Great Ideas in Theoretical CS

Lecture 9: Time Complexity

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The big O

Adding two n-bit numbers

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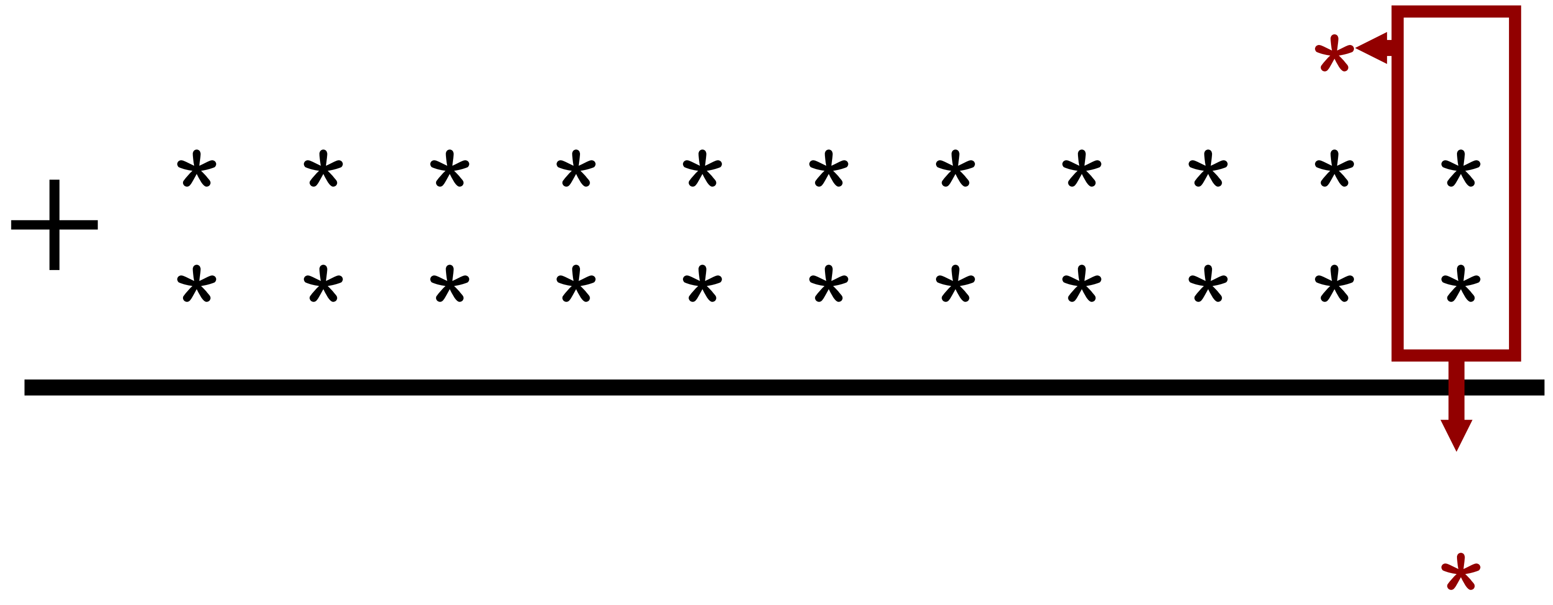
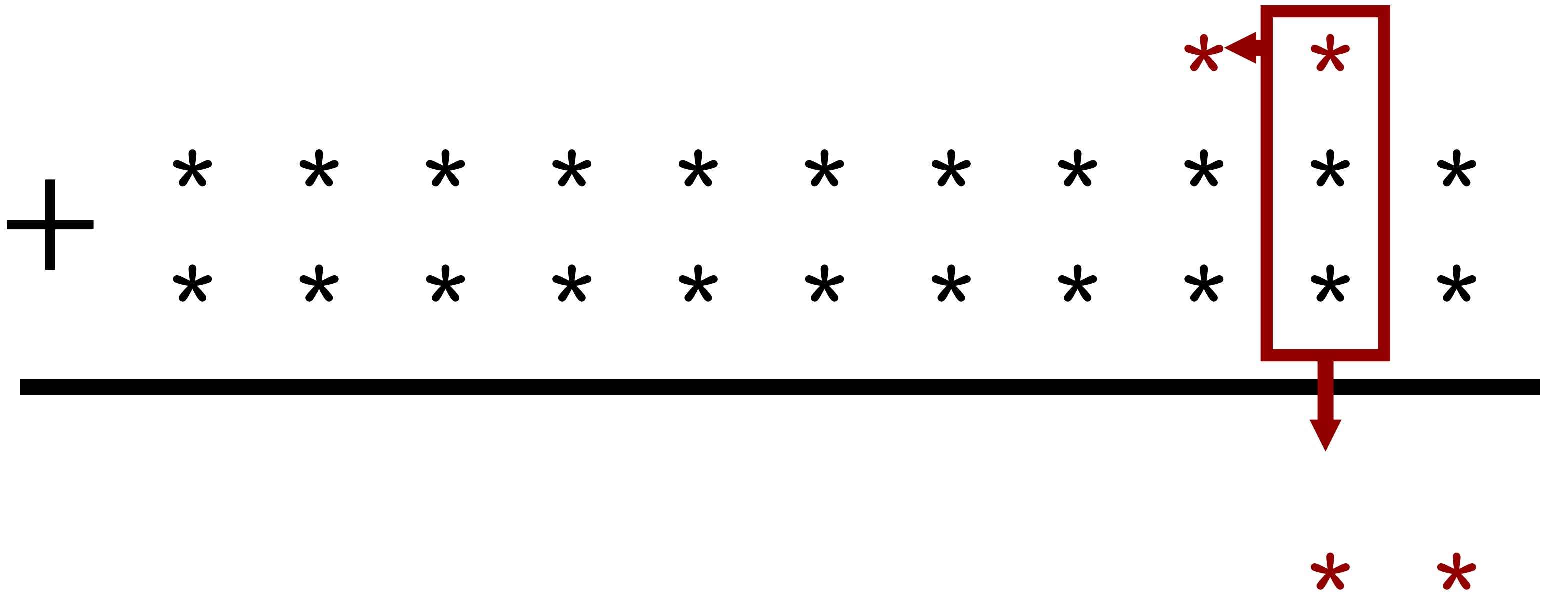
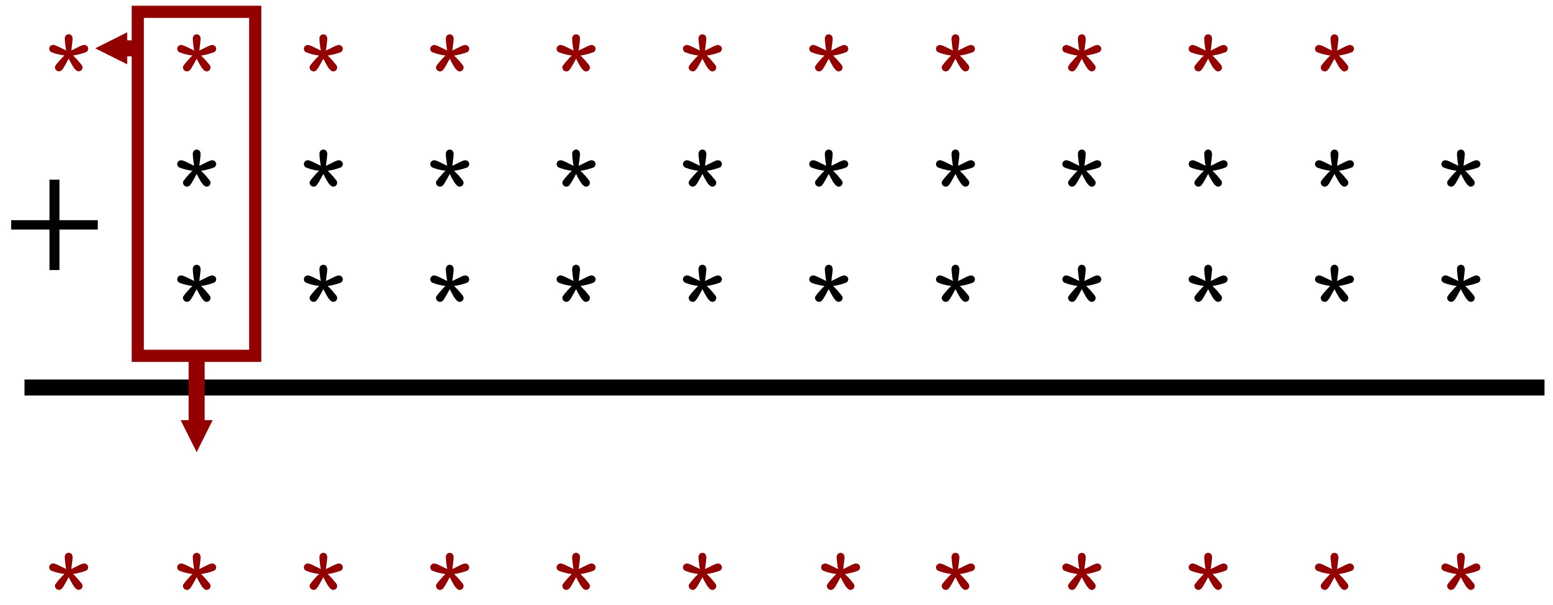
Adding two n-bit numbers



