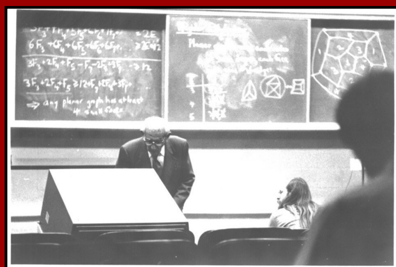
15-251: Great Theoretical Ideas in Computer Science
Lecture 12

Graph Algorithms



L.F.O.A.

Lecture Full Of Acronyms

LFOA Fun Poll:.

Which acronym(s) will we not learn about today

AFS
BFS
CFS
DFS
MST
AFSOC

The most basic graph algorithms:

BFS: Breadth-first search

DFS: Depth-first search

AFS: Arbitrary-first search

What problems do these algorithms solve?

Graph Search Algorithms

Given a graph $G = (V,E)$...

- Check if vertex s can reach vertex t.
- Decide if G is connected.
- Identify connected components of G.

All reduce to:

"Given $s \in V$, identify all nodes reachable from s."
(We'll call this set $\textsc{ConnComp}(s)$.)

Algorithm $AFS(G,s)$ does exactly this.

Bonus of $AFS(G,s)$:

Finds a **spanning tree** of $\textsc{ConnComp}(s)$ rooted at s.

Given $G = (V,E)$, a **spanning tree** is
a tree $T = (V,E')$ such that $E' \subseteq E$.

More informally, a minimal set of edges
connecting up all vertices of G.

Bonus of AFS(G,s):

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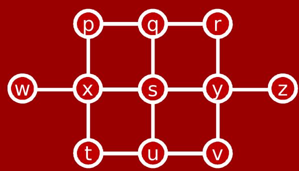

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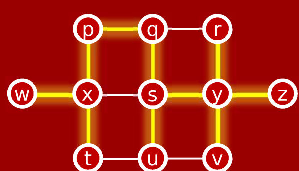

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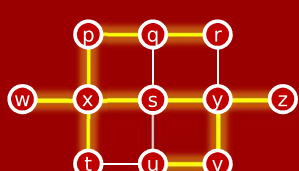
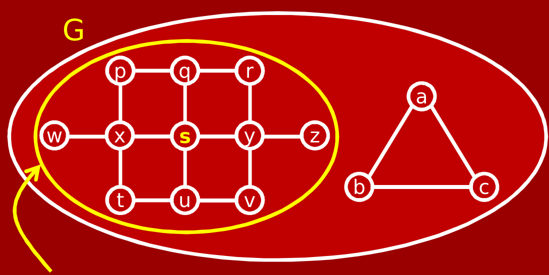


AFS(G,s): Finding all nodes reachable from s



"Duh, it's these ones."

But it's not so obvious when the input looks like…

AFS(G,s): Finding all nodes reachable from s

V = { a,b,c,p,q,r,s,t,u,v,w,x,y,z }

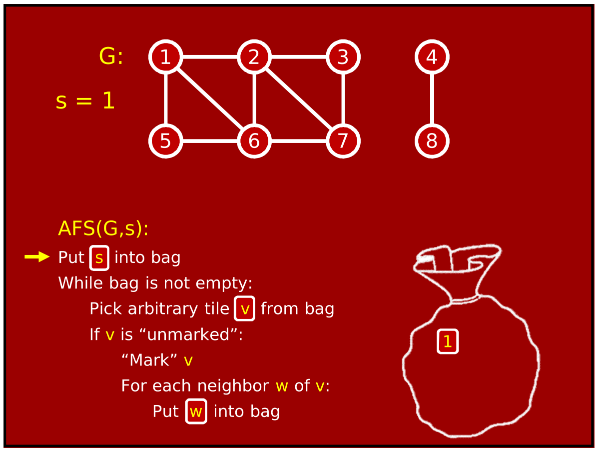
E = { {a,b},{a,c},{b,c},{p,q},{p,x},{q,r},
{q,s},{r,y},{s,u},{s,x},{s,y},{t,u},
{t,x},{u,v},{v,y},{w,x},{y,z} }

AFS(G,s):

// Has a "bag" data structure holding tiles
// Each tile has a vertex name written on it

Put s into bag
While bag is not empty:
    Pick an Arbitrary tile v from bag
    If v is "unmarked":
        "Mark" v
        For each neighbor w of v:
            Put w into bag

