

# **15-252**

## **More Great Ideas in Theoretical Computer Science**

### **Lecture I: Sorting Pancakes**



*September 1st, 2017*

# Question

If there are  $n$  pancakes in total (all in different sizes), what is the max number of flips that we would ever have to use to sort them?



$P_n =$  the number described above

What is  $P_n$  ?

# Understanding the question

$$P_n = \max_S \min_A \# \text{ flips when sorting } S \text{ by } A$$



over all strategies/algorithms for sorting

over all pancake stacks of size **n**

Number of flips necessary to sort the **worst** stack of size **n**.

# Is it always possible to sort the pancakes?

Yes!

A sorting strategy (algorithm):

- Move the largest pancake to the bottom.
- Recurse on the other  $n-1$  pancakes.

# Playing around with an example

## Introducing notation:

- represent a pancake with a number from 1 to  $n$ .
- represent a stack as a permutation of  $\{1, 2, \dots, n\}$

e.g. (5 2 3 4 1)

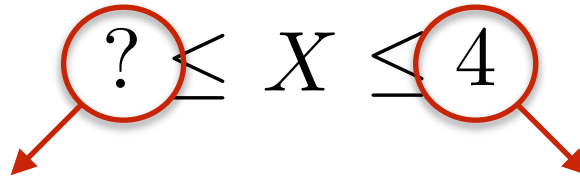
↓  
top

↓  
bottom

Let  $X$  = min number of flips to sort (5 2 3 4 1)

What is  $X$  ?

# Playing around with (5 2 3 4 1)

$$\textcircled{?} \leq X \leq \textcircled{4}$$


Need an argument  
for a lower bound.

A strategy/algorithm  
for sorting gives us  
an upper bound.

$$0 \leq X ?$$

$$1 \leq X ?$$

$$2 \leq X ?$$

$$3 \leq X ?$$

$$4 \leq X ?$$

# Playing around with (5 2 3 4 1)

**Proposition:**  $X = 4$

**Proof:** We already showed  $X \leq 4$ .

We now show  $X \geq 4$ . The proof is by contradiction.

So suppose we can sort the pancakes using 3 or less flips.

Observation: Right before a pancake is placed at the bottom of the stack, it must be at the top.

Claim: The first flip must put 5 on the bottom of the stack.

Proof: If the first flip does not put 5 on the bottom of the stack, then it puts it somewhere in the middle of the stack.

After 3 flips, 5 must be placed at the bottom.

Using the observation above, 2nd flip must send 5 to the top.

Then after 2 flips, we end up with the original stack.

But there is no way to sort the original stack in 1 flip.

The claim follows. □

# Playing around with (5 2 3 4 1)

**Proposition:**  $X = 4$

**Proof continued:**

So we know the first flip must be:  $(5\ 2\ 3\ 4\ 1) \rightarrow (1\ 4\ 3\ 2\ 5)$ .

In the remaining 2 flips, we must put 4 next to 5.

Obviously 5 cannot be touched.

So we can ignore 5 and just consider the stack  $(1\ 4\ 3\ 2)$ .

We need to put 4 at the bottom of this stack in 2 flips.

Again, using the observation stated above,  
the next two moves must be:

$$(1\ 4\ 3\ 2) \rightarrow (4\ 1\ 3\ 2) \rightarrow (2\ 3\ 1\ 4)$$

This does not lead to a sorted stack,

which is a contradiction since we assumed we could sort the stack  
in 3 flips. □



# Playing around with (5 2 3 4 1)

$$X = 4$$

What does this say about  $P_n$  ?

Pick the one that you think is true:

$$P_n = 4$$

$$P_n \leq 4$$

$$P_n \geq 4$$

$$P_5 = 4$$

$$P_5 \leq 4$$

$$P_5 \geq 4$$

None of the above.

Beats me.

# Playing around with (5 2 3 4 1)

$$X = 4$$

What does this say about  $P_n$  ?

$$P_5 = \max_S \min_A \# \text{ flips when sorting } S \text{ by } A$$

↳ all stacks of size 5

all stacks: (5 2 3 4 1) (5 4 3 2 1) (1 2 3 4 5) (5 4 1 2 3) ...

min # flips: 4 1 0 2

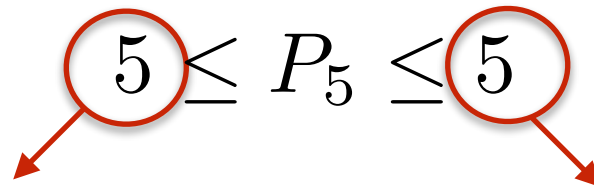
$P_5 =$  max among these numbers

$P_5 =$  min # flips to sort the “hardest” stack

So:  $X = 4 \implies P_5 \geq 4$

# Playing around with (5 2 3 4 1)

**In fact:** (will not prove)

$$\textcircled{5} \leq P_5 \leq \textcircled{5}$$


Find a specific “hard” stack.  
Show any method  
must use 5 flips.