Adding two n-bit numbers



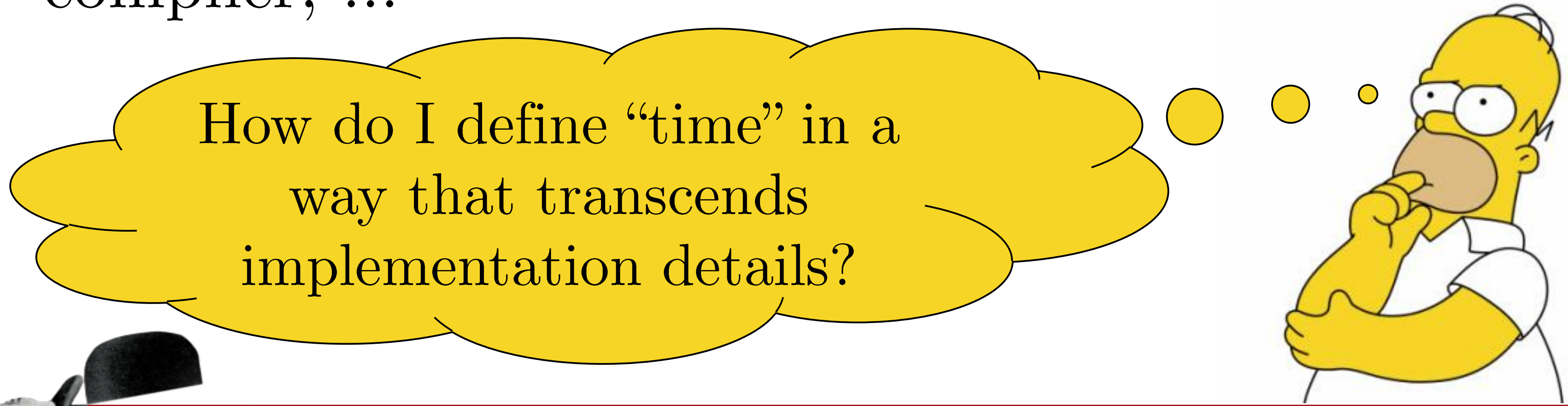
Adding two n-bit numbers



Grade school addition

Time complexity

- $T(n)$ = amount of time grade school addition takes to add two n-bit numbers
- What do we mean by "time"?
- Given algorithm will take different amounts of time on the same input depending on hardware, compiler, ...

