15-251: Great Theoretical Ideas In Computer Science

Recitation 6

Announcements

- Midterm 2 next **Wednesday, October 10**! It will be held in **DH 2315** from **6.30pm to 9.30pm** in place of the writing session. (Note the later end time.)
- Practice problems have been posted!
- We will be holding topical reviews with the venues and times to be confirmed. Watch Diderot and the course calendar for updates!
- Homework solution sessions will be Saturday and Sunday 1:30pm to 2:30pm

Recap of some definitions and facts

- Regular graph
- The tree-nity (the three salient features of a tree)
- The handshake lemma

"Clearly" Correct

A connected graph with no cycles is a tree. Consider the following claim and its proof.

> **Claim:** Any graph with n vertices and $n-1$ edges is a tree.
> **Proof:** We prove the claim by induction. The claim is clearly true for $n = 1$ and $n = 2$. Now suppose the claim holds for $n = k$. We'll prove that it also holds for $n = k+1$. Let G be a graph with k vertices and $k-1$ edges. By the induction hypothesis, G is a tree (and therefore clearly connected). Add a new vertex v to G by connecting it with any other vertex in G. So we create a new graph G' with $k+1$ vertices and k edges. The new vertex we added is clearly a leaf, so it clearly does not create a cycle. Also, since G was connected, G' is clearly also connected. A connected graph with no cycles is a tree, so G' is also a tree. So the claim follows by induction.

Explain why the given proof is incorrect.

2 3 Proofs 4 You

Give two proofs (one using induction and another using a degree counting argument) for the following claim: the number of leaves in a tree with $n \geq 2$ vertices is

$$2 + \sum_{\substack{v \in V \\ \deg(v) \geq 3}} (\deg(v) - 2)$$

Counting Colors 1, 2, 3, ...

Let $G = (V, E)$ be an undirected graph. Let $k \in \mathbb{N}^+$. A *k-coloring* of V is just a map $\chi : V \to C$ where C is a set of cardinality k. (Usually the elements of C are called *colors*. If $k = 3$ then {red, green, blue} is a popular choice. If k is large, we often just call the colors $1, 2, \ldots, k$.) A k-coloring is said to be *legal* for G if every edge in E is *bichromatic*, meaning that its two endpoints have different colors. (I.e., for all $\{u, v\} \in E$ it is required that $\chi(u) \neq \chi(v)$.) Finally, we say that G is k*-colorable* if it has a legal k-coloring.

(a) Suppose G has no cycles of length greater than 251. Prove that G is 251-colorable. Hint: DFS.

(b) Give an example to show that the above is tight, i.e., find a graph G with no cycles of length greater than 251 that is not 250-colorable.

(Extra) Long Walks

Suppose a graph G has minimum degree δ (so the vertex of lowest degree has degree δ). Show that G contains a path of length (at least) δ.

(Bonus) Graphitti

How many colors do you need to color the vertices of this graph?