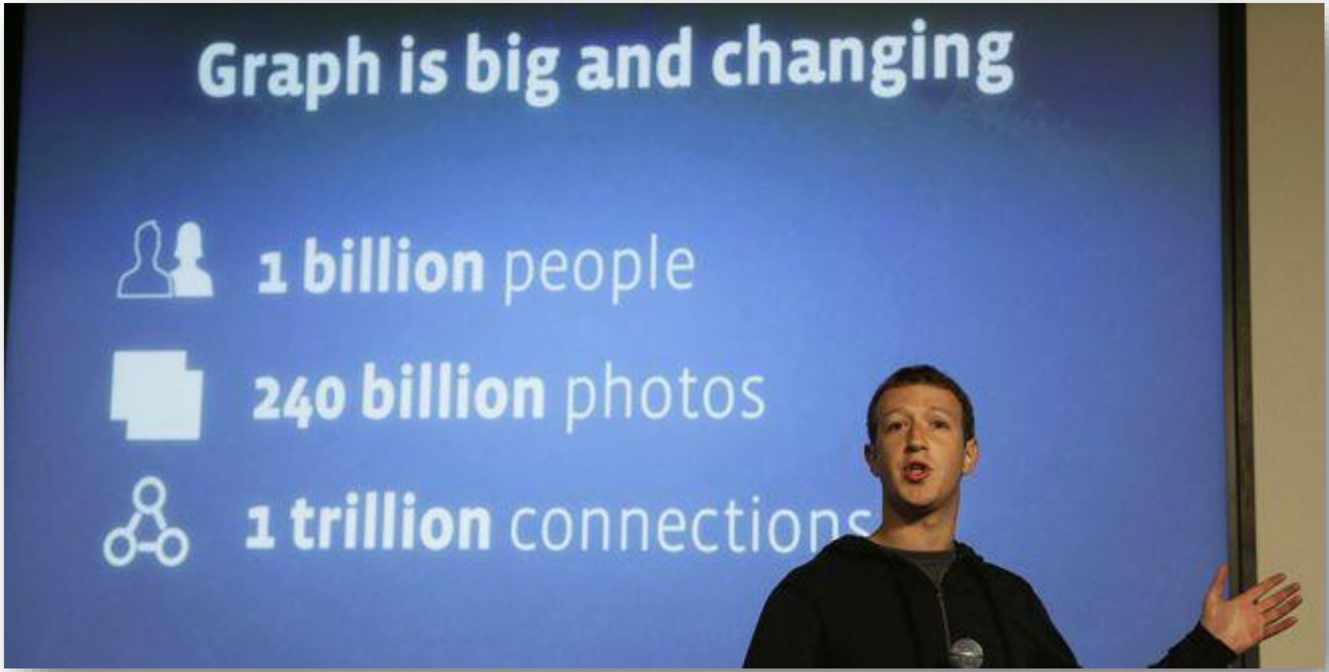
$N(b) = \{a, c\}$
$\deg(b) = 2$

- **Theorem:** $\sum_{u \in V} \deg(u) = 2m$
- **Proof:**
  - Each vertex places a token on each of its edges
  - The number of tokens is $\sum_{u \in V} \deg(u)$
  - Each edge has exactly two tokens placed on it
  - The number of tokens is $2m$ ■



$$2 + 2 + 3 + 1 = 2 \cdot 4$$

Facebook, revisited

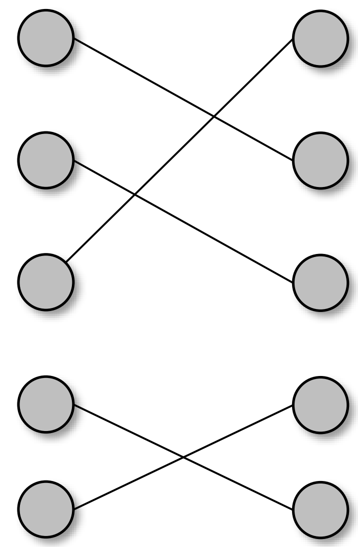


$$\#\text{vertices } n = 10^9, \ \#\text{edges } m = 10^{12}$$

Regular Graphs

- A graph is *d*-regular if all nodes have degree d
- The empty graph is 0-regular
- 1-regular graph is called a perfect matching
- Poll 1: How many 2-regular graphs with $V = \{a, b, c, d\}$ are there?