Intent:
"Marked" vertices should be those reachable from s.
w in bag means we want to keep exploring from w.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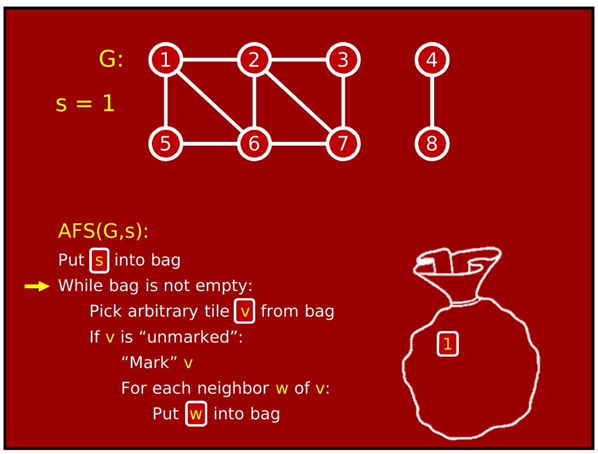
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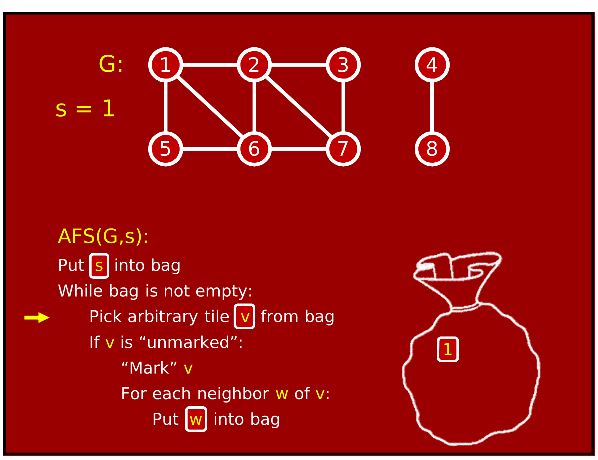
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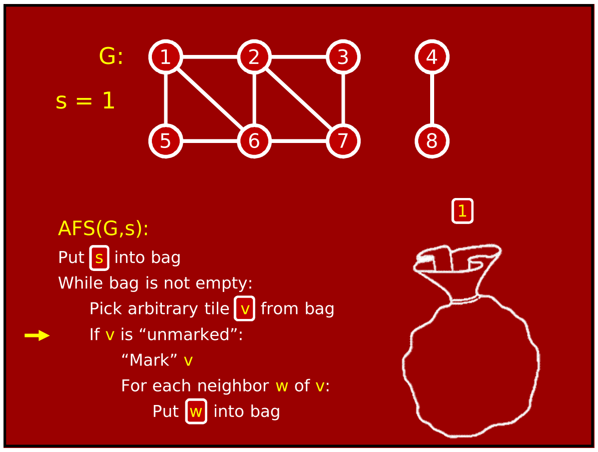
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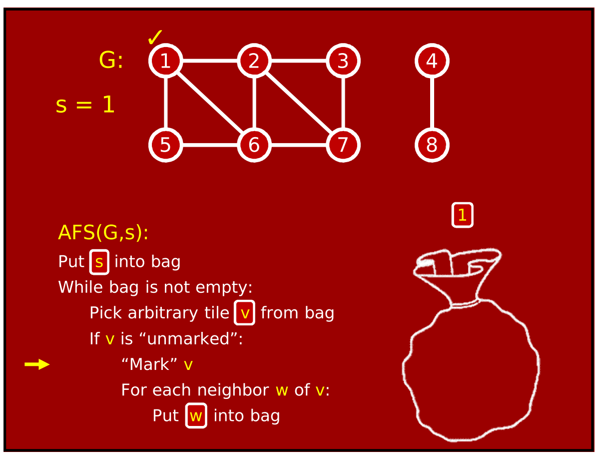
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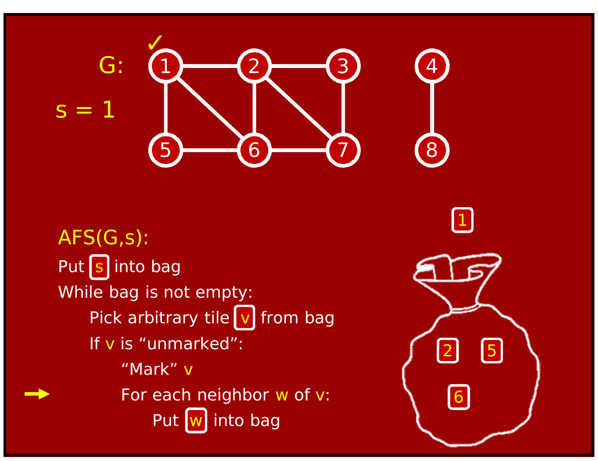
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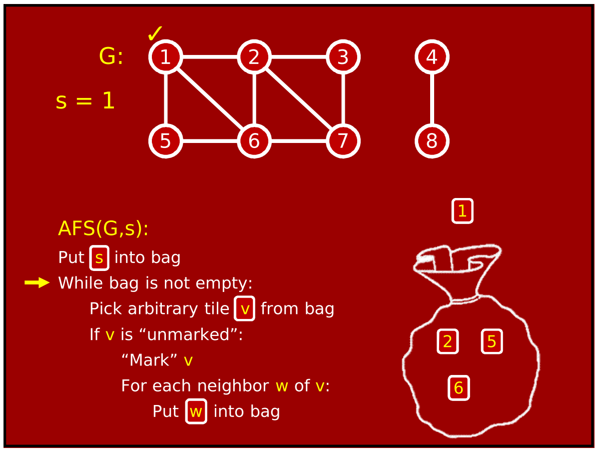
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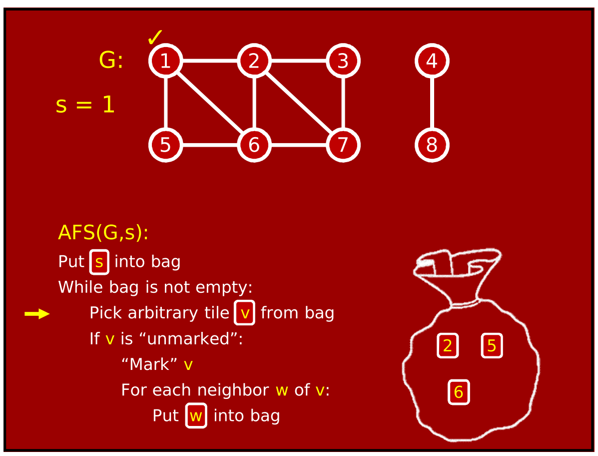
