NORMAL-FORM GAME

• A game in normal form consists of:
  ◦ Set of players $N = \{1,...,n\}$
  ◦ Strategy set $S$
  ◦ For each $i \in N$, utility function $u_i : S^n \to \mathbb{R}$; if each $j \in N$ plays the strategy $s_j \in S$, the utility of player $i$ is $u_i(s_1,\ldots,s_n)$

• Next example created by taking screenshots of http://youtu.be/jILgxeNBK_8
THE ICE CREAM WARS

- \( N = \{1,2\} \)
- \( S = [0,1] \)
- \( u_i(s_i, s_j) = \begin{cases} \frac{s_i + s_j}{2}, & s_i < s_j \\ 1 - \frac{s_i + s_j}{2}, & s_i > s_j \\ \frac{1}{2}, & s_i = s_j \end{cases} \)
- To be continued...

THE PRISONER’S DILEMMA

- Two men are charged with a crime
- They are told that:
  - If one rats out and the other does not, the rat will be freed, other jailed for nine years
  - If both rat out, both will be jailed for six years
- They also know that if neither rats out, both will be jailed for one year

<table>
<thead>
<tr>
<th></th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>-1,-1</td>
<td>-9,0</td>
</tr>
<tr>
<td>Defect</td>
<td>0,-9</td>
<td>-6,-6</td>
</tr>
</tbody>
</table>

What would you do?
UNDERSTANDING THE DILEMMA

• Defection is a dominant strategy
• But the players can do much better by cooperating
• Related to the tragedy of the commons

IN REAL LIFE

• Presidential elections
  • Cooperate = positive ads
  • Defect = negative ads
• Nuclear arms race
  • Cooperate = destroy arsenal
  • Defect = build arsenal
• Climate change
  • Cooperate = curb CO₂ emissions
  • Defect = do not curb

ON TV

http://youtu.be/S0qjK3TwZE8
THE PROFESSOR’S DILEMMA

<table>
<thead>
<tr>
<th>Class</th>
<th>Listen</th>
<th>Sleep</th>
</tr>
</thead>
<tbody>
<tr>
<td>Make effort</td>
<td>$10^6, 10^6$</td>
<td>$-10, 0$</td>
</tr>
<tr>
<td>Slack off</td>
<td>$0, -10$</td>
<td>$0, 0$</td>
</tr>
</tbody>
</table>

Dominant strategies?

NASH EQUILIBRIUM

- Each player’s strategy is a best response to strategies of others
- Formally, a Nash equilibrium is a vector of strategies $s = (s_1, ..., s_n) \in S^n$ such that
  \[
  \forall i \in N, \forall s'_i \in S, u_i(s) \geq u_i(s'_i, s_{-i})
  \]
NASH EQUILIBRIUM

http://youtu.be/CemLiSI5ox8

11/21/2017

RUSSELL CROWE WAS WRONG

END OF THE ICE CREAM WARS
Does NE Make Sense?

• Two players, strategies are \{2, ..., 100\}
• If both choose the same number, that is what they get
• If one chooses \(s\), the other \(t\), and \(s < t\), the former player gets \(s + 2\), and the latter gets \(s - 2\)
• Poll 2: what would you choose?

Back to Prison

• The only Nash equilibrium in Prisoner’s dilemma is bad; but how bad is it?
• Objective function: social cost = sum of costs
• NE is six times worse than the optimum
ANARCHY AND STABILITY

- Fix a class of games, an objective function, and an equilibrium concept
- The price of anarchy (stability) is the worst-case ratio between the worst (best) objective function value of an equilibrium of the game, and that of the optimal solution
- In this lecture:
  - Objective function = social cost (sum of costs)
  - Equilibrium concept = Nash equilibrium

EXAMPLE: COST SHARING

- $n$ players in weighted directed graph $G$
- Player $i$ wants to get from $s_i$ to $t_i$; strategy space is $s_i \rightarrow t_i$ paths
- Each edge $e$ has cost $c_e$
- Cost of edge is split between all players using edge
- Cost of player is sum of costs over edges on path

EXAMPLE: COST SHARING

- With $n$ players, the example on the right has an NE with social cost $n$
  - Optimal social cost is 1
  - $\Rightarrow$ Price of anarchy $\geq n$
  - Price of anarchy is also $\leq n$
    - Each player can always deviate to his strategy at the optimal solution, and pay for it alone; the cost is at most OPT
    - At equilibrium, no player wants to deviate, so each player pays at most OPT
**Example: Cost Sharing**

- Think of the 1 edges as cars, and the 2 edge as mass transit
- Bad Nash equilibrium with cost \( n \)
- Good Nash equilibrium with cost 2
- Now let’s modify the example...

**Example: Cost Sharing**

- \( \text{OPT} = 2 \)
- Poll 3: What is the social cost at Nash equilibrium?
- \( \Rightarrow \) price of stability is at least this cost \( f(n)/2 \)
- Theorem: The price of stability of cost sharing games is \( \leq f(n) \)

**Cost Sharing Summary**

- In every cost sharing game
  - \( \forall \text{NE } s, \text{cost}(s) \leq n \cdot \text{OPT} \)
  - \( \exists \text{NE } s \text{ such that } \text{cost}(s) \leq f(n) \cdot \text{OPT} \)
- There exist cost sharing games s.t.
  - \( \exists \text{NE } s \text{ such that } \text{cost}(s) \geq n \cdot \text{OPT} \)
  - \( \forall \text{NE } s, \text{cost}(s) \geq \Omega(f(n)) \cdot \text{OPT} \)
Summary

• Terminology:
  - Normal-form game
  - Nash equilibrium
  - Price of anarchy/stability
  - Cost sharing games

• Nobel-prize-winning ideas:
  - Nash equilibrium 😊