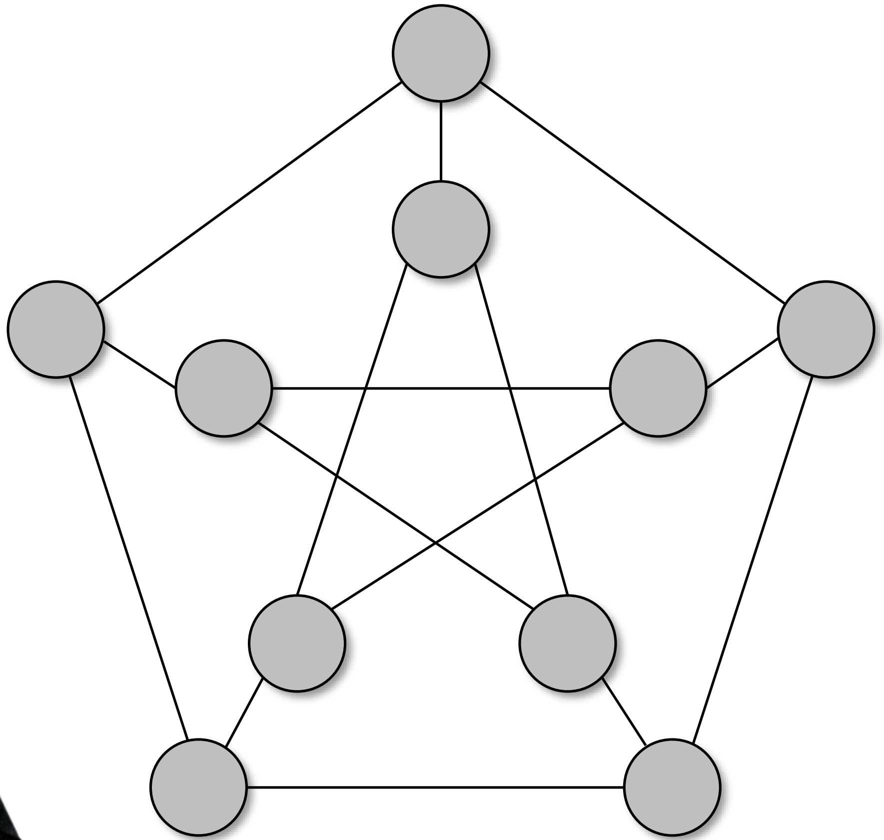
1 3 6 12

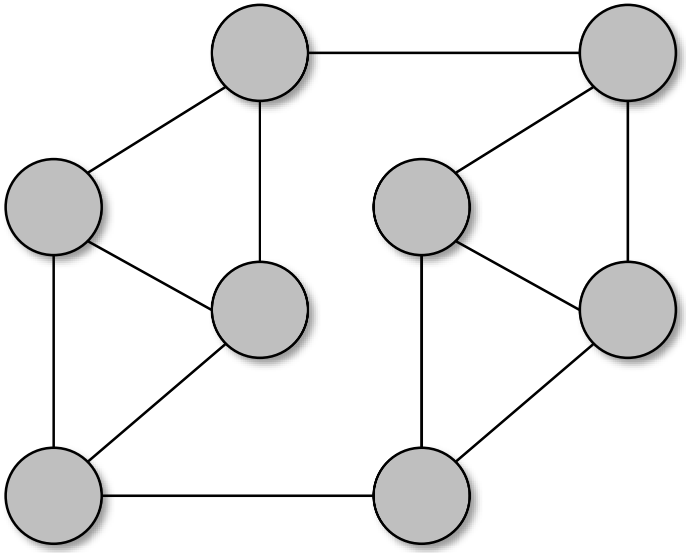


1-regular graph

3-REGULAR GRAPHS

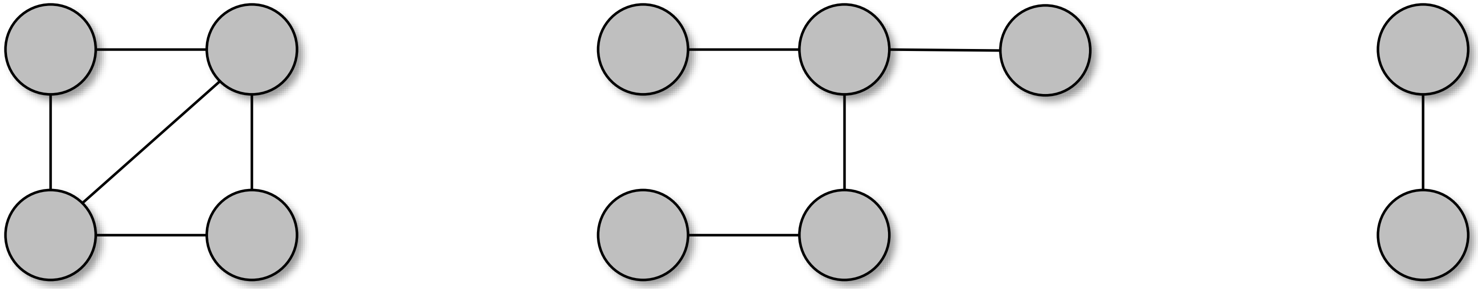
There are lots and lots of possibilities





Connectedness

- Graph G is connected if for all $u, v \in V$ there is a path between u and v

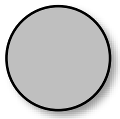


This 11-vertex graph is not connected
It has 3 connected components

Connectedness

What is the minimum number of edges needed to make a connected 27-vertex graph?