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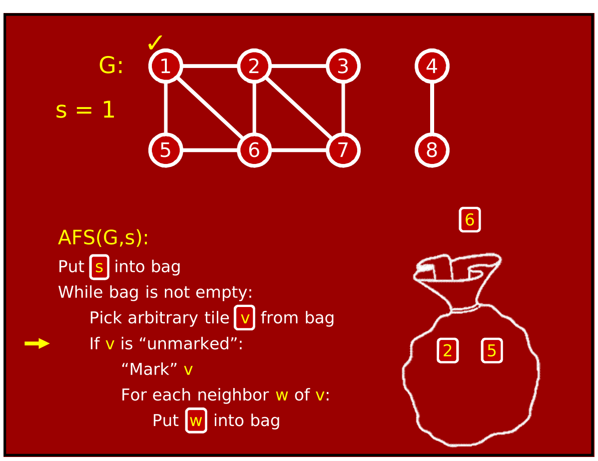
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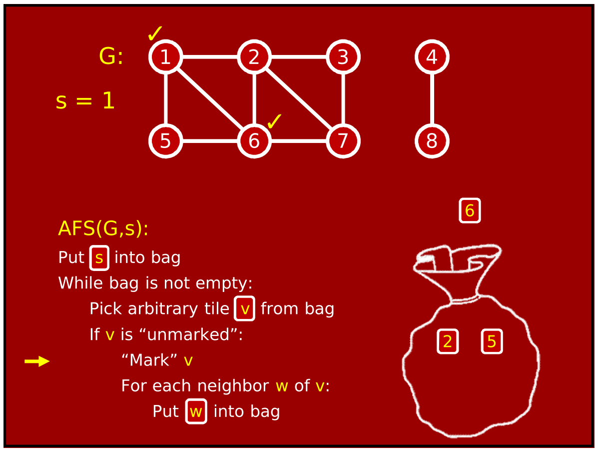
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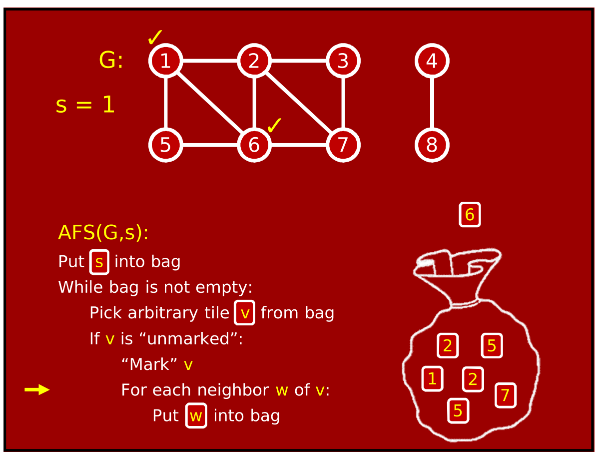
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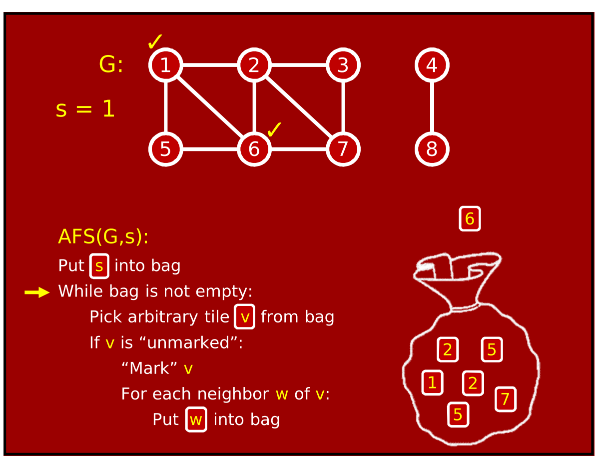
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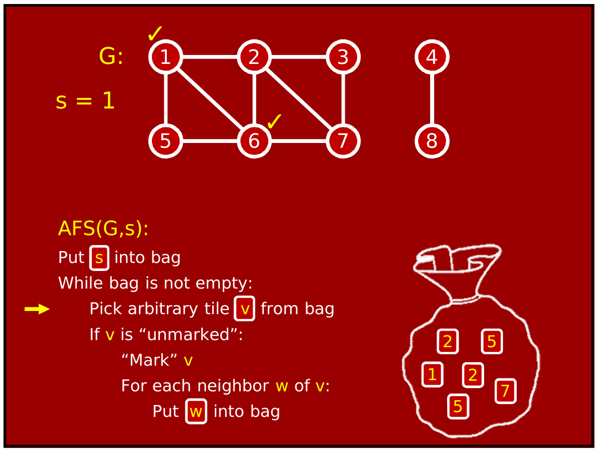
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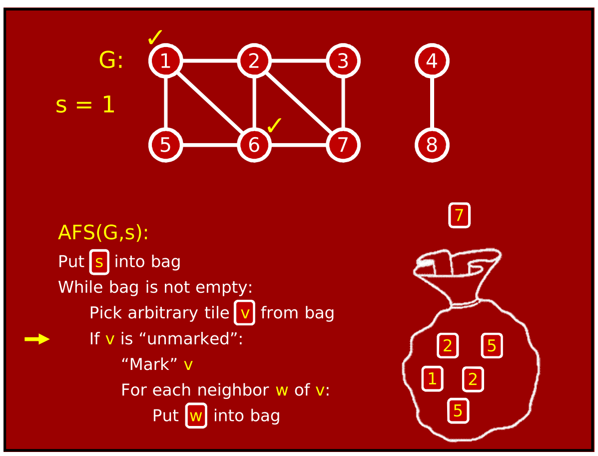
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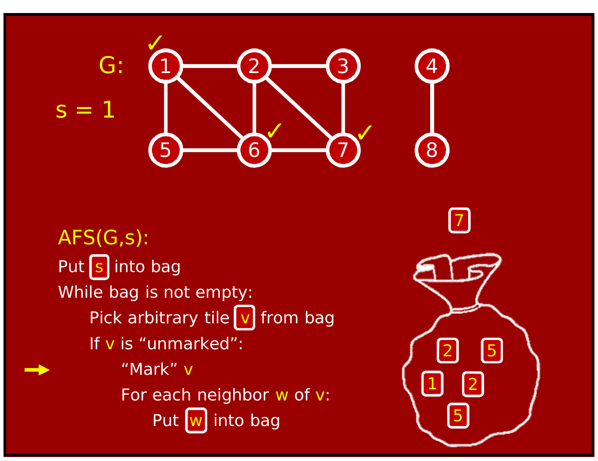
