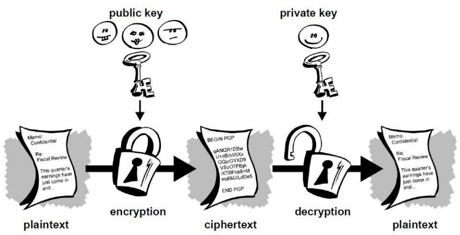
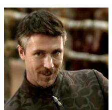
15-251
Great Ideas in
Theoretical Computer Science

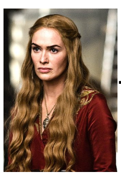
Lecture 27:
Cryptography

November 30th, 2017



What is cryptography about?





"loru23n8uladjkfb!#@"

"I will cut your throat"
↓ encryption
"loru23n8uladjkfb!#@"

"loru23n8uladjkfb!#@"
↓ decryption
"I will cut your throat"

What is cryptography about?

Study of protocols that avoid the bad affects of adversaries.

- Can two parties who have never met before share a secret by only communicating publicly?
- Can we have secure online voting schemes?
- Can we use digital signatures.
- Can we do computation on encrypted data?
- Can I convince you that I have proved P=NP without giving you any information about the proof?

⋮

Reasons to like cryptography

Can do pretty cool and unexpected things.

Has many important real-world applications.

Is fundamentally related to computational complexity.

In fact, computational complexity revolutionized crypto.
(exploit computationally hard problems)

There is good math (e.g. number theory).

The plan

Recall important things from **modular arithmetic**.

Private (secret) key cryptography.

Secret key sharing.

Public key cryptography.

Important Things to Remember from Last Time

$\mathbb{Z}_4$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\mathbb{Z}_N = \{0, 1, 2, \ldots, N-1\}$

behaves nicely with respect to *addition*

$\mathbb{Z}_8^*$

•	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

$\mathbb{Z}_N^* = \{A \in \mathbb{Z}_N : \gcd(A, N) = 1\}$

behaves nicely with respect to *multiplication*

$\varphi(N) = |\mathbb{Z}_N^*|$

if P prime, $\varphi(P) = P - 1$

if P, Q distinct primes, $\varphi(PQ) = (P-1)(Q-1)$

$\mathbb{Z}_5^*$

•	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\varphi(5) = 4$

1^0	1^1	1^2	1^3	1^4	1^5	1^6	1^7	1^8
1	1	1	1	1	1	1	1	1
2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8
1	2	4	3	1	2	4	3	1
3^0	3^1	3^2	3^3	3^4	3^5	3^6	3^7	3^8
1	3	4	2	1	3	4	2	1
4^0	4^1	4^2	4^3	4^4	4^5	4^6	4^7	4^8
1	4	1	4	1	4	1	4	1

2 and 3 are called generators.

$\mathbb{Z}_5^*$

•	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$\varphi(5) = 4$

1^0	1^1	1^2	1^3	1^4	1^5	1^6	1^7	1^8
1	1	1	1	1	1	1	1	1
2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8
1	2	4	3	1	2	4	3	1
3^0	3^1	3^2	3^3	3^4	3^5	3^6	3^7	3^8
1	3	4	2	1	3	4	2	1
4^0	4^1	4^2	4^3	4^4	4^5	4^6	4^7	4^8
1	4	1	4	1	4	1	4	1

$$\forall A, \quad A^4 = 1 \qquad \Longrightarrow \quad A^{4k} = (A^4)^k = 1$$

Euler's Theorem:

For any $A \in \mathbb{Z}_N^*$, $A^{\varphi(N)} = 1$.

1			
$\Vert$			
A^0	A^1	A^2	$\cdots$
$\Vert$	$\Vert$	$\Vert$	
$A^{\varphi(N)}$	$A^{\varphi(N)+1}$	$A^{\varphi(N)+2}$	$\cdots$
$\Vert$	$\Vert$	$\Vert$	
$A^{2\varphi(N)}$	$A^{2\varphi(N)+1}$	$A^{2\varphi(N)+2}$	$\cdots$

IMPORTANT!!!

