Great Ideas in Theoretical CS

Lecture 21:
Probability I

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Gambling 101

- You choose a die first, I choose second
- We both throw; higher number wins
- Which die would you choose?

	5			7			8			3	
5	5	5	1	7	1	8	2	8	3	3	3
	5			7			2			9	
	5			7			2			9	
	A			B			C			D	

Gambling 101

- Antoine Gombaud (1607-1684) made history for being a loser

  I will roll a die four times; I win if I get a 1

- After a while no one would take the bet
- $1-\left(\frac{5}{6}\right)^4=0.518$

Gambling 101

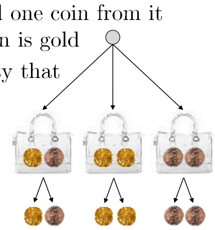
- Gombaud invented a new scam:

  I will roll two dice 24 times;
  I win if I get a double 1

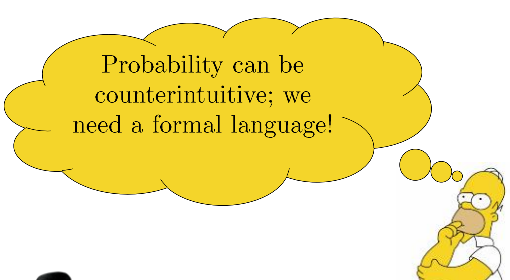
- Why was he losing money?
- $1 - \left(\frac{35}{36}\right)^{24} = 0.491$
- Gombaud wrote to Pascal and Fermat, who subsequently created probability theory

Pennies and gold

- Three bags contain two gold coins, two pennies, and one of each
- Bag is chosen at random, and one coin from it is selected at random; the coin is gold
- Poll 1: What is the probability that the other coin is gold?
  1. 1/6
  2. 1/3
  3. 2/3
  4. 1



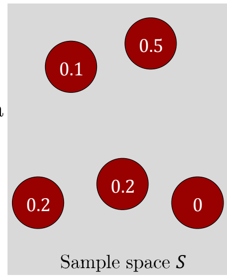
Language of probability



Language of probability

- The sample space is a finite set of elements S
- A probability distribution p assigns a non-negative real probability to each element, such that

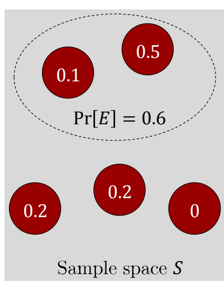
$$\sum_{x \in S} p(x) = 1$$



Language of probability

- An event is a subset $E \subseteq S$
- $\Pr[E] = \sum_{x \in E} p(x)$
- If each element $x \in S$ has equal probability, the distribution is uniform:

$$\Pr[E] = \sum_{x \in E} p(x) = \frac{|E|}{|S|}$$

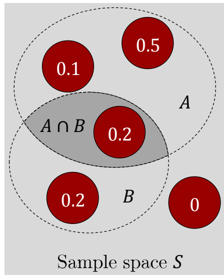


Language of probability

- We roll a white die and black die
- $S =$ $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
  $(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
  $(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
  $(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
  $(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
  $(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$
- Poll 2: Probability that the sum is 7 or 11?
  1. $1/9$
  2. $2/9$
  3. $3/9$
  4. $4/9$

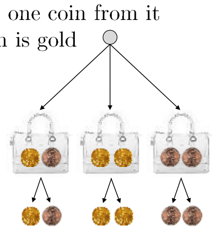
Conditional probability

- The probability of event A given event B is defined as $\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$
- Think of it as the proportion of $A \cap B$ to B



Pennies and gold, revisited

- Three bags contain two gold coins, two pennies, and one of each
- Bag is chosen at random, and one coin from it is selected at random; the coin is gold
- G_i: coin $i \in \{1,2\}$ is gold
- $\Pr[G_1] = \frac{1}{2}, \Pr[G_1 \cap G_2] = \frac{1}{3}$
- $\Pr[G_2|G_1] = \frac{1/3}{1/2} = \frac{2}{3}$



Conditional probability

- $\Pr[A \cap B] = \Pr[B] \times \Pr[A|B]$
- Interpretation: For A and B to occur, B must occur, and A must occur given that B occurred
- Applying iteratively: $\Pr[A_1 \cap \cdots \cap A_n] = \Pr[A_1] \times \Pr[A_2|A_1] \times \cdots \Pr[A_n|A_1, \cdots, A_{n-1}]$

This is called the chain rule

BAYES' RULE

- $\Pr[B] \times \Pr[A|B] = \Pr[A \cap B] = \Pr[A] \times \Pr[B|A]$

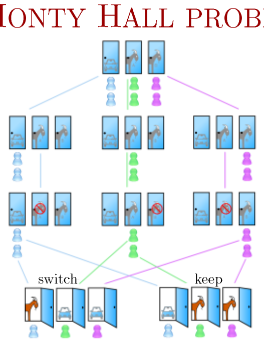
Bayes' rule:

$$\Pr[A|B] = \frac{\Pr[A]\Pr[B|A]}{\Pr[B]}$$

MONTY HALL PROBLEM

- Announcer hides prize behind one of three doors at random
- You choose a door
- Announcer opens a door with no prize
- Should you stay with your choice or switch?