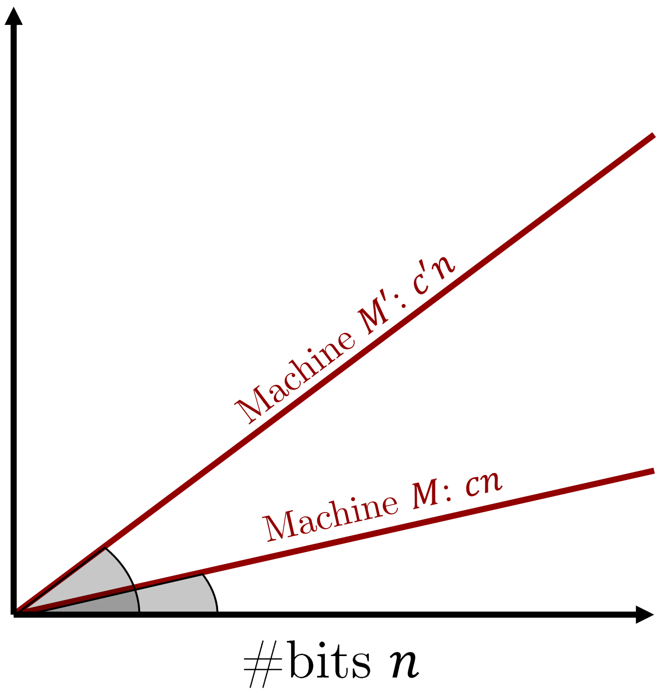

How do I define "time" in a way that transcends implementation details?

A great idea

- On any reasonable computer, adding 3 bits and writing down the 2 bit answer can be done in constant time
- For a computer M, let c be the time it takes to perform on M
- The total time to add two n-bit numbers using grade school addition on M is $c \cdot n$
- On M', the time to perform could be c'
- The total time on M' is $c'n$

A great idea

- The fact that we get a line is invariant under different implementations
- Different machines result in different slopes, but the running time grows linearly