When exponentiating elements $A \in \mathbb{Z}_N^*$

can think of the exponent living in the universe $\mathbb{Z}_{\varphi(N)}$.

Complexity of Arithmetic Operations

> addition $A +_N B$
Do regular addition. Then take mod *N*.

> subtraction $A -_N B$
-B = *N-B*. Then do addition.

> multiplication $A \cdot_N B$
Do regular multiplication. Then take mod *N*.

> division $A /_N B$
Find B^{-1}. Then do multiplication.

> exponentiation $A^B \bmod N$
Fast modular exponentiation: repeatedly square and mod.

> taking roots

> logarithm

No known efficient algorithm exists.

In $\mathbb{Z}$

$(B, E) \rightarrow$ EXP $\rightarrow B^E$ **hard**

Two inverse functions:

$(B^E, E) \rightarrow \mathrm{ROOT}_E \rightarrow B$ **easy**

$(B^E, B) \rightarrow \mathrm{LOG}_B \rightarrow E$ **easy**

In $\mathbb{Z}_N^*$

$(B, E, N) \rightarrow$ EXP $\rightarrow B^E \mod N$ **easy**

Two inverse functions:

$(B^E, E, N) \rightarrow \mathrm{ROOT}_E \rightarrow B$ **seems hard**

$(B^E, B, N) \rightarrow \mathrm{LOG}_B \rightarrow E$ **seems hard**

One-way function: easy to compute, hard to invert.
EXP seems to be one-way.

Private Key Cryptography
(Cryptography Before WW2)

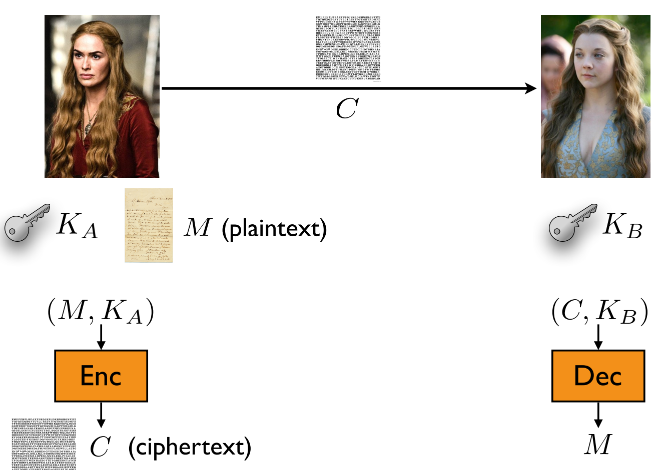
Private key cryptography

Parties must agree on a key pair beforehand.

Private key cryptography

there must be a secure way of exchanging the key

Private key cryptography



A note about security

Better to consider worst-case conditions.

Assume the adversary knows everything except the key(s) and the message:

Completely sees cipher text C.

Completely knows the algorithms Enc and Dec.

Caesar shift

<u>Example</u>: shift by 3

```
a b c d e f g h i j k l m n o p q r s t u v w x y z
↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓
d e f g h i j k l m n o p q r s t u v w x y z a b c
```

(similarly for capital letters)

"Dear Math, please grow up and solve your own problems."

↓

"Ghdu Pdwk, sohdvh jurz xs dqg vroyh brxu rzq sureohpv."

 : the shift number

Easy to break!

Substitution cipher

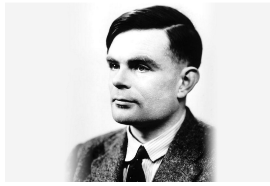
```
a b c d e f g h i j k l m n o p q r s t u v w x y z
↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓
j k b d e l m c f g n o x y r s v w z a t u p q h i
```

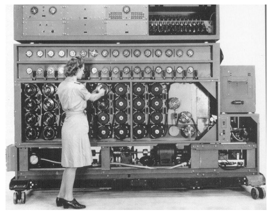
 : permutation of the alphabet

Easy to break by looking at letter frequencies!