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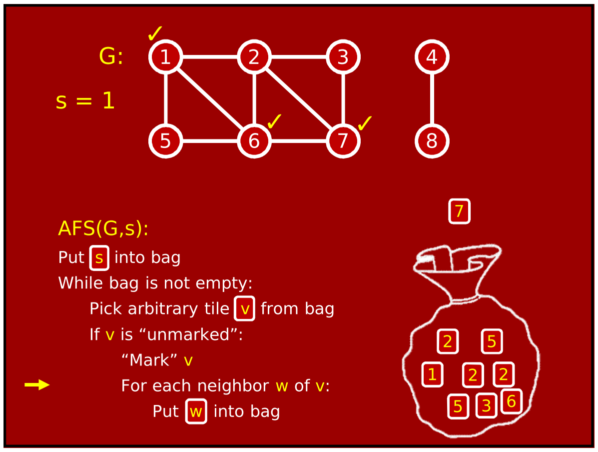
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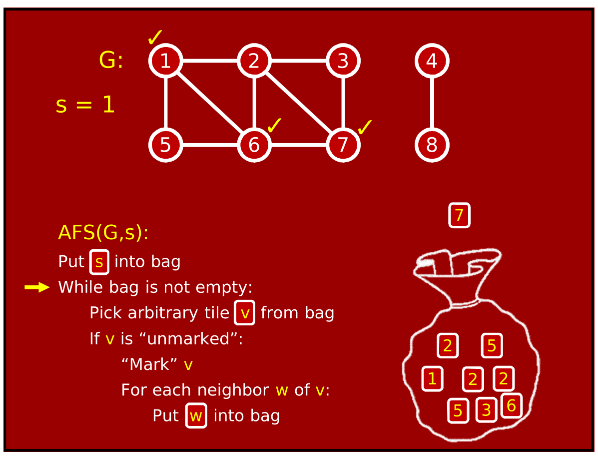
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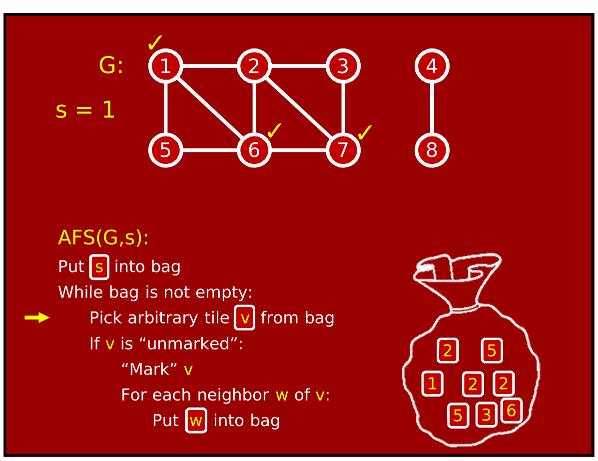
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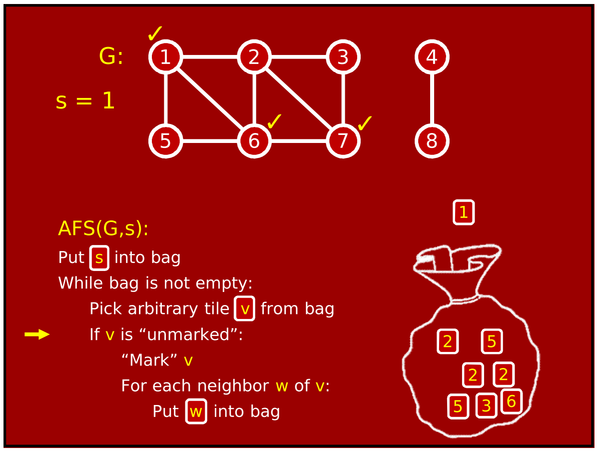
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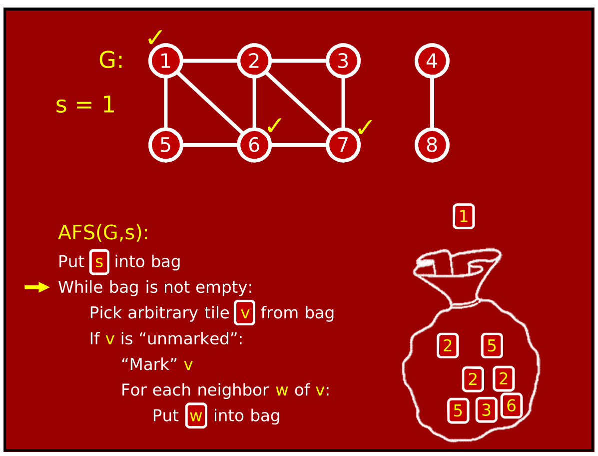
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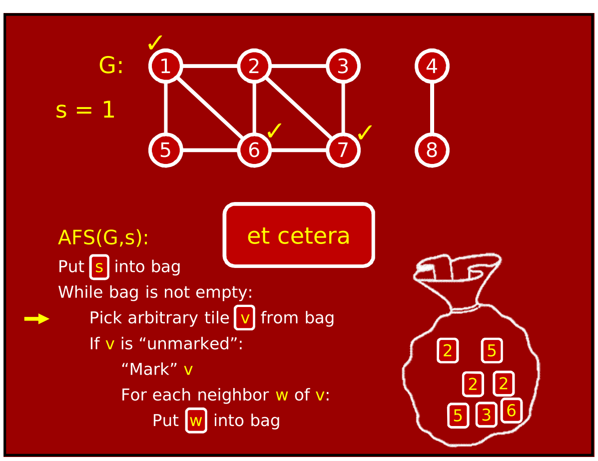
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For each neighbor w of v:
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2
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5
3
6