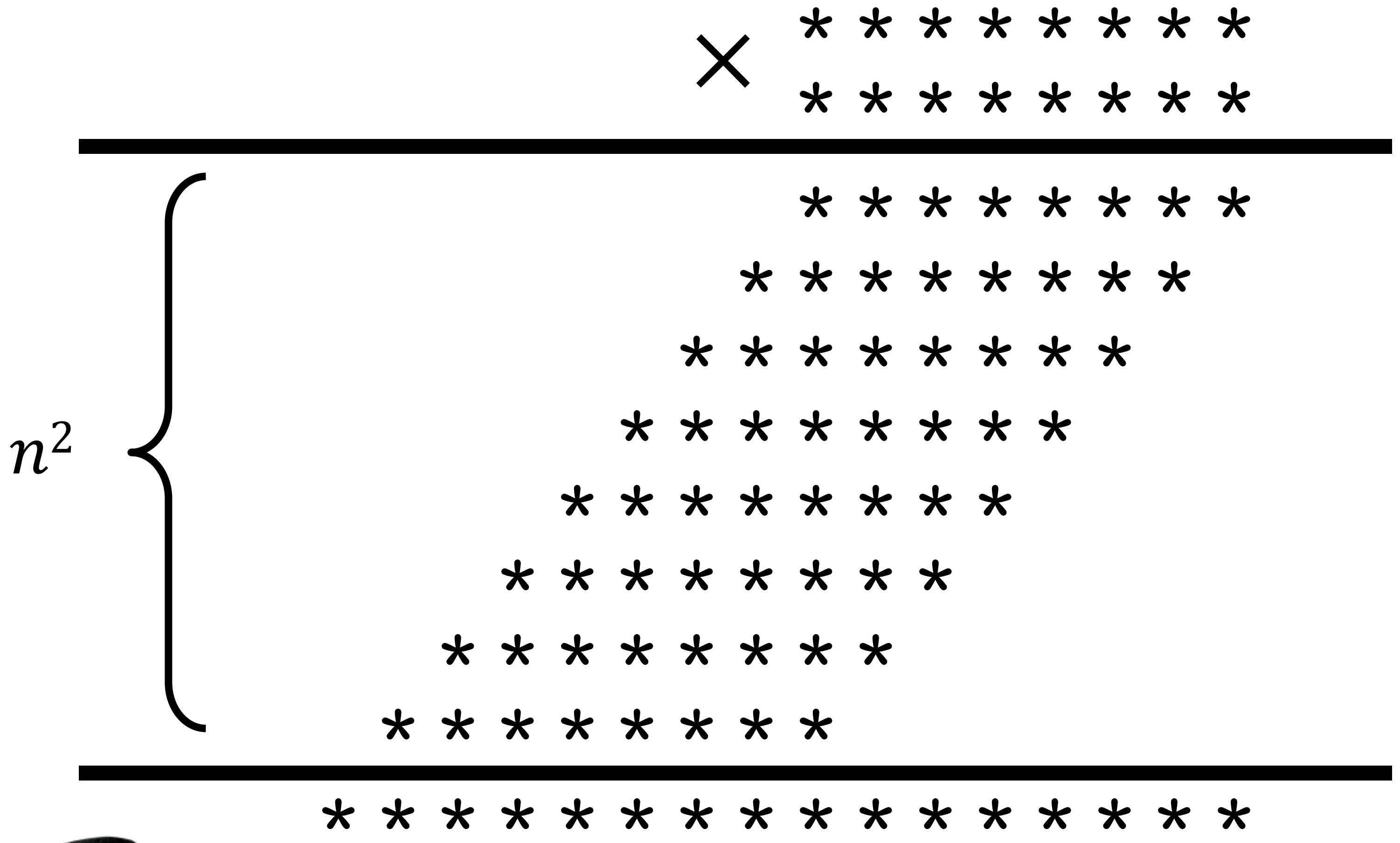


A great idea

- Conclusion: Grade school addition is a linear time algorithm

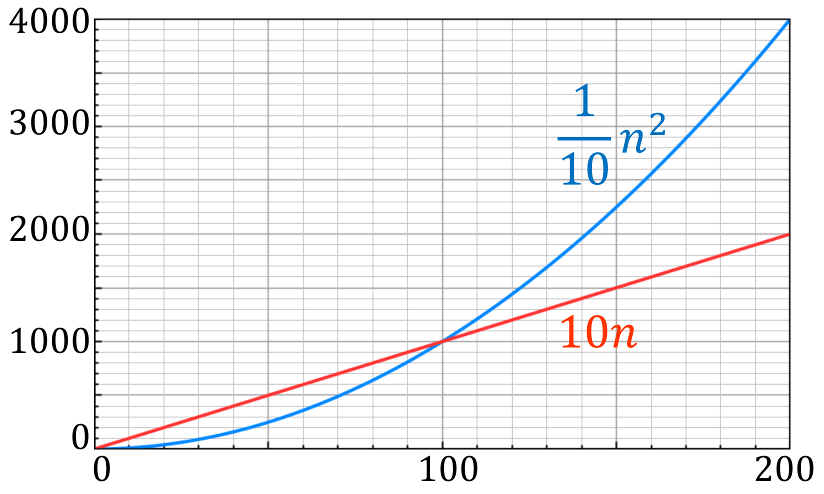
This process of abstracting away details and determining the rate of resource usage in terms of the problem size n is one of the fundamental ideas in computer science

Multiplying two n-bit numbers



Linear vs. quadratic

- Total time to multiply: cn^2
- Addition is linear time, multiplication is quadratic time
- Regardless of the constants, the quadratic curve will eventually dominate the linear curve



Nursery school addition

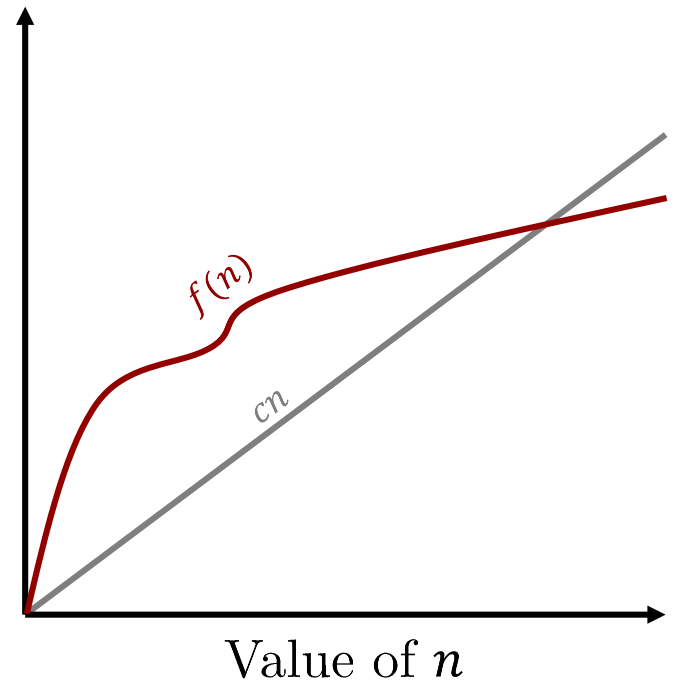
- To add two n-bit numbers a and b, start at a and increment (by 1) b times
- What is $T(n)$?
- If $b = 00\cdots0$, NSA takes almost no time
- Poll 1: If $b = 11\cdots1$, NSA takes time
  1. $c(\log n)^2$
  2. $cn\log n$
  3. cn^2
  4. $cn2^n$

Worst case time

Worst-case running time $T(n)$ of algorithm $A =$ the maximum over all feasible inputs x of size n of the running time of A on x

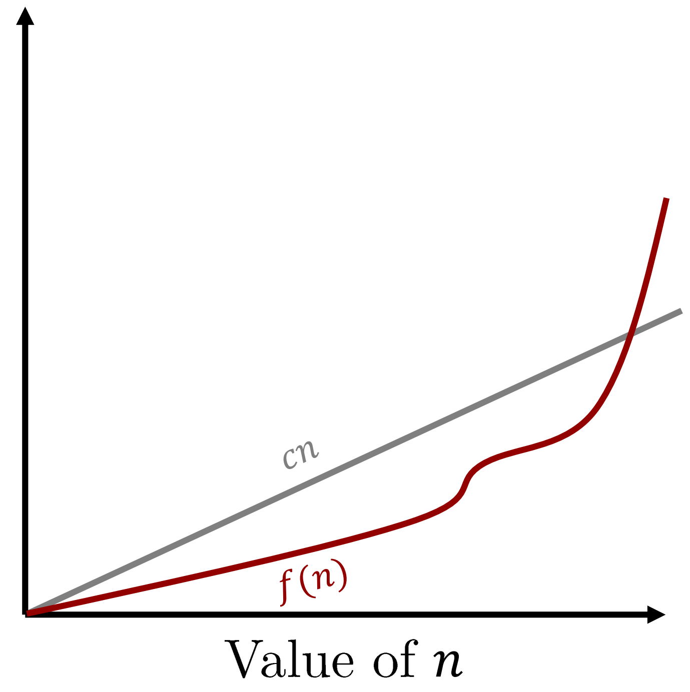
More formally: O

- For a function $f: \mathbb{N} \to \mathbb{N}$, $f(n) = O(n)$ if there exists a constant c such that for all sufficiently large n, $f(n) \leq cn$
- Informally: There is a line that can be drawn that stays above f from some point on