Enigma

A much more complex cipher.





One-time pad

M = message K = key C = encrypted message

(everything in binary)

Encryption:

$$
\begin{array}{rl}
M = & 01011010111010100000111 \\
\oplus \quad K = & 11001100010101111000101 \\
\hline
C = & 10010110101111011000010
\end{array}
$$

$C = M \oplus K$ (bit-wise XOR)

For all i: $C[i] = M[i] + K[i] \pmod{2}$

One-time pad

M = message K = key C = encrypted message

(everything in binary)

Decryption:

$$
\begin{array}{rl}
C = & 10010110101111011000010 \\
\oplus \quad K = & 11001100010101111000101 \\
\hline
M = & 01011010111010100000111
\end{array}
$$

Encryption: $C = M \oplus K$

Decryption: $C \oplus K = (M \oplus K) \oplus K = M \oplus (K \oplus K) = M$

(because $K \oplus K = 0$)

One-time pad

$$
\begin{array}{lrl}
 & M = & 01011010111010100000111 \\
\oplus & K = & 11001100010101111000101 \\
\hline
 & C = & 10010110101111011000010
\end{array}
$$

One-time pad is perfectly secure:

For **any** M, if K is chosen uniformly at random, then C is uniformly at random.

So adversary learns nothing about M by seeing C.

One-time pad

$$
\begin{array}{lrl}
 & M = & 01011010111010100000111 \\
\oplus & K = & 11001100010101111000101 \\
\hline
 & C = & 10010110101111011000010
\end{array}
$$

Could we reuse the key?

One-time only:

Suppose you encrypt two messages M_1 and M_2 with K.

$C_1 = M_1 \oplus K$

$C_2 = M_2 \oplus K$

Then $C_1 \oplus C_2 = M_1 \oplus M_2$

Shannon's Theorem

Is it possible to have a secure system like one-time pad with a smaller key size?

Shannon proved "no".

If K is shorter than M:

An adversary with unlimited computational power could learn some information about M.

Question

What if we relax the assumption that the adversary is computationally unbounded?

Answers

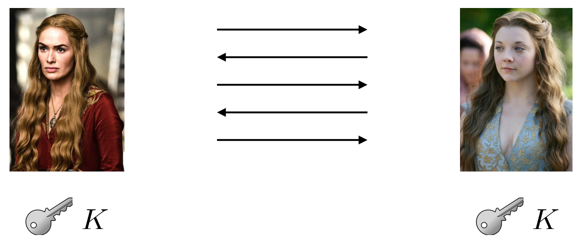
We can find a way to share a random secret key.
(over an insecure channel)

We can get rid of the secret key sharing part.
(public key cryptography)

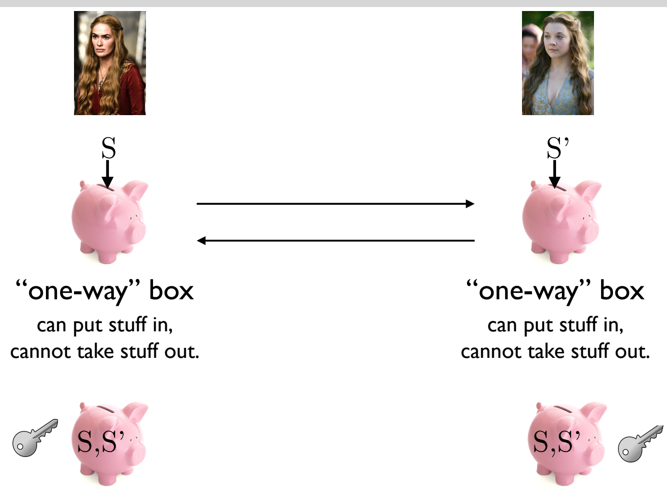
And do much more!!!