Find a generic method  
that sorts any 5-stack  
with 5 flips.

Good progress so far:

- we understand the problem better
- we made some interesting observations

Ok what about  $P_n$  for general  $n$  ?

# $P_n$ for small $n$

$$P_0 = 0$$

$$P_1 = 0$$

$$P_2 = 1$$

$$P_3 = 3$$

lower bound:

(1 3 2) requires 3 flips.

upper bound:

- bring largest to the bottom in 2 flips
- sort the other 2 in 1 flip (if needed)

# A general upper bound: “Bring-to-top” alg.

**if**  $n = 1$ : do nothing

**else:**

- bring the largest pancake to bottom in 2 flips
- recurse on the remaining  $n-1$  pancakes

# A general upper bound: “Bring-to-top” alg.

**if**  $n = 1$ : do nothing

**else if**  $n = 2$ : sort using at most 1 flip

**else:**

- bring the largest pancake to bottom in 2 flips
- recurse on the remaining  $n-1$  pancakes

$T(n)$  = max # flips for this algorithm

$$T(1) = 0$$

$$T(2) \leq 1$$

$$T(n) \leq 2 + T(n-1) \quad \text{for } n \geq 3$$

$$\implies T(n) \leq 2n - 3 \quad \text{for } n \geq 2$$

# A general upper bound: “Bring-to-top” alg.

**Theorem:**  $P_n \leq 2n - 3$  for  $n \geq 2$ .

**Corollary:**  $P_3 \leq 3$ .

**Corollary:**  $P_5 \leq 7$ .

(So this is a **loose** upper bound, i.e. not tight.)

# A general lower bound

How about a lower bound?

You must argue against **all** possible strategies.

What is the worst initial stack?



# A general lower bound

## Observation:

Given an initial stack, suppose pancakes  $i$  and  $j$  are adjacent. They will remain adjacent if we never insert the spatula in between them.

(5 2 3 4 1)

## So:

If  $i$  and  $j$  are adjacent and  $|i - j| > 1$ , then we **must** insert the spatula in between them.

## Definition:

We call  $i$  and  $j$  a **bad** pair if

- they are adjacent
- $|i - j| > 1$

# A general lower bound

## Lemma (Breaking-apart argument):

A stack with  $b$  **bad** pairs needs at least  $b$  flips to be sorted.

e.g. (5 2 3 4 1) requires at least 2 flips.

In fact, we can conclude it requires at least 3 flips. Why?

Bottom pancake and plate can also form a **bad** pair.

# A general lower bound

**Theorem:**  $P_n \geq n$  for  $n \geq 4$ .

## **Proof:**

Take cases on the parity of  $n$ .

If  $n$  is even, the following stack has  $n$  bad pairs:

$$(2\ 4\ 6\ \cdots\ n-2\ n\ 1\ 3\ 5\ \cdots\ n-1)$$

If  $n$  is odd, the following stack has  $n$  bad pairs:

$$(1\ 3\ 5\ \cdots\ n-2\ n\ 2\ 4\ 6\ \cdots\ n-1)$$

By the previous lemma, both need  $n$  flips to be sorted.

So  $P_n \geq n$  for  $n \geq 4$ .



Where did we use the assumption  $n \geq 4$  ?

So what were we able to prove about  $P_n$  ?

**Theorem:**  $n \leq P_n \leq 2n - 3$  for  $n \geq 4$ .

# Best known bounds for $P_n$

**Jacob Goodman 1975:** what we saw



published under pseudonym Harry Dweighter

**William Gates and Christos Papadimitriou 1979:**



$$\frac{17}{16}n \leq P_n \leq \frac{5}{3}(n + 1)$$

**Currently best known:**  $\frac{15}{14}n \leq P_n \leq \frac{18}{11}n$

## William Gates and Christos Papadimitriou 1979:

Introduced “Burnt pancakes” problem.

$$\frac{3}{2}n - 1 \leq BP_n \leq 2n + 3$$

## David Cohen and Manuel Blum 1995:



Carnegie  
Mellon  
University

$$\frac{3}{2}n \leq BP_n \leq 2n - 2$$

**David Samuel Cohen** (born July 13, 1966), better known as **David X. Cohen**, is an American television writer. He has written for *The Simpsons* and served as the [head writer](#) and [executive producer](#) of *Futurama*.