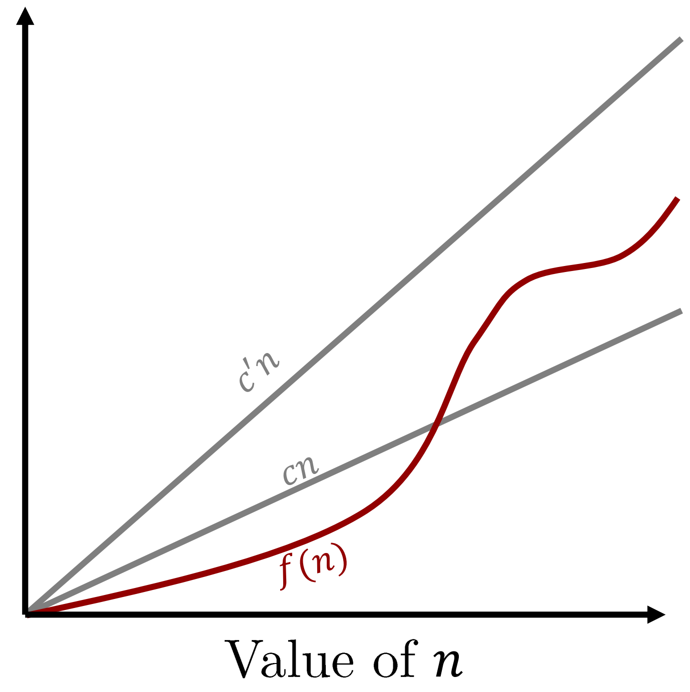
More formally: Ω

- For a function $f\colon \mathbb{N} \rightarrow \mathbb{N}$, $f(n) = \Omega(n)$ if there exists a constant c such that for all sufficiently large n, $f(n) \geq cn$
- Informally: There is a line that can be drawn that stays below f from some point on



More formally: Θ

- For a function $f: \mathbb{N} \to \mathbb{N}$, $f(n) = \Theta(n)$ if $f(n) = O(n)$ and $f(n) = \Omega(n)$
- Informally: f can be sandwiched between two lines from some point on



More formally and generally

- $f(n) = O(g(n))$ if there exists a constant c such that for all sufficiently large n, $f(n) \leq c \cdot g(n)$
- $f(n) = \Omega(g(n))$ if there exists a constant c such that for all sufficiently large n, $f(n) \geq c \cdot g(n)$
- $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Exercises

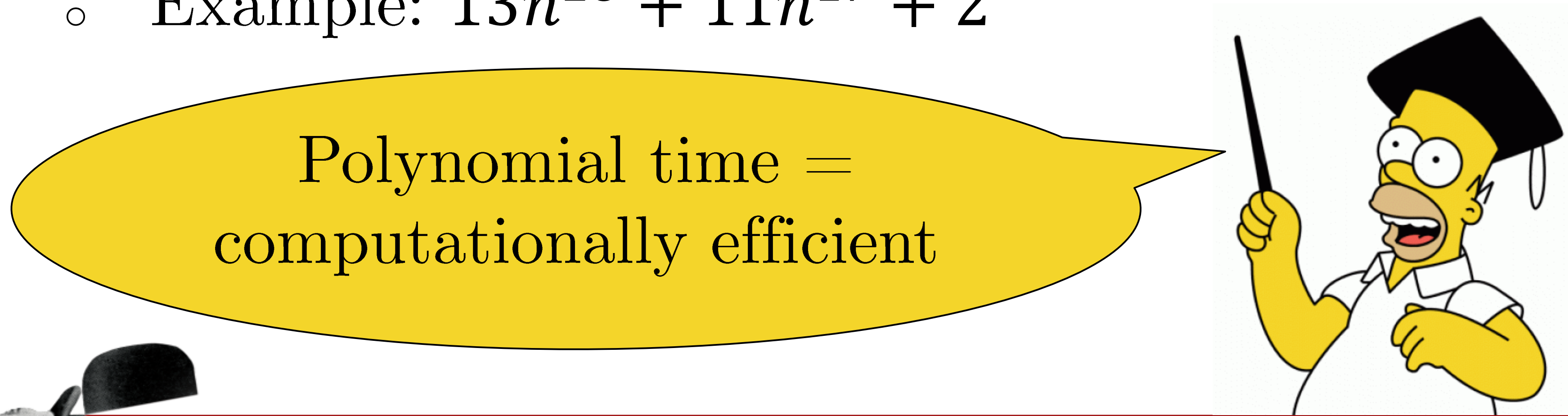
- $n^4 + 3n + 22 = O(n^4)$?
- $n^4 + 3n + 22 = \Omega(n^4 \log n)$?
- Poll 2: Which of the following statements is true:
  1. $\ln n = O(\log_2 n)$
  2. $\ln n = \Omega(\log_2 n)$
  3. Both
  4. Neither

Exercises

- Poll 3: $\log(n!) = ?$
  1. $\Theta(n)$
  2. $\Theta(n \log n)$
  3. $\Theta(n^2)$
  4. $\Theta(2^n)$
- Poll 4: Which of the following statements is true:
  1. $f = O(g)$ and $g = O(h) \Rightarrow f = O(h)$
  2. $f = O(h)$ and $g = O(h) \Rightarrow f = O(g)$
  3. Both
  4. Neither