Analysis of AFS

Want to show: When this algorithm halts,

{ marked vertices }
=
{ vertices reachable from s }.

{ marked } ⊆ { reachable }: This is clear.

{ reachable } ⊆ { marked }:

Wait, why does the algorithm even halt?!

Why does AFS halt?

Every time a bunch of tiles is added to bag,
it's because some vertex v just got marked.

♦ we add at most |V| bunches of tiles to the bag (since each vertex is marked ≤ 1 time).

♦ at most finitely many tiles ever go into the bag.

Each iteration through loop removes 1 tile.

♦ AFS halts after finitely many iterations.

```
AFS(G,s):
Put [s] into bag
While bag is not empty:
    Pick arbitrary tile [v] from bag
    If v is "unmarked":
        "Mark" v
  →     For each neighbor w of v:
            Put [w] into bag
```

A more careful analysis

Every time a bunch of tiles is added to bag,
it's because some vertex v just got marked.

In this case, we add deg(v) tiles to the bag.

♦ total number of tiles that ever enter the bag is

$$\leq \sum_{v \in V} \deg(v) = 2|E|$$

Each iteration through loop removes 1 tile.

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Each iteration through loop removes 1 tile.

♦ AFS halts after ≤ 2|E| many iterations.

AFS(G,s):
Put $\boxed{s}$ into bag
While bag is not empty:
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Each iterati[…] loop re[…]

we forgot about this line

♦ AFS halts after ≤ 2|E|+1 many iterations.

AFS(G,s):
Put $\boxed{s}$ into bag
While bag is not empty:
  Pick arbitrary tile $\boxed{v}$ from bag
  If v is "unmarked":
    "Mark" v
→   For each neighbor w of v:
      Put $\boxed{w}$ into bag

When a tile $\boxed{w}$ is added to the bag, it gets there "because of" a neighbor v that was just marked.

(Except for the initial $\boxed{s}$.)

Let's actually record this info on the tile, writing $\boxed{v\to w}$.

Meaning: "We want to keep exploring from w. By the way, we got to w from v."

(And we'll write $\boxed{\perp\to s}$ initially.)

AFS(G,s):

Put $\boxed{s}$ into bag
While bag is not empty:
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    If v is “unmarked”:
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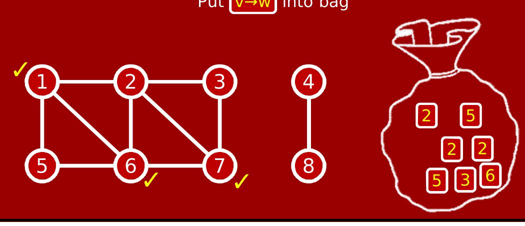
        For each neighbor w of v:
            Put $\boxed{w}$ into bag

AFS(G,s):

Put $\boxed{\perp\to s}$ into bag
While bag is not empty:
    Pick an Arbitrary tile $\boxed{p\to v}$ from bag
    If v is “unmarked”:
        “Mark” v
        For each neighbor w of v:
            Put $\boxed{v\to w}$ into bag

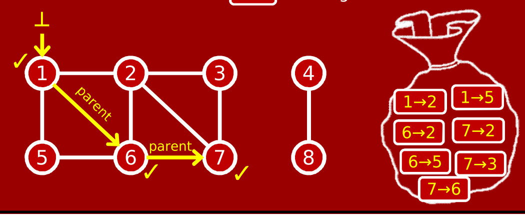
AFS(G,s):

Put $\boxed{\perp\to s}$ into bag
While bag is not empty:
    Pick an Arbitrary tile $\boxed{p\to v}$ from bag
    If v is “unmarked”:
        “Mark” v and record parent(v) := p
        For each neighbor w of v:
            Put $\boxed{v\to w}$ into bag

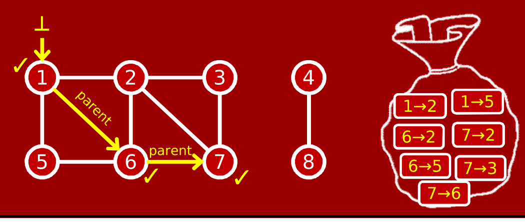


AFS(G,s):

Put ⊥→s into bag
While bag is not empty:
    Pick an Arbitrary tile p→v from bag
    If v is “unmarked”:
        “Mark” v and record parent(v) := p
        For each neighbor w of v:
            Put v→w into bag



Suppose the next few tiles pulled are
6→2, 6→5, 7→3.
Then AFS would reach the following state…



Suppose the next few tiles pulled are
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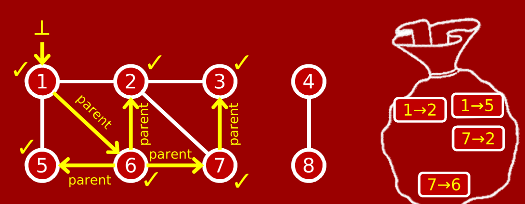
Then AFS would reach the following state…

Then remaining tiles would be pulled & discarded.



AFS(G,s):

Put $\boxed{\perp \to s}$ into bag
While bag is not empty:
    Pick an Arbitrary tile $\boxed{p \to v}$ from bag
    If v is "unmarked":
        "Mark" v and record parent(v) := p
        For each neighbor w of v:
            Put $\boxed{v \to w}$ into bag

Theorem: Every vertex in ConnComp(s) gets marked.

Theorem: Every vertex in ConnComp(s) gets marked.

Equivalently: For all vertices y, if there's a path from s to y of length k, then y gets marked.

