15-251
Great Ideas in
Theoretical Computer Science

Lecture 14:
Graphs IV: Stable Matchings

October 12th, 2017

Hall’s Theorem

Characterization for perfect matchings

Often we are interested in perfect matchings.

\[ |X| \neq |Y| \]
Characterization for perfect matchings

Often we are interested in perfect matchings.

An obstruction:
If $|X| > |Y|$, we cannot “cover” all the nodes in $X$.
If $|X| > |N(X)|$, we cannot “cover” all the nodes in $X$.

Characterization for perfect matchings

Often we are interested in perfect matchings.

An obstruction:
For $S \subseteq X$:
if $|S| > |N(S)|$, we cannot “cover” all the nodes in $S$.

Characterization for perfect matchings

Is this the only type of obstruction?

Theorem [Hall’s Theorem]:

Corollary:
Suppose a deck of cards is dealt into 13 piles of 4 cards each.

**Claim**: there is always a way to select one card from each pile so that you have one card from each rank.

**So we want to show**:

For any $S \subseteq X$, $|S| \leq |N(S)|$.

For any $S \subseteq X$, total weight coming out $= 4|S|$.

All this weight is absorbed by $N(S)$.

Each $y \in N(S)$ absorbs $\leq 4$ units of this weight.

$$\implies N(S) \text{ absorbs } \leq 4|N(S)| \implies 4|S| \leq 4|N(S)|$$
**Stable matching problem**

**2-Sided Markets**

A market with 2 distinct groups of participants each with their own preferences.

**Other examples:**
- medical residents - hospitals
- students - colleges
- professors - colleges

| 1. Alice | 1. Bob |
| 2. Bob | 2. David |
| 3. Charlie | 3. Alice |
Aspiration: A Good Centralized System

What can go wrong?

Alice  Macrosoft
Bob  Moogle
Charlie  Umbrella
David  KLG

Formalizing the problem

An instance of the problem can be represented as a complete bipartite graph + preference list of each node.

Students  Companies

(a, b, c, d)  (a, b, c, d)
(a, b, c, d)  (a, b, c, d)
(a, b, c, d)  (a, b, c, d)

Goal:

Formalizing the problem

What is a stable matching?

X  Y
(a, b)  (a, b)
(a, b)  (a, b)
A variant: Roommate problem

A non-bipartite version

\[(c,b,d) \quad (a,c,d) \quad (b,a,d) \quad (a,c,b)\]

Does this have a stable matching?

Stable matching: Is there a trivial algorithm?

\[
\begin{align*}
X & \quad Y \\
(e,f,h,g) & \quad (a,b,c,d) \\
(e,g,h,f) & \quad (a,b,c,d) \\
(e,h,f,g) & \quad (a,b,c,d) \\
(e,f,g,h) & \quad (a,b,c,d)
\end{align*}
\]

Trivial algorithm:

The Gale-Shapley proposal algorithm

While there is a man \(m\) who is not matched:

- Let \(w\) be the highest ranked woman in \(m\)'s list to whom \(m\) has not proposed yet.
- If \(w\) is unmatched, or \(w\) prefers \(m\) over her current match:
  - Match \(m\) and \(w\).
  (The previous match of \(w\) is now unmatched.)

Cool, but does it work correctly?

- Does it always terminate?
- Does it always find a stable matching?
  (Does a stable matching always exist?)
The Gale-Shapley proposal algorithm always terminates with a stable matching after at most $n^2$ iterations.

A constructive proof that a stable matching always exists.

3 things to show:

1. Number of iterations is at most $n^2$.

2. The algorithm terminates with a perfect matching.
   If we don’t have a perfect matching:
   A man is not matched
   $\Rightarrow$ All women must be matched
   $\Rightarrow$ All men must be matched. Contradiction
Gale-Shapley algorithm analysis

2. The algorithm terminates with a perfect matching.

If we don’t have a perfect matching:

A man is not matched

$\implies$ All women must be matched

$\implies$ All men must be matched.

**Contradiction**

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Gale-Shapley algorithm analysis

3. The matching has no unstable pairs.

“Improvement” Lemma:

(i) A man can only go down in his preference list.
(ii) A woman can only go up in her preference list.

**Unstable pair:**

$(m,w)$ unmatched

**but** they prefer each other.

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Further questions

**Theorem:**

The Gale-Shapley proposal algorithm always terminates with a stable matching after at most $n^2$ iterations.

Does the order of how we pick men matter?

Would it lead to different matchings?

Is the algorithm “fair”?

Does this algorithm favor men or women or neither?
Further questions

\textbf{m} and \textbf{w} are \textit{valid partners} if there is a stable matching in which they are matched.

\textbf{best}(\textbf{m}) = \text{highest ranked valid partner of} \textbf{m}

\textbf{Theorem:}

Further questions

\textbf{worst}(\textbf{w}) = \text{lowest ranked valid partner of} \textbf{w}

\textbf{Theorem:}

Real-world applications

Variants of the Gale-Shapley algorithm is used for:

- matching medical students and hospitals
- matching students to high schools (e.g. in New York)
- matching students to universities (e.g. in Hungary)
- matching users to servers
The Gale-Shapley Proposal Algorithm (1962)

Nobel Prize in Economics  2012

"for the theory of stable allocations and the practice of market design."