Names for growth rates

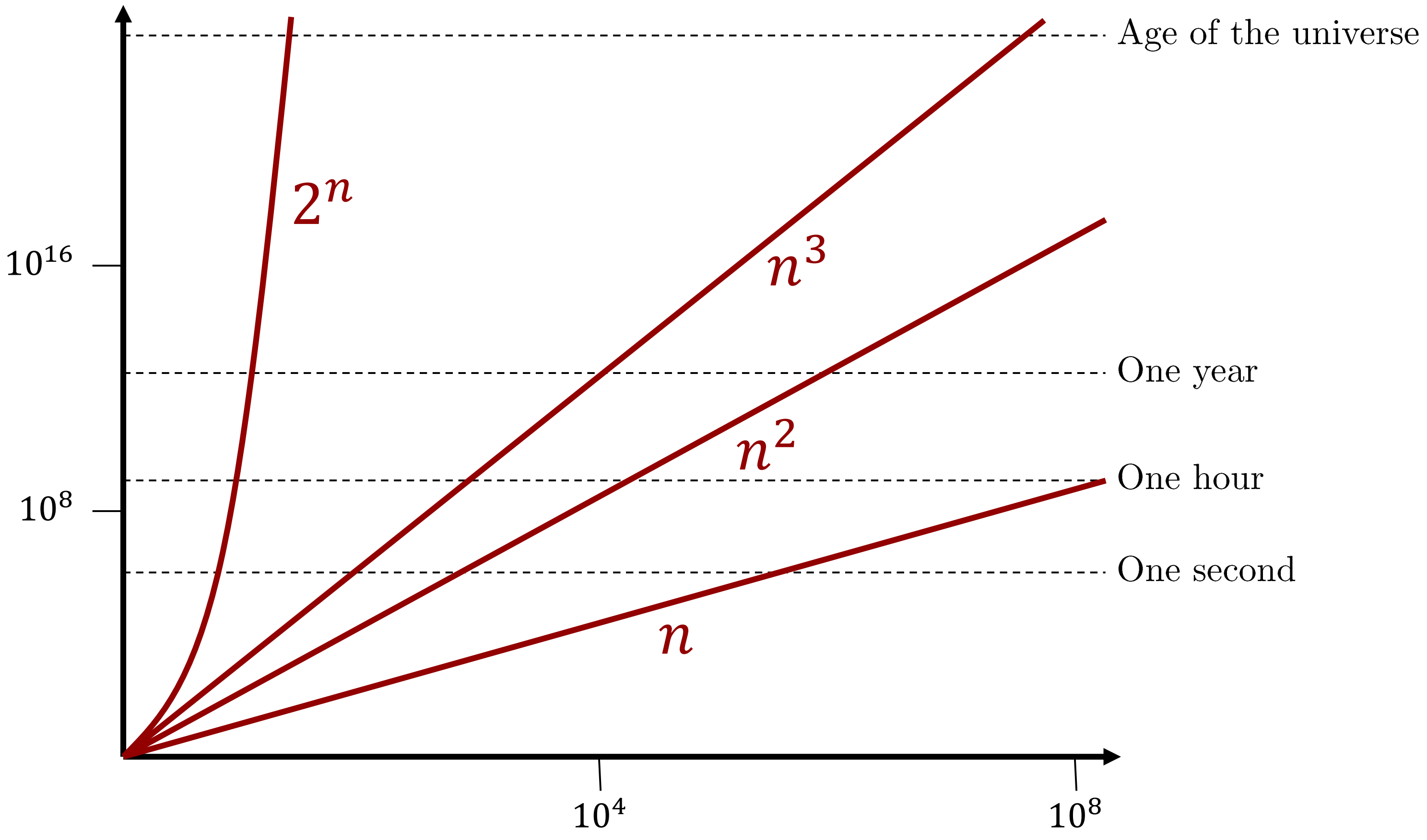
- Linear time: $T(n) = O(n)$
- Quadratic time: $T(n) = O(n^2)$
- Polynomial time: there exists $k \in \mathbb{N}$ such that $T(n) = O(n^k)$
    - Example: $13n^{28} + 11n^{17} + 2$



Names of growth rates

- Exponential time: there exists $k \in \mathbb{N}$ such that $T(n) = O(k^n)$
  - Example: $T(n) = n2^n = O(3^n)$
- Logarithmic time: $T(n) = O(\log n)$
  - A logarithmic-time algorithm can't read all of its input
  - The running time of binary search is logarithmic

