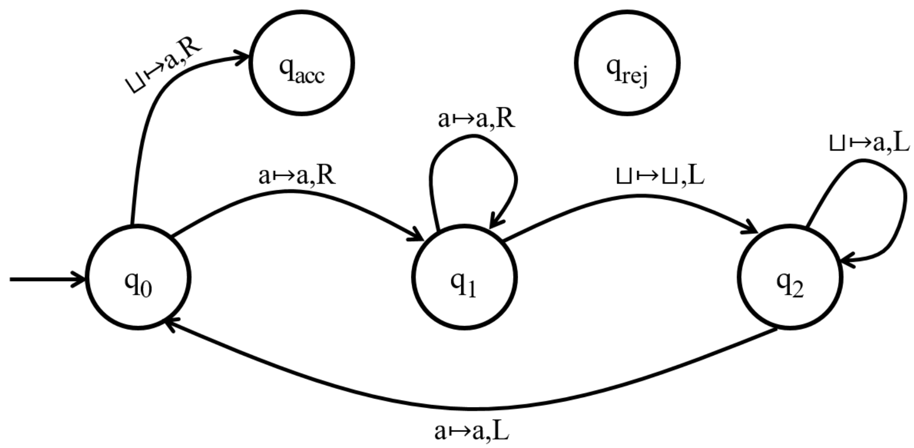
Midterm 1 Practice Problems

1. Consider the following TM called M, which has input alphabet $\{a\}$ and tape alphabet $\{a, \sqcup\}$.



Prove that M does not halt on input aaa. Your proof should use the notion of a *configuration*.

2. (a) Let $\Sigma = \{a, b\}$ and $L = \{w \in \{a, b\}^* \mid w$ has the same number of a's and b's$\}$. Show that L is decidable by giving a low-level description of a TM deciding L. You can do so by drawing the state diagram. You can use any finite tape alphabet Γ containing Σ and $\sqcup$; just be sure to specify what it is. Proof of correctness is not required, but please specify the "meaning" of the states you use in order to make your construction easy to understand.

   (b) Let $\Sigma = \{a, b\}$ and $K = \{ww^R \mid w \in \Sigma^*\}$. Show that K is decidable by giving a low-level description of a TM deciding K as in the previous part.

   (c) Given a word $w \in \Sigma^*$, a permutation of w is a word obtained by reordering the symbols of w. For example, the set of all permutations of 101 is $\{011, 101, 110\}$. For a language $L \subseteq \{0, 1\}^*$, define $\text{perm}(L)$ to be the set of all permutations of all the words in L. Prove that if L is decidable, then $\text{perm}(L)$ is also decidable. You can present your algorithm using a high-level description.

3. Fix the alphabet $\Sigma = \{0, 1\}$. In this problem, we are looking for high-level descriptions of deciders/algorithms.

   (a) Let $K = \{\langle D \rangle :$ there is no DFA D' with fewer states than D such that $L(D) = L(D')\}$. Show that K is decidable by arguing that deciding K reduces to deciding EQ_{DFA}.

   (b) Let

$$L = \{\langle D \rangle : D \text{ is a DFA that does not accept any string that starts and ends with the same symbol}\}.$$

   Show that L is decidable by arguing that deciding L reduces to deciding $\text{EMPTY}_{\text{DFA}}$.

4. Let L be a language over the alphabet $\Sigma = \{0, 1\}$ that is defined recursively as follows:

   - $\epsilon \in L$;
   - $u, v \in L \implies 0u1v \in L$;
   - $u, v \in L \implies 1u0v \in L$.

   The only strings in L are the ones that can be constructed inductively by applying the above rules. Prove that L is the language consisting of all strings that have an equal number of 0 symbols and 1 symbols. In other words, if K denotes the language consisting of all strings with an equal number of 0's and 1's, then you need to show that $L = K$. To do this, you should argue $L \subseteq K$ **and** $K \subseteq L$.

5. A *world-state* in Game of Life consists of an assignment of "alive" or "dead" to each cell (x, y) in the infinite 2-d grid $\mathbb{Z} \times \mathbb{Z}$. Determine, with proof, whether the set of all possible world-states is countable.

6. Consider the set of functions $f : \mathbb{N} \to \mathbb{N}$ which: (i) are increasing (meaning $f(x) > f(y)$ whenever $x > y$); and, (ii) satisfy $f(n) \leq 2n$ for all $n \in \mathbb{N}$. Determine with proof whether this set is countable.

7. Consider the following languages over the alphabet $\{a, b\}$:

   $$L = \{a^m b^n : n, m \in \mathbb{N}, n - m \leq 251\},$$
   $$K = \{a^m b^n : n, m \in \mathbb{N}, n + m \leq 251\}.$$

   Show that one of these languages is regular and the other is not. For the proof of regularity, all you need to do is draw or describe a DFA; a proof of correctness is not required. For the proof of irregularity, present all the details.

8. Let's say that a DFA D is "interesting" if it accepts at least one string x with $k \leq |x| \leq 2k$, where k denotes the number of states in D.

   (a) Show that $L = \{\langle D \rangle : D \text{ is interesting}\}$ is decidable.

   (b) Show that D is interesting if and only if D accepts infinitely many strings.

9. Given a language $L \subseteq \Sigma^*$, define

   $$\text{SUB}(L) = \{x \in \Sigma^* : \exists u, v \in \Sigma^* \text{ such that } uxv \in L\}.$$

   Show that if L is regular, then SUB(L) must also be regular. You may use any of the closure properties of regular languages proved in the notes.

10. Given a language $L \subseteq \{0, 1\}^*$, define

    $$\text{ONLY-ONES}(L) = \{1^{|x|} : x \in L\} \subseteq \{1\}^*.$$

    Show that if L is regular, then ONLY-ONES(L) must also be regular.