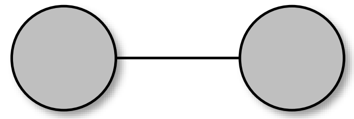
$n = 1$



Done

$m = 0$

$n = 2$

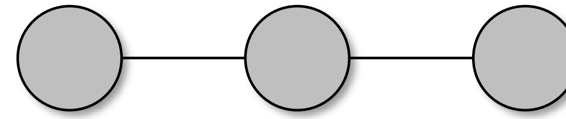


$m = 1$

necessary

and sufficient

$n = 3$

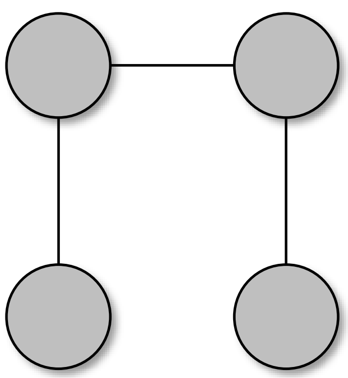


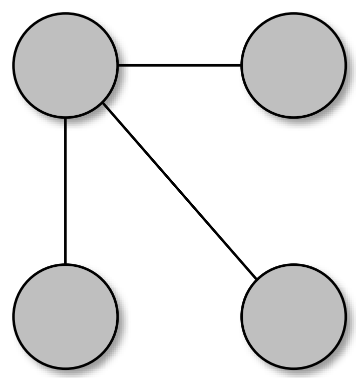
$m = 2$

necessary

and sufficient

$n = 4$