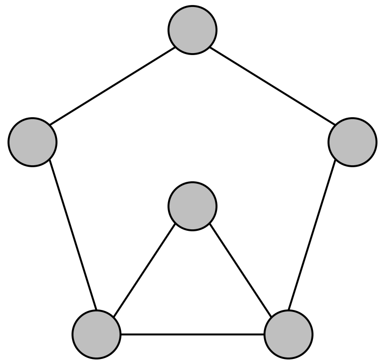
Limits of the possible



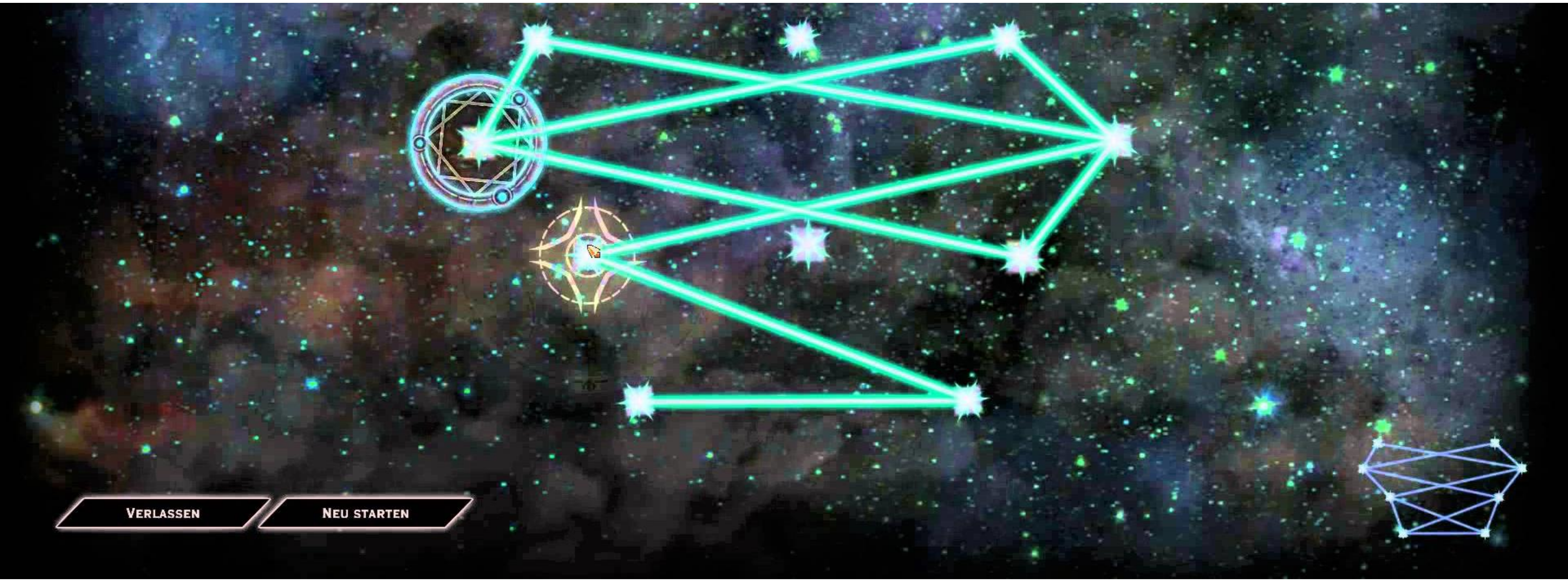
Log-log plot with 1 step = 1μs

Two similar problems

- Eulerian-Cycle:
  - Instance: A connected graph
  - Input size: Number of vertices
  - Question: Is there a tour visiting each edge exactly once?
- Algorithm: The answer is "yes" if and only if each vertex has even degree; complexity $O(n^2)$
- Theorem (Euler): The algorithm correctly solves Eulerian-Cycle



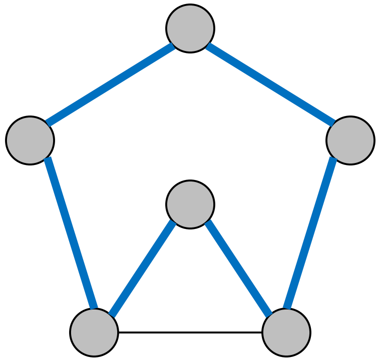
Application: Dragon Age



This is nothing but the EULERIAN-PATH problem!