Secret Key Sharing

Secret Key Sharing



Secret Key Sharing



DH key exchange

In $\mathbb{Z}_N^*$

$(B, E, N) \rightarrow$ EXP $\rightarrow B^E \mod N$ **easy**

$(B^E, B, N) \rightarrow$ LOG_B $\rightarrow E$ **seems hard**

Want to make sure for the inputs we pick, LOG is hard.

e.g. we don't want $B^0\ B^1\ B^2\ B^3\ B^4 \ldots$ $=$ $1\ B\ 1\ B\ 1 \ldots$

Much better to have a *generator* B.

DH key exchange

In $\mathbb{Z}_N^*$

$(B, E, N) \rightarrow$ EXP $\rightarrow B^E \mod N$ **easy**

$(B^E, B, N) \rightarrow \mathrm{LOG}_B \rightarrow E$ **seems hard**

We'll pick $N = P$ a prime number.

(This ensures there is a generator in $\mathbb{Z}_P^*$.)

We'll pick $B \in \mathbb{Z}_P^*$ so that it is a *generator*.

$$\{B^0,\ B^1,\ B^2,\ B^3,\ \cdots,\ B^{P-2}\} = \mathbb{Z}_P^*$$

DH key exchange

Secure?

Adversary sees: P, B, B^{E_1}, B^{E_2}

Hopefully he can't compute E_1 from B^{E_1}.
(our hope that LOG_B is hard)

Good news: No one knows how to compute LOG_B efficiently.

Bad news: Proving that it cannot be computed efficiently is at least as hard as the **P** vs **NP** problem.

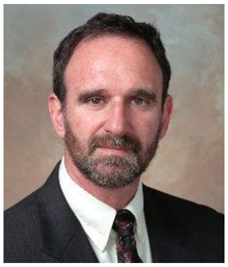
DH assumption:
Computing $B^{E_1 E_2}$ from P, B, B^{E_1}, B^{E_2} is hard.

Decisional DH assumption:
You actually learn no information about $B^{E_1 E_2}$.

Diffie-Hellman key exchange

1976

Whitfield Diffie



Martin Hellman

To send a private message, one can use:

Diffie-Hellman
(to share a secret key)

+

One-time Pad

Note

This is only as secure as its weakest link, i.e. Diffie-Hellman.

Answers

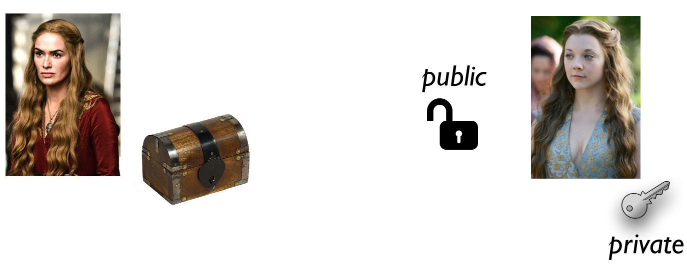
We can find a way to share a random secret key.
(over an insecure channel)

We can get rid of the secret key sharing part.
(public key cryptography)

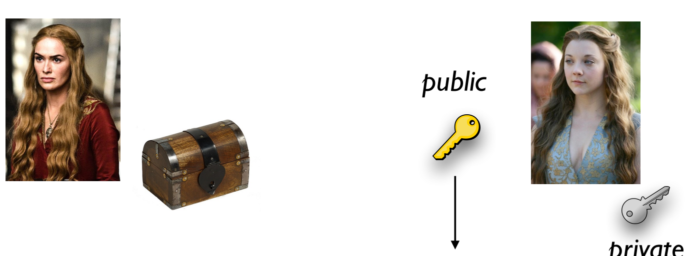
And do much more!!!