Proof: By induction on k.
Base case k = 0: Indeed, s gets marked.

Induction step: Suppose it's true for some $k \in \mathbb{N}$.
Now suppose ∃ a length-(k+1) path from s to some y.
Write it as (s, …, x, y). So (s, …, x) is a length-k path.
By induction, x gets marked.
When x gets marked by the algorithm, $\boxed{x \to y}$ goes in bag.
We proved the bag eventually empties.
Thus $\boxed{x \to y}$ will come out, and the algorithm will mark y. □

So we've proved AFS(G,s) indeed marks ConnComp(s).

From now on, let's assume ConnComp(s) is all of G.

Corollary: The parent() information recorded by AFS encodes a spanning tree of G rooted at s.

Proof:
It certainly records a bunch of edges.
Each vertex in G, except s, has exactly one parent edge.
Thus there are $|V|-1$ edges.
Further, it's clear that for all vertices v,
parent(parent(⋯parent(v)⋯)) must reach s.
♦ all vertices are connected to s, hence to each other.
♦ parent edges form a tree ($|V|-1$ edges, connected). □

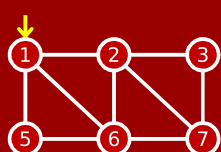
Instantiations of AFS

DFS: Depth-First Search

When the bag is a "**stack**".
LIFO: Last-In First-Out.

(Assume sorted adjacency list representation.)

(actually implemented using an array)



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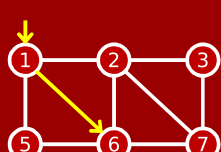

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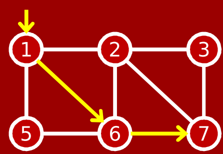

(actually implemented using an array)

DFS: Depth-First Search

When the bag is a “**stack**”.
LIFO: Last-In First-Out.

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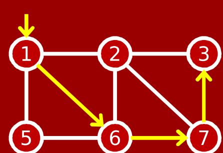

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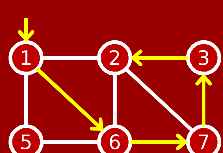

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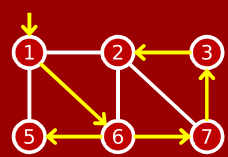


(actually implemented using an array)

DFS: Depth-First Search

When the bag is a "**stack**".
LIFO: Last-In First-Out.

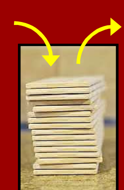
DFS is cute because many programming languages allow recursion, which means the compiler takes care of implementing the stack for you!

(actually implemented using an array)

DFS: Depth-First Search

When the bag is a "**stack**".
LIFO: Last-In First-Out.

```
RecursiveDFS(v)
  if v unmarked
    mark v
    for each w ∈ N(v)
      RecursiveDFS(w)
```



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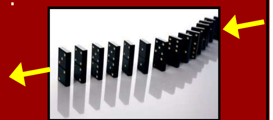
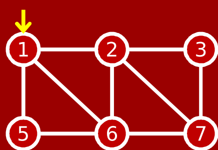
BFS: Breadth-First Search

When the bag is a "**queue**".
FIFO: First-In First-Out.

(Assume sorted adjacency list representation.)



(usually implemented using a linked list)



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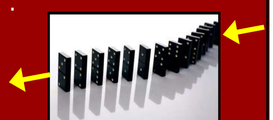

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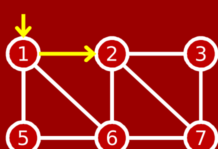

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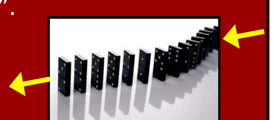

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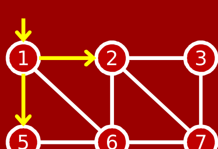

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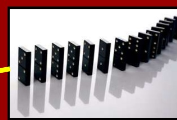

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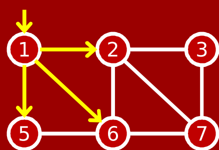

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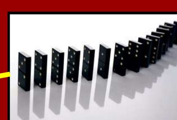

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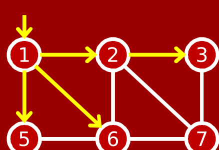

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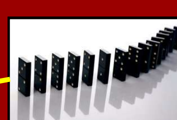

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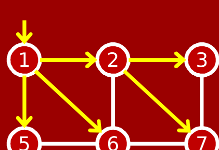

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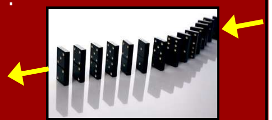
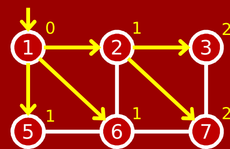


(usually implemented using a linked list)

BFS bonus property:
Vertices marked in increasing order of distance from s.

BFS(G,s)
…
parent(v) := p
dist(v) := dist(parent(v))+1
…



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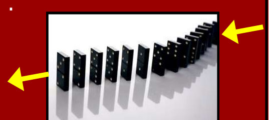
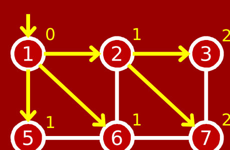


(usually implemented using a linked list)

BFS bonus property:
Vertices marked in increasing order of distance from s.

Exercise: Prove this.

So path from s to any v in BFS tree is a shortest path.



BFS & DFS: Running time

Put $\boxed{\perp\to s}$ into bag
While bag is not empty:
  Pick an Arbitrary tile $\boxed{p\to v}$ from bag
  If v is “unmarked”:
    “Mark” v and record parent(v) := p
    For each neighbor w of v:
      Put $\boxed{v\to w}$ into bag

Recall: # of tiles put in bag is ≤ 2|E|+1.
Actually, exactly 2|E|+1, assuming G connected.
Bag operations are O(1) time for stack/queue.
Each tile engenders O(1) work.
♦ Total run-time: O(|E|).