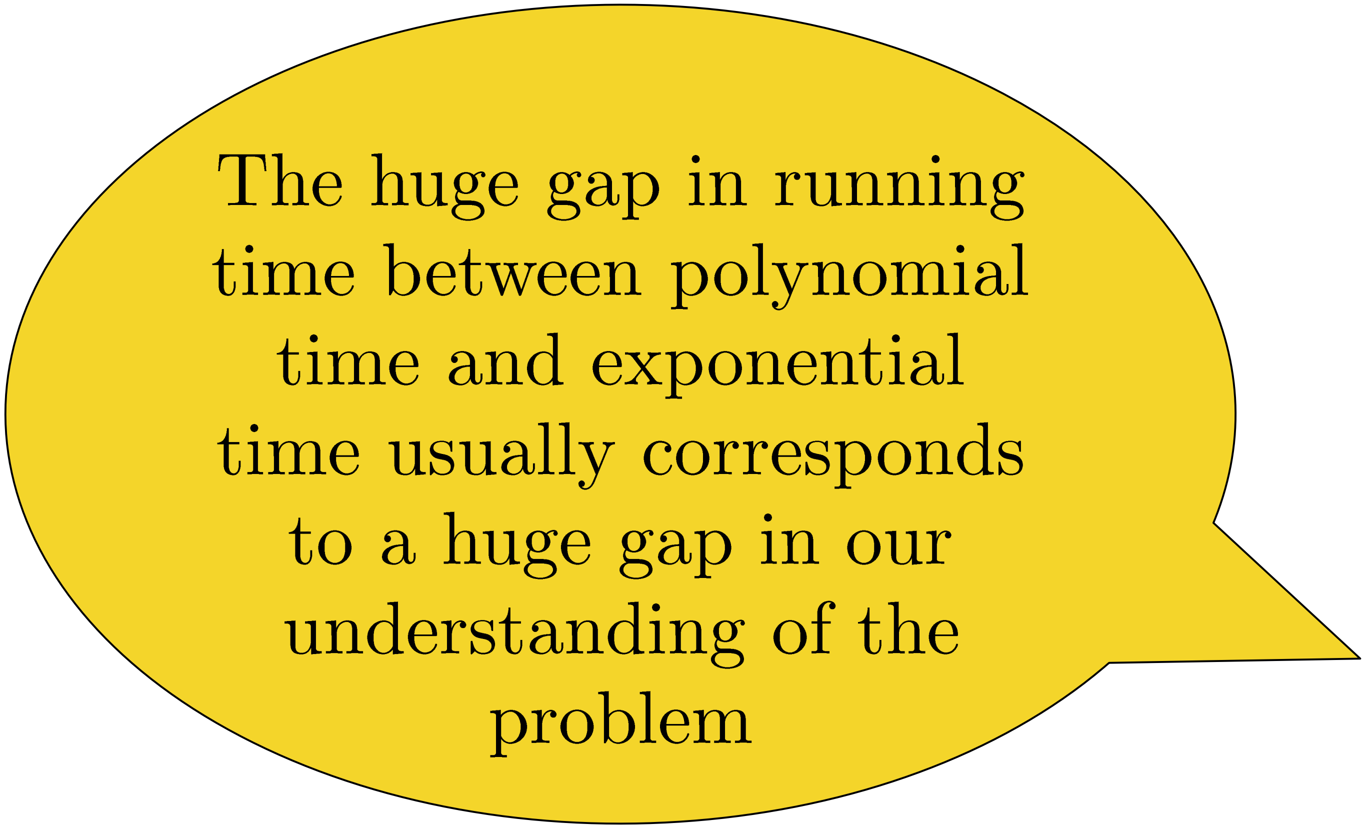
Two similar problems

- Hamiltonian-Cycle:
  - Instance: A connected graph
  - Input size: Number of vertices
  - Question: Is there a tour visiting each vertex exactly once?



- Complexity:
  - Brute force algorithm: $n!$
  - 1970: 2^n
  - 2010: 1.657^n

Polynomial time


The huge gap in running time between polynomial time and exponential time usually corresponds to a huge gap in our understanding of the problem

Representation

- The way a problem is represented can have a huge impact on its complexity
- KNAPSACK:
  - Instance: m items $1, \dots, m$ with values $v_1, \dots, v_m$ and weights $w_1, \dots, w_m$, capacity B, value V
  - Input size: We'll talk about this later
  - Question: Is there a subset of items S such that $\sum_{i \in S} w_i \leq B$ and $\sum_{i \in S} v_i \geq V$

Representation

- Dynamic programming algorithm for Knapsack:
    - $m \times B$ matrix A
    - $A(i,j) = \max\{A(i-1,j), A(i-1,j-w_i)+v_i\}$

Item	value	weight
1	4	5
2	3	2
3	5	2

$\boxed{B = 5}$

	1	2	3	4	5
1	0	0	0	0	4
2	0	3	3	3	4
3	0	5	5	8	8

Capacity allowed

Items allowed

Representation

- Running time of the dynamic programming algorithm: $\Theta(mB)$
- Binary representation for Knapsack:
  - Input size: $n \approx 2m \cdot \max\{\log B, \log V\}$
  - Exponential running time!
- Unary representation for Knapsack:
  - Input size: $n \approx 2m \cdot \max\{B, V\}$
  - Linear running time!

Cool growth rates: 2stack

- $\mathrm{2STACK}(0) = 1$
- $\mathrm{2STACK}(n) = 2^{\mathrm{2STACK}(n-1)}$
- Examples:
  - $\mathrm{2STACK}(1) = 2$
  - $\mathrm{2STACK}(2) = 4$
  - $\mathrm{2STACK}(3) = 16$
  - $\mathrm{2STACK}(4) = 65536$
  - $\mathrm{2STACK}(5) =$ yikes!

$$2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}$$

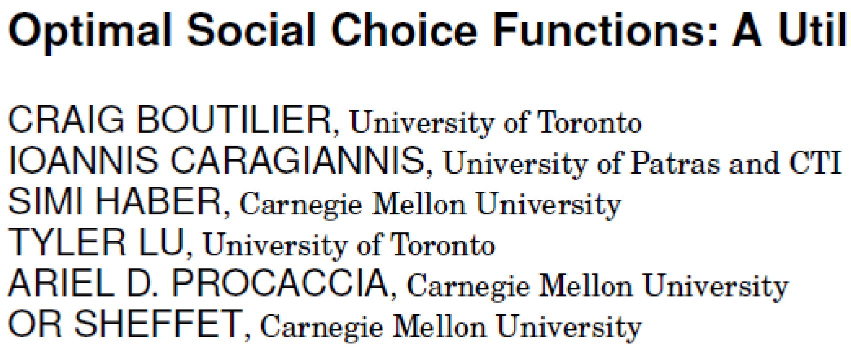
Cool growth rates: log*

- $\log^*(n) = \#$times you have to apply the log function to n to make it ≤ 1
- Examples:

$\text{2STACK}(1) = 2$	$\log^*(2) = 1$
$\text{2STACK}(2) = 4$	$\log^*(4) = 2$
$\text{2STACK}(3) = 16$	$\log^*(16) = 3$
$\text{2STACK}(4) = 65536$	$\log^*(65536) = 4$
$\text{2STACK}(5) = \text{yikes!}$	$\log^*(\text{yikes!}) = 5$

Cool growth rates: log*

- There's no way **log*** is actually useful, right?
- Multiplication takes $O(n \log n \, 2^{\log^* n})$



Optimal Social Choice Functions: A Utilitarian View

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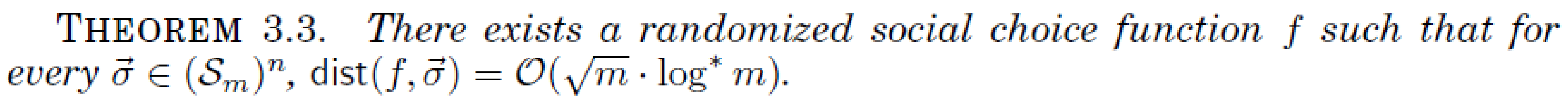

THEOREM 3.3. *There exists a randomized social choice function f such that for every $\vec{\sigma} \in (\mathcal{S}_m)^n$*, $\mathrm{dist}(f, \vec{\sigma}) = \mathcal{O}(\sqrt{m} \cdot \log^* m)$.

Summary

- Terminology:
  - Big O notation
  - Names for growth rates
- Principles:
  - Why polynomial time?
  - Representation matters