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## Early life [edit]

Cohen was born in [New York City](#). He changed his middle initial around the time *Futurama* debuted due to [Writers' Guild](#) policies prohibiting more than one member from having the same name.<sup>[1]</sup> Both of his parents were [biologists](#), and growing up Cohen had always planned to be a scientist, though he also enjoyed writing and drawing cartoons.<sup>[2]</sup>

Cohen graduated from [Dwight Morrow High School](#) in [Englewood, New Jersey](#), where he wrote the humor column for the high school paper and was a member of the school's state champion mathematics team.<sup>[3]</sup> From there, Cohen went on to attend [Harvard University](#), graduating with a [B.A.](#) in [physics](#), and the [University of California, Berkeley](#), with a [M.S.](#) in [computer science](#).<sup>[4]</sup> At Harvard, he wrote for and served as President of the *[Harvard Lampoon](#)*.

Cohen's most notable academic publication concerned the theoretical computer science problem of [pancake sorting](#),<sup>[5]</sup> which was also the subject of an academic publication by [Bill Gates](#).<sup>[6]</sup>

## David X. Cohen



Cohen at the 2010 [San Diego Comic-Con International](#).

<b>Born</b>	<div>David Samuel Cohen</div> July 13, 1966 (age 50) <div><a href="#">New York, New York, United States</a></div>
<b>Occupation</b>	Television writer
<b>Period</b>	1992–present
<b>Genre</b>	<a href="#">Comedy</a>
<b>Spouse</b>	<a href="#">Patty Cohen</a>
<b>Children</b>	1

**Manuel Blum** ([Caracas](#), 26 April 1938) is a Venezuelan [computer scientist](#) who received the [Turing Award](#) in 1995 "In recognition of his contributions to the foundations of [computational complexity theory](#) and its application to [cryptography](#) and program checking".<sup>[2][3][4][5][6][7][8]</sup>

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## Education [[edit](#)]

Blum was educated at [MIT](#), where he received his bachelor's degree and his master's degree in EECS in 1959 and 1961 respectively, and his [Ph.D.](#) in mathematics in 1964 supervised by [Marvin Minsky](#).<sup>[1][7]</sup>

## Career [[edit](#)]

He worked as a professor of computer science at the [University of California, Berkeley](#) until 1999. In 2002 he was elected to the [United States National Academy of Sciences](#).

He is currently the Bruce Nelson Professor of Computer Science at [Carnegie Mellon University](#), where his wife, [Lenore Blum](#),<sup>[9]</sup> and son, [Avrim Blum](#), are also professors of Computer Science.

## Research [[edit](#)]

In the 60s he developed an axiomatic complexity theory which was independent of concrete machine models. The theory is based on [Gödel numberings](#) and the [Blum axioms](#). Even though the theory is not based on any machine model it yields concrete results like the [compression theorem](#), the [gap theorem](#), the honesty theorem and the [Blum speedup theorem](#).

### Manuel Blum



Manuel Blum (left) with his wife [Lenore Blum](#) and their son [Avrim Blum](#), 1973

<b>Born</b>	April 26, 1938 (age 79) <a href="#">Caracas</a> , <a href="#">Venezuela</a>
<b>Residence</b>	<a href="#">Pittsburgh</a>
<b>Fields</b>	<a href="#">Computer Science</a>
<b>Institutions</b>	<a href="#">University of California, Berkeley</a> <a href="#">Carnegie Mellon University</a>
<b>Alma mater</b>	<a href="#">Massachusetts Institute of Technology</a>
<b>Thesis</b>	<i>A Machine-Independent Theory of the Complexity of Recursive Functions</i> <span><span><span></span></span></span> (1964)
<b>Doctoral advisor</b>	<a href="#">Marvin Minsky</a> <sup>[1]</sup>
<b>Doctoral students</b>	<a href="#">Leonard Adleman</a> <a href="#">Dana Angluin</a> <a href="#">C. Eric Bach</a> <a href="#">Joan Boyar</a> <a href="#">William Evans</a> <a href="#">Peter Gemmell</a> <a href="#">John Gill, III</a> <a href="#">Shafi Goldwasser</a>



# Best known bounds for $P_n$

$n$	$P_n$
4	4
5	5
6	7
7	8
8	9
9	10
10	11
11	13
12	14
13	15
14	16
15	17
16	18
17	19
18	20
19	22

$P_{20} = ?$

23 or 24

# Why study pancake numbers?

Perhaps surprisingly, it has interesting applications.

- In designing efficient networks that are resilient to failures of links.

Google: pancake network

- In biology.

Can think of chromosomes as permutations.

Interested in mutations in which some portion of the chromosome gets flipped.

# Lessons

Simple problems may be hard to solve.

Simple problems may have far-reaching applications.

By studying pancakes, you can be a billionaire.



# Analogy with computation

**input:** initial stack

**output:** sorted stack

**computational problem:** (input, output) pairs  
*pancake sorting problem*

**computational model:** specified by the allowed operations on the input.

**algorithm:** a precise description of how to obtain the output from the input.

**computability:** is it always possible to sort the stack?

**complexity:** how many flips are needed?