BFS & DFS: Running time

AFS(G,s) just finds the connected component of s.

What if we want to find all connected components?

FullAFS(G):
  For all vertices v:
    If v is unmarked
      AFS(G,v)

Overall run-time: O(|V|+|E|) (Why?)

We have seen AFS, BFS, DFS

Looks like we're missing something…

CFS! **Cheapest-First Search**

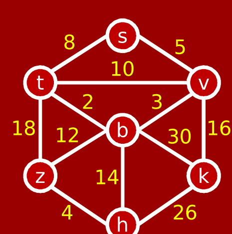
The goal of CFS is more ambitious than just finding connected components.

Its goal is to find a **minimum spanning tree** (MST).

Weighted Graphs

Often in life, each edge of a graph G = (V,E) will have a real number associated to it.

Variously called:
- **weight**
- **length**
- **distance**
- or **cost**.



"Cost function", $c : E \to \mathbb{R}^+$

Positive values only, unless otherwise specified.

MST

The year: 1926
The place: Brno, Moravia
Our hero: Otakar Borůvka

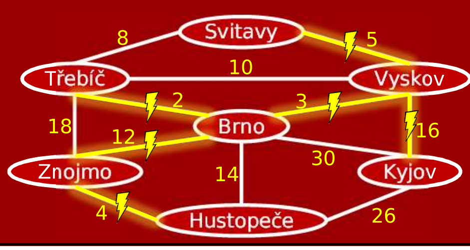
Borůvka's had a pal called Jindřich Saxel who worked for Západomoravské elektrárny (the West Moravian Power Plant company).

Saxel asked him how to figure out the most efficient way to electrify southwest Moravia.

MST

Edge exists if it's feasible to connect two towns by power lines.

Edge weights might be distance in km, or cost in 1000's of koruna to install lines.



MST

Minimum Spanning Tree (MST) problem:

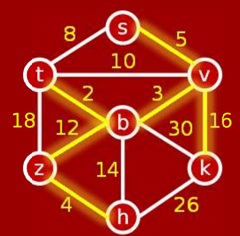
Input: A weighted graph $G = (V,E)$, with cost function $c : E \to \mathbb{R}^+$.

Output: Subset of edges of minimum total cost such that all vertices connected.

The edges will form a tree:
If you had a cycle, you could delete any edge on it and still be connected, but cheaper.

MST

Example:



In this case, there's a unique solution,
of cost 5+2+3+12+16+4=**42**.

MST

Convenient assumption: Edges have distinct costs.

In this case, not hard to show the MST is unique.

Thus we can speak of **the** MST, not just **an** MST.

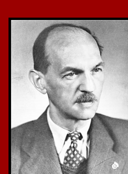
A hint for the little trick
that shows this is WLOG:

"Whether [the] distance from
Brno to Břeclav is 50 km
or 50 km and 1 cm
is a matter of conjecture."

MST via Cheapest-First Search

Often known as **Prim's Algorithm**,
due to a 1957 publication by
Robert C. Prim.

Actually first discovered by
Vojtěch Jarník, who described it
in a letter to Borůvka, and
published it in 1930.



Jarník

Borůvka himself had published a
different algorithm in 1926.

MST via Cheapest-First Search

```
Put [⊥→s] into bag
While bag is not empty:
    Pick an Arbitrary edge [p→v] from bag
    If v is "unmarked":
        "Mark" v, record parent(v) := p
        For each neighbor w of v:
            Put [v→w] into bag
```

MST via Cheapest-First Search

```
JARNÍK-PRIM(G): Let s be any vertex
    Put [⊥→s] into bag
    While bag is not empty:
        Pick the cheapest edge [p→v] from bag
        If v is "unmarked":
            "Mark" v, record parent(v) := p
            For each neighbor w of v:
                Put [v→w] into bag
```

Naive implementation: Unsorted list.

O(|E|) time to scan for cheapest edge.

$O(|E|^2)$ total run-time.

MST via Cheapest-First Search

```
JARNÍK-PRIM(G): Let s be any vertex
    Put [⊥→s] into bag
    While bag is not empty:
        Pick the cheapest edge [p→v] from bag
        If v is "unmarked":
            "Mark" v, record parent(v) := p
            For each neighbor w of v:
                Put [v→w] into bag
```

Sophisticated implementation: "Priority Queue".

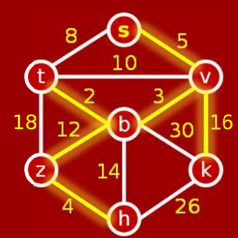
O(log |E|) time for both bag operations.

O(|E| log |E|) total run-time.

MST via Cheapest-First Search

Effectively: CFS grows a tree from s, always adding the cheapest edge next.

Example:



MST via Cheapest-First Search

Theorem: JARNÍK–PRIM finds the MST.

MST via Cheapest-First Search

Theorem: For each $0 \le k \le n-1$, the first k edges added are all in the MST.

Proof: By induction on k.

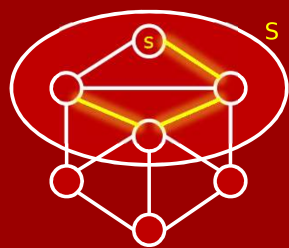
Base case $k=0$: Vacuously true.

Induction step: Suppose CFS has added k edges so far ($0 \le k < n-1$), and all are in MST.

We need to show next added edge is also in MST.

MST via Cheapest-First Search

Let S be the set of vertices connected to s so far,



MST via Cheapest-First Search

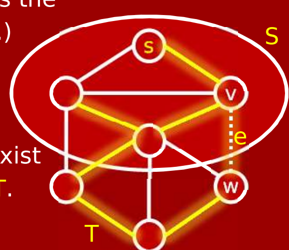
Let S be the set of vertices connected to s so far, and let e = {v,w} be next edge added by CFS.