MONTY HALL PROBLEM



Monty Hall problem

- Choose door 1, door 2 opens
- $\Pr[P_3|O_2] = \frac{\Pr[P_3]\Pr[O_2|P_3]}{\Pr[O_2]}$
- $\Pr[P_3] = \frac{1}{3}, \Pr[O_2|P_3] = 1,$ $\Pr[O_2] = 1/2$
- Therefore, $\Pr[P_3|O_2] = 2/3$
- Poll 3: Assuming there are five doors, what is the probability of winning when switching?
  1. $3/15$
  2. $4/15$
  3. $5/15$
  4. $6/15$

Independence

- Events A and B are independent if and only if $\Pr[A|B] = \Pr[A]$
- Poll 4: Which of the following events are independent when rolling black die and white die?
  1. Black die is 1, white die is 1
  2. Sum is 2, sum is 3
  3. Black die is 1, product is 2
  4. Black die is 1, sum is 2

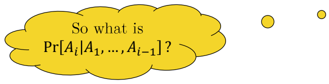
The birthday paradox

- m people in a room; suppose all birthdays are equally likely (excluding Feb 29); what is the probability that two people have the same birthday?
- $S = \{1, \dots, 365\}^m$, sample $\vec{x} = (x_1, \dots, x_m)$
- $E = \{\vec{x} \in S \mid \exists i, j, \text{s.t. } x_i = x_j\}$

The birthday paradox

- E is the event that two people share a birthday
- We will compute $\bar{E}$
- Let A_i be the event that person i's birthday differs from the birthdays of $1, \ldots, i-1$
- $\bar{E} = A_1 \cap \cdots \cap A_n$
- Using the chain rule:

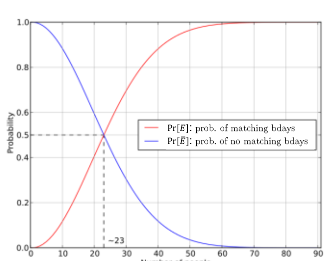
$$\Pr[\bar{E}] = \Pr[A_1] \times \Pr[A_2|A_1] \times \cdots \Pr[A_n|A_1, \cdots, A_{n-1}]$$



The birthday paradox

- $A_1 \cap \cdots \cap A_{i-1}$ means first $i-1$ students had different birthdays
- $i-1$ out of 365 occupied when ith birthday is chosen
- $\Pr[A_i|A_1, \ldots, A_{i-1}] = \frac{365-(i-1)}{365} = 1 - \frac{i-1}{365}$
- $\Pr[\bar{E}] = 1 \times \left(1 - \frac{1}{365}\right) \times \cdots \times \left(1 - \frac{m-1}{365}\right)$
- $\Pr[E] = 1 - \Pr[\bar{E}]$

The birthday paradox



The birthday paradox

- Poll 5: What is the probability that two people have the same birthday if there are 730 people?
  1. 1/2
  2. 0.75
  3. 0.99999999999997
  4. 1

Birthday attack*

- A cryptographic hash function 'scrambles' a string S into a k-bit hash $f(S)$
- It should be hard to find a collision: two strings S_1, S_2 such that $f(S_1) = f(S_2)$
- Application: digital signatures
  - Alice wants Bob to sign a message m
  - They compute $f(m)$ and it is signed using Bob's secret key
  - Bad collision: Alice can find a fair contract m and a fraudulent contract m' such that $f(m) = f(m')$

Birthday attack*

- The SHA-1 cryptographic hash function uses 160 bits
- To find a collision for SHA-1, take a huge number of strings, hash them all, and hope that two hash to the same string
- If SHA-1 is really safe, each $f(S)$ should be uniform in $\{1, \ldots, 2^{160}\}$
- This is like the birthday problem with 2^{160} days of the year!

Birthday attack*

- To find a collision you would need roughly $\sqrt{2^{160}} = 2^{80}$ strings
- A crypto hash function is considered broken if you can beat the birthday attack
- SHA-1 collisions can be found using "only" 2^{63} strings
- On 2/23/2017, Google and CWI announced that they had generated two different PDF files with the same SHA-1 hash

* Just for fun

Summary

- Terminology:
  - Language of probability
  - Conditional probability
  - Independence
- Principles:
  - Chain rule
  - Bayes' rule