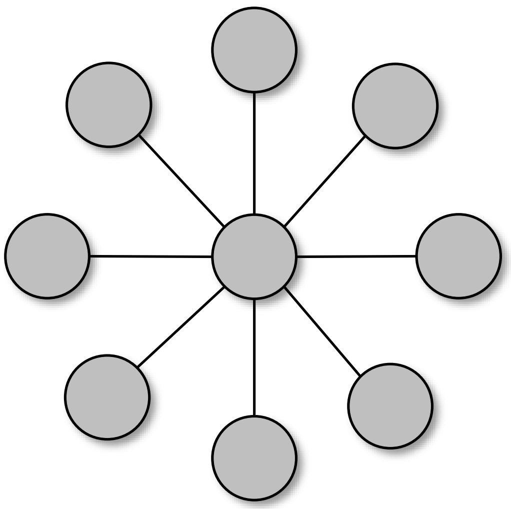
$m = 3$

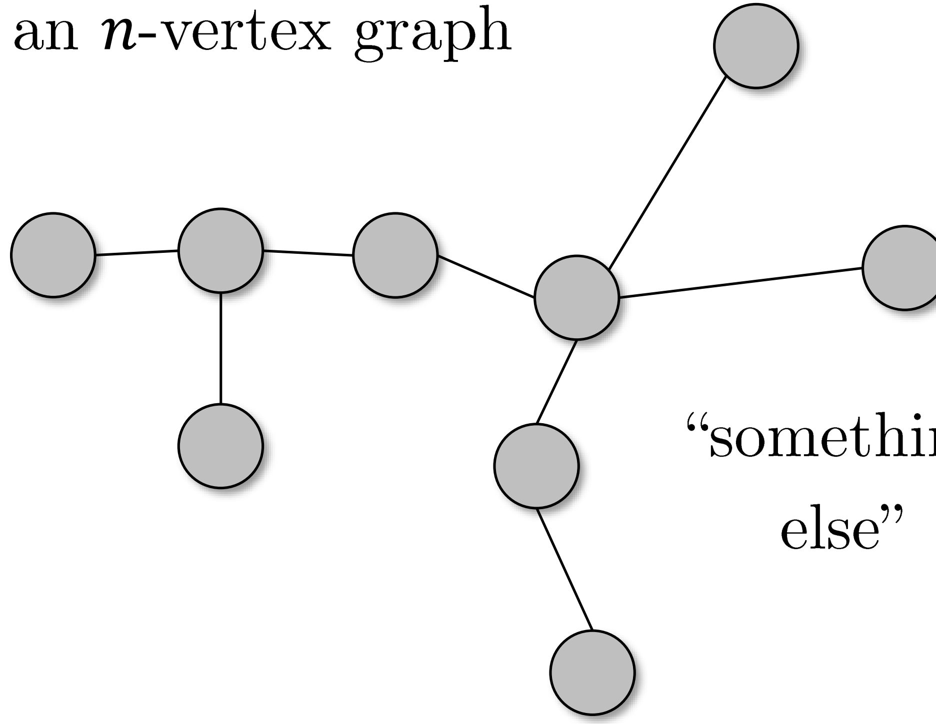
necessary

and sufficient

$n - 1$ edges are always sufficient to connect an n-vertex graph

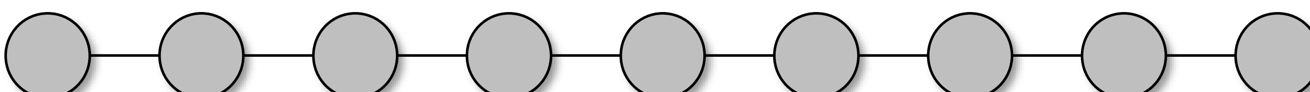
“star graph”



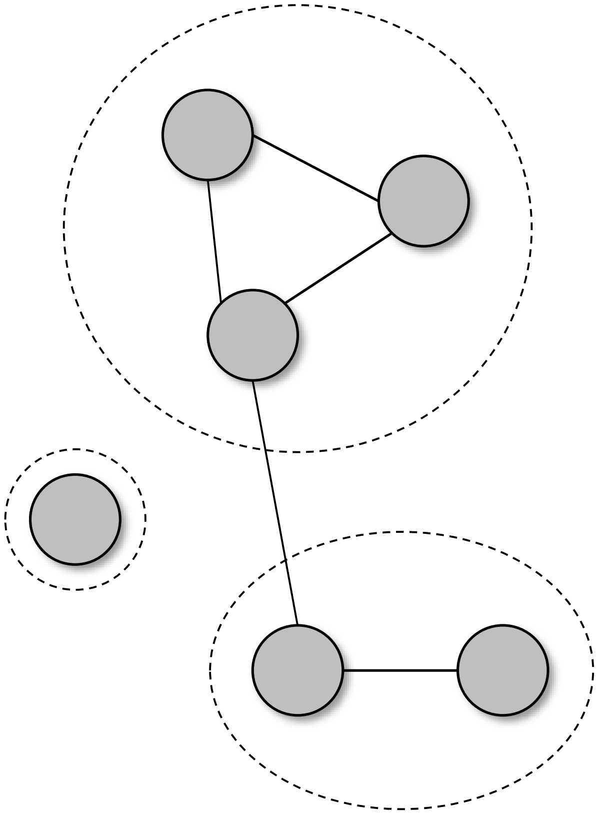


“something else”