(By definition of CFS, e is the cheapest edge out of S.)

Let T be the MST for G.

AFSOC that e ∉ T.

Since T spans G, must exist a path from v to w in T.



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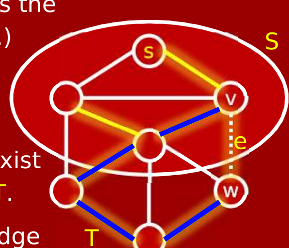
AFSOC that e ∉ T.

Since T spans G, must exist a path from v to w in T.

Let e'={v',w'} be first edge on that path which exits S.



MST via Cheapest-First Search

Let S be the set of vertices connected to s so far, and let e = {v,w} be next edge added by CFS.

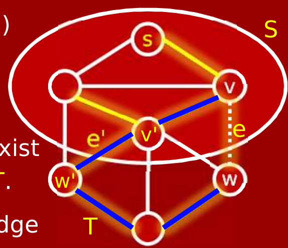
(By definition of CFS, e is the cheapest edge out of S.)

Let T be the MST for G.

AFSOC that $e \notin T$.

Since T spans G, must exist a path from v to w in T.

Let e'={v',w'} be first edge on that path which exits S.



MST via Cheapest-First Search

Claim: T' := T − e' ∪ {e} is a spanning tree.
If true, we have a contradiction because cost(e') > cost(e) (why?) and so cost(T') > cost(T).

T' has |V|−1 edges, so we just need to check it's still connected.

Any walk in T formerly using e' = {v,w} can now take path from v' to v, then take e, then take path from w to w'. □

Look carefully at our proof that $e \in$ MST.

We didn't actually use the fact that the edges inside S were part of the MST.

All we used: e was the cheapest edge out of S.

Thus we more generally proved…

MST Cut Property:

Let $G=(V,E)$ be a graph with distinct edge costs.

Let $S \subseteq V$ (with $S \neq \emptyset$, $S \neq V$).

Let $e \in E$ be the cheapest edge with one endpoint in S and the other not in S.

Then a minimum spanning tree **must** contain e.

MST Cut Property

Using this, it's not hard to show that practically any natural "greedy" MST algorithm works.

Kruskal's Algorithm:

Go through edges in order of cheapness.
Add edge as long as it doesn't make a cycle.

Borůvka's Algorithm:

Start with each vertex a connected component.
Repeatedly: add the cheapest edge coming out of each connected component.

Run-time Race for MST (an amusing story)

The "classical" (pre-1960) MST algorithms, Borůvka, Jarník–Prim, Kruskal, all run in time $O(m \log m)$.

That is very good.

In practice, these algorithms are great.

Nevertheless, algorithms & data structures wizards tried to do better.

Run-time Race for MST

1984: Fredman & Tarjan invent the "Fibonacci heap" data structure.

Run-time improved from O(m log(m)) to O(m log*(m)).

Remember log*(m)?

It is the number of times you need to take log to get down to 2.

For all real-world purposes, log*(m) ≤ 5.

Run-time Race for MST

1984: Fredman & Tarjan invent the "Fibonacci heap" data structure.

Run-time improved from O(m log(m)) to O(m log*(m)).

Run-time Race for MST

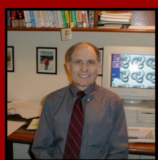
1986: Gabow, Galil, T. Spencer, Tarjan improved the algorithm.

Run-time improved from O(m log*(m)) to…
O(m log (log*(m))).

Run-time Race for MST

1986: Gabow, Galil, T. Spencer, Tarjan improved the algorithm.

Run-time improved from $O(m \log^*(m))$ to… $O(m \log(\log^*(m)))$.


Gabow


Galil


Tarjan & Not-Spencer

Run-time Race for MST

1997: Chazelle invents "soft heap" data structure.

Run-time improved from $O(m \log(\log^*(m)))$ to… $O(m\, \alpha(m) \log(\alpha(m)))$.

I will tell you what function $\alpha(m)$ is in a second.
I assure you, it's comically slow-growing.


Chazelle

Run-time Race for MST

2000: Chazelle improves it down to $O(m\, \alpha(m))$.

$\alpha(m)$ is called the Inverse-Ackermann function.

$\log^*(m)$ = # of times you need to do log to get down to 2

$\log^{**}(m)$ = # of times you need to do $\log^*$ to get down to 2

$\log^{***}(m)$ = # of times you need to do $\log^{**}$ to get down to 2

…

$\alpha(m)$ = # of $*$'s you need so that $\log^{***\cdots***}(m) \le 2$

It's incomprehensibly, preposterously slow-growing!

Run-time Race for MST

1995: Meanwhile, Karger, Klein, and Tarjan give an algorithm with run-time O(m).

It's a **randomized** algorithm:
O(m) is the expected value of the running time.


Karger


Klein


Tarjan

Run-time Race for MST

2002: Pettie and Ramachandran gave a new **deterministic** MST algorithm.

They proved its running time is O(**optimal**).


Pettie


Ramachandran

Run-time Race for MST

2002: Pettie and Ramachandran gave a new **deterministic** MST algorithm.

They proved its running time is O(**optimal**).

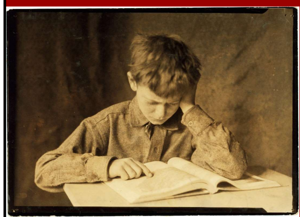
Would you like to know its running time?

So would we.

Its running time is **unknown**.

All we know is: whatever it is, it's optimal.

Study Guide



Definition:

Minimum Spanning Tree

Algorithms and analysis:

AFS
BFS
DFS
CFS (Jarník–Prim algorithm)