Public Key Cryptography
(Cryptography After WW2)

Public Key Cryptography

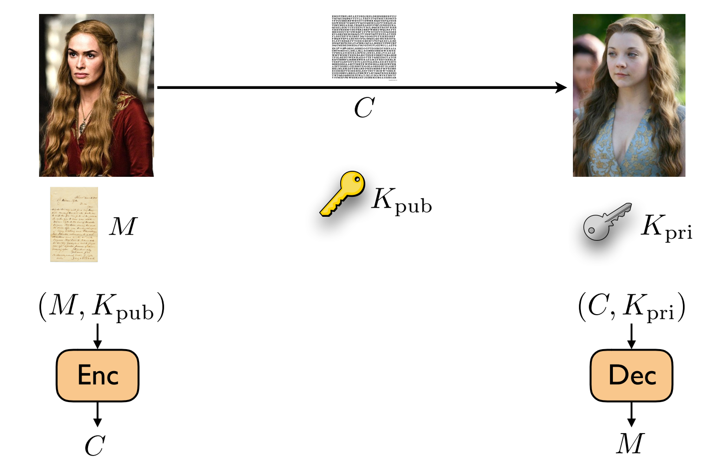


Public Key Cryptography



Can be used to lock.
But **can't** be used to unlock.

Public key cryptography



RSA crypto system

In $\mathbb{Z}_N^*$

$(B, E, N) \rightarrow$ EXP $\rightarrow B^E \mod N$ **easy**

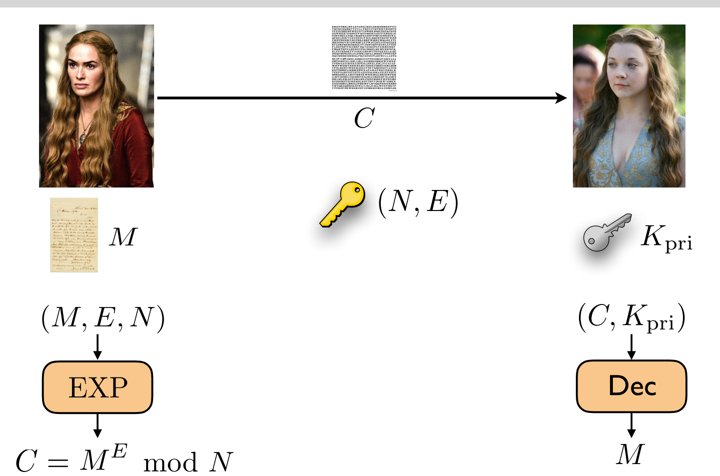
$(B^E, E, N) \rightarrow$ ROOT_E $\rightarrow B$ **seems hard**

What if we encode using EXP ? $(M = B)$

Public key can be (E, N) .

$(M, K_{\text{pub}}) = (M, E, N) \rightarrow$ Enc $\rightarrow M^E \mod N = C$

RSA crypto system



RSA crypto system

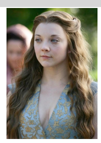
(M, E, N) → EXP → $C = M^E \mod N$

$M \in \mathbb{Z}_N^*$
$E \in \mathbb{Z}_{\varphi(N)}$

(C, K_{pri}) → Dec → M

RSA crypto system

C



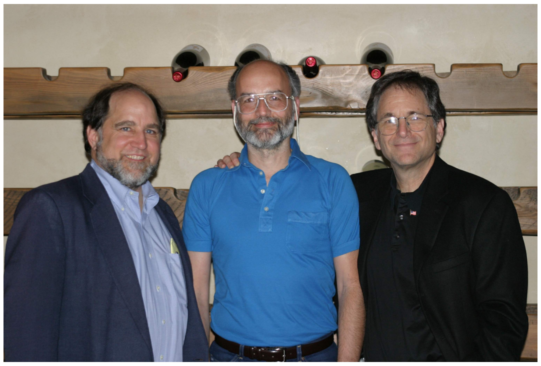
M

(M, E, N) → EXP → $M^E = C$

Secure?

RSA crypto system

1977



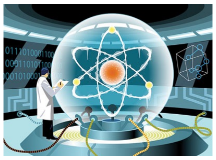
Ron Rivest   Adi Shamir   Leonard Adleman

Concluding remarks

A variant of this is widely used in practice.

From N, if we can efficiently compute $\varphi(N)$, we can crack RSA.

If we can factor N, we can compute $\varphi(N)$.



Quantum computers can factor efficiently.

Is this the only way to crack RSA?

We don't know!

So we are really hoping it is secure.