“path graph”



- Theorem: $n-1$ edges are also necessary to connect an n-vertex graph
- Proof:
  - If G has k connected components, and G' is formed from G by adding an edge, then G' has at least $k-1$ components
  - Add edges one by one; to obtain a single connected component, need at least $n-1$ steps ■



Acyclic graphs

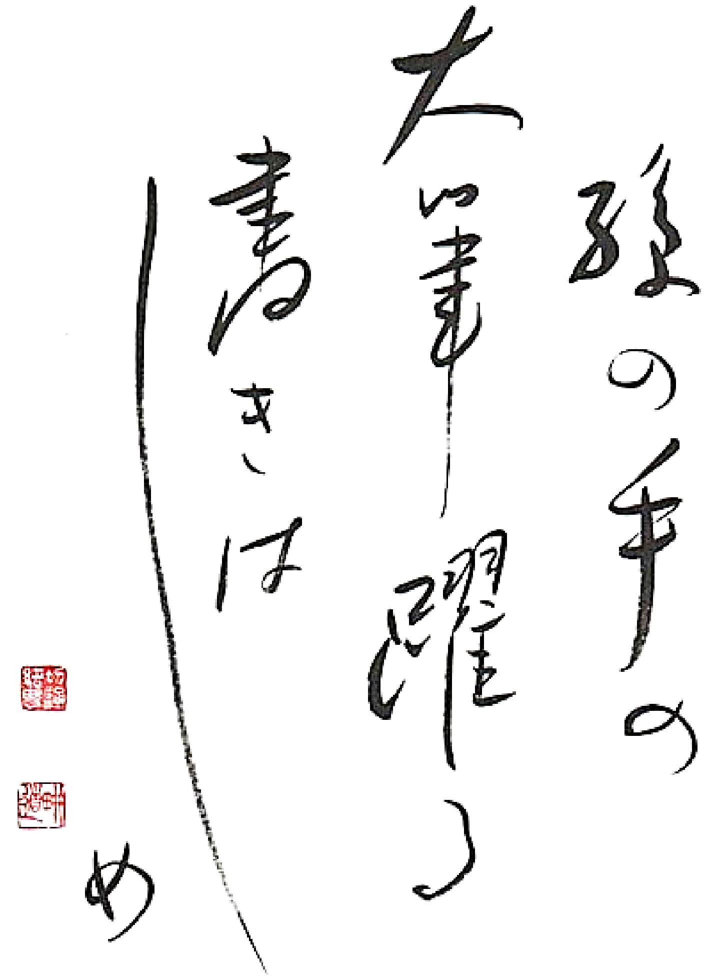
- Poll 2: Assume that G is connected. Then:
  1. $m = n - 1 \Rightarrow G$ is acyclic
  2. G is acyclic $\Rightarrow m = n - 1$
  3. G is acyclic $\Leftrightarrow m = n - 1$
  4. Incomparable

Trees

A tree is a connected acyclic graph

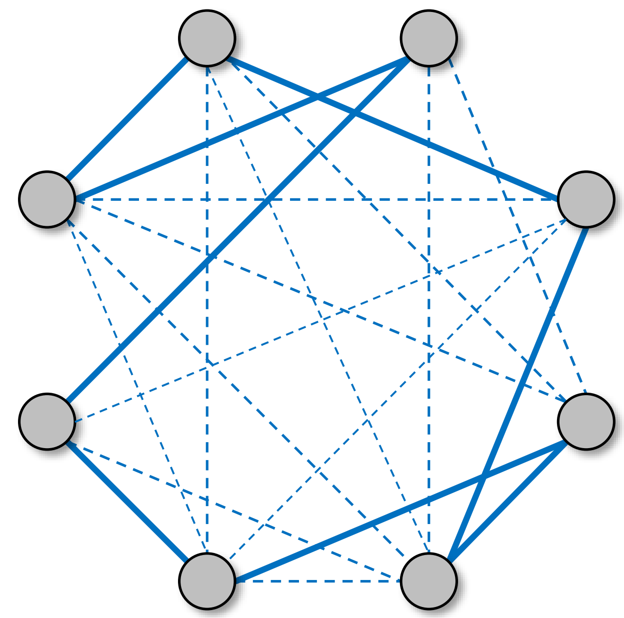
"Tree graph"

