15-251: Great Theoretical Ideas In Computer Science

Recitation 7

Announcements

- We'll be going over midterm solutions this weekend at the normal times and locations.
- Remember to complete the weekly quiz by Sunday 9pm.

Definitions

- A **matching** in G is a subset of G's edges which share no vertices.
  A **maximal** matching is one which isn't a subset of any other matching.
  A **maximum** matching is a matching which is at least as large as any possible matching.
  A **perfect** matching is a matching such that every vertex is contained in one of its edges.
- An **alternating path** (with respect to some matching M) is one which alternates between edges in M and edges not in M.
  An **augmenting path** is an alternating path that begins and ends with vertices unmatched in M.
- An **unstable pair** is a pair who prefer each other to their assigned partners.
  A **stable matching** is a perfect matching (includes all vertices) which contains no unstable pairs.
- Gale Shapley algorithm on sets A (men) and B (women), as follows...
  While there is a man $m \in A$ who is unmatched:
  (a) Let $w \in B$ be the highest ranked woman in m's list whom he hasn't proposed to yet.
  (b) If w is unmatched or prefers m to her current match, match w and m.

Corridor's Theorem

(a) Recall and prove Hall's Theorem, restated below:

Let $G = (X, Y, E)$ be a bipartite graph, and for any $S \subseteq X$ let $N(S) = \{y \in Y \mid \exists x \in S \text{ s.t. } \{x, y\} \in E\}$, i.e. the neighbors of S. Then G has a matching covering all of X iff for every $S \subseteq X$, $|S| \leq |N(S)|$.

A Misogynist Algorithm

(a) Prove that the Gale-Shapley algorithm always matches every guy with his best valid partner. That is, show that every guy prefers the girl he is paired with by the Gale-Shapley algorithm at least as much as any girl he is paired with in any other stable matching.

(b) Prove that the Gale-Shapley algorithm always matches every girl with her worst valid partner. That is, show that in any other stable matching, each girl is paired with a guy she likes at least as much as the one she is paired with by Gale Shapley.

(Extra) Soulmates

Call a man m and a woman w "soulmates" if they are paired with each other in every stable matching.

(a) Given a man m and a woman w, design a polynomial-time algorithm to determine if they are soulmates.

(b) Give a polynomial time algorithm to determine if an instance of the stable matching problem has a *unique* stable matching.

(Extra) Counting in a Couple Ways

(a) Find, with proof, the maximum possible number of perfect matchings in a graph on n vertices.

(b) Find, with proof, the maximum possible number of perfect matchings in a *bipartite* graph on n vertices.

(c) Find a way to construct an instance of the stable marriage problem with n men and n women which has at least n stable matchings (tight bounds on the number of stable matchings for n pairs of men and women are not known).

(Bonus) Pharaoh's Theorem

(a) Given $G = (V, E)$ and $S \subseteq V$, let $G \setminus S$ be the graph formed by removing the vertices in S and all the edges incident to them. Also, define $o(G)$ to be the number of connected components in G of odd size. Then prove the following theorem:

Let $G = (V, E)$ be any graph. Then G has a perfect matching iff for every $S \subseteq V$, $o(G \setminus S) \leq |S|$.