Graph Theory Haiku



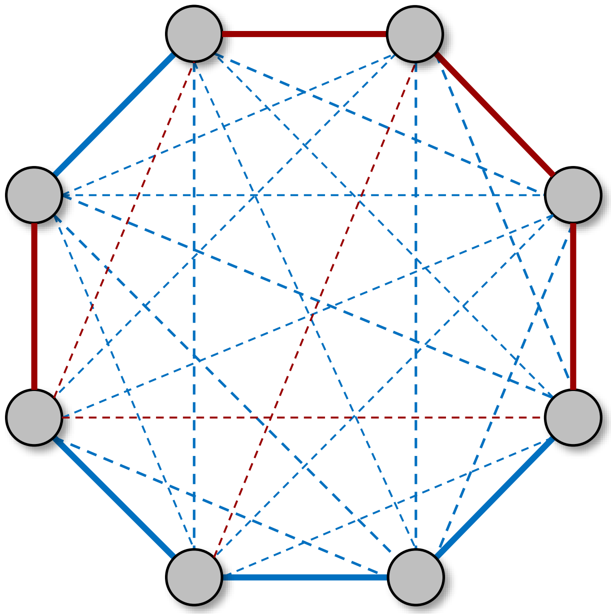
Ore's Theorem

- A Hamiltonian cycle in G is a cycle that visits every $v \in V$ exactly once (see Lect. 9)
- Theorem [Ore, 1960]: Let G be a graph on $n \geq 3$ vertices such that $\deg(u) + \deg(v) \geq n$ for any $u, v \in V$ that are not neighbors, then G contains a Hamiltonian Cycle



Proof of Ore's Theorem

- Color the edges of G blue, add red edges to form a complete graph, and choose a Hamiltonian Cycle C
- If C is not completely blue, will find C' with more blue edges

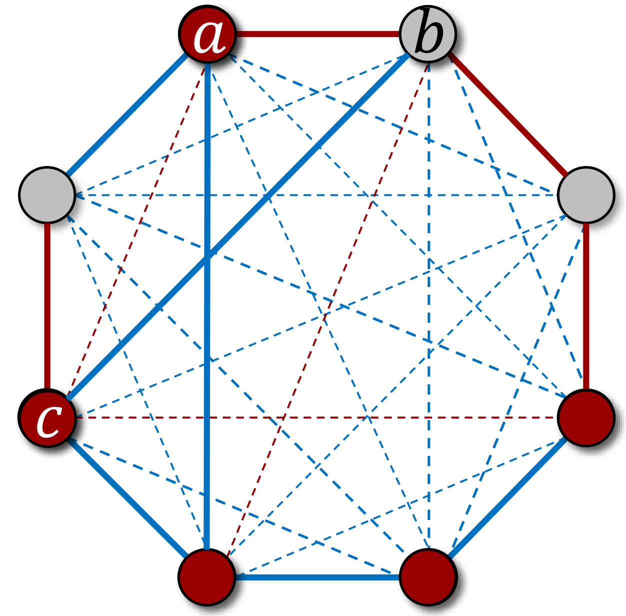


Proof of Ore's Theorem

- Let $\{a, b\}$ be a red edge in C
- Let S be the successors of $N(a)$ on C
- $\deg(b) \geq n - \deg(a)$

$$
\begin{aligned}
&= |V| - |N(a)| \\
&= |V| - |S| \\
&> |V \setminus (S \cup \{b\})|
\end{aligned}
$$

- So b is a neighbor of $c \in S$
- We can find a bluer cycle ■



SUMMARY

- Terminology:
  - Regular graph
  - Connected graph
  - Neighborhood, degree
  - Hamiltonian cycle
- Theorems:
  - If G is connected,
    $|E| = n - 1 \Leftrightarrow$ acyclic
  - $\sum_{u \in V} \deg(u) = 2m$