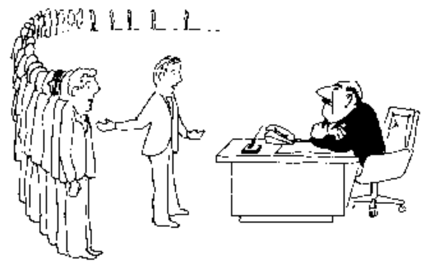
15-251: Great Ideas in Theoretical Computer Science

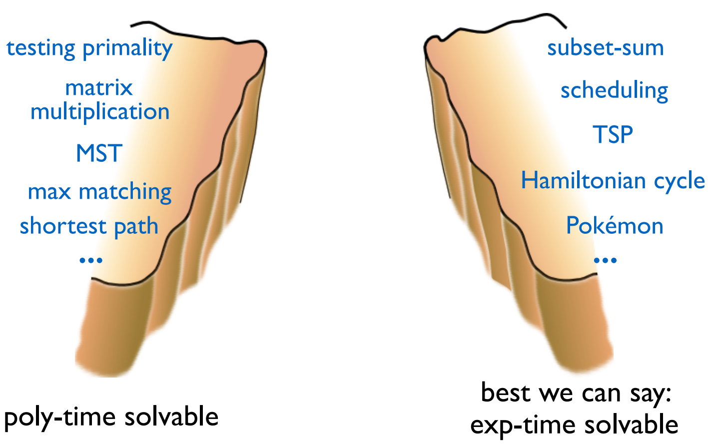
Lecture 15: NP and NP-completeness I

October 18th, 2018



I can't find an efficient algorithm, but neither can all these famous people.

The chasm between poly-time and exp-time.



Exponential running time examples

Subset Sum Problem

Theorem Proving Problem

Traveling Salesperon Problem (TSP)

Satisfiability Problem (SAT)

Circuit Satisfiability Problem (Circuit-SAT)

Sudoku Problem

In our quest to understand efficient computation,
we come across:

P vs NP problem

Biggest open problem in all of Computer Science.

One of the biggest open problems in all of Mathematics.

So what is the **P** vs **NP** question?

The P vs NP question is the following:

An important goal for a computer scientist

Identifying and dealing with intractable problems

After decades of research and billions of dollars of funding, **no** poly-time algs for:

Subset Sum, SAT, Theorem Proving, TSP, Sudoku, …

Can we prove there is no poly-time alg?

Goal

Revisiting reductions

A central concept for comparing the "difficulty" of problems.

differs based on context

Right now we are interested in poly-time decidability **vs** **not** poly-time decidability

Want to define: $A \leq B$ (B is at least as hard as A w.r.t. poly-time decidability.)

Revisiting reductions

Revisiting reductions

Example

A:

Given a graph and an integer k, does there exist at least k pairs of vertices connected to each other?

(by a path)

B:

Given a graph and a pair of vertices (s,t), are s and t connected?

Revisiting reductions

The 2 sides of reductions

1. Expand the landscape of tractable problems.

Revisiting reductions

The 2 sides of reductions

2. Expand the landscape of intractable problems.

Gathering evidence for intractability

including some that we think should not be in **P**

If we can show $L \leq_T^P A$ for **many** L

then that would be good evidence that $A \notin$ **P**.

Definition of **C**-hard

Definition of **C**-complete

Definitions of **C**-hard and **C**-complete

Observation:

Suppose A is **C**-complete.

2 possible worlds

Recall the goal

Good evidence for $A \notin$ **P** :

- A is **C**-complete for a really rich/large set **C**
  (a set **C** such that we believe **C** ≠ **P**)

So what is a good choice for **C** ?

(if we want to show *SAT, Theorem Proving, TSP,* … are **C**-complete?)

Main Goal Reduces to:

Finding the right complexity class **C**

<u>Try 1:</u>

<u>Try 2:</u>

A complexity class for BFS?

What would be a reasonable definition for:

"class of problems decidable using BFS" ?

What is common about

SAT, Theorem Proving, TSP, Sudoku, etc…?

The complexity class **NP**

<u>Informally:</u>

Poll: Test your intuition

Which of these are in **NP**?

- Subset Sum
- TSP
- SAT
- Circuit-SAT
- Sudoku
- HALTS
- $\{0^k 1^k : k \in \mathbb{N}\}$

Formal definition of **NP**

Examples of languages in **NP**

CLIQUE

Input: $\langle G, c \rangle$ where G is a graph and c is a positive int.

Output: Yes iff G contains a clique of size c.

Fact: CLIQUE is in **NP**.

Examples of languages in **NP**

Proof: We need to show a verifier TM V exists as specified in the definition of **NP**.

def $V(x,u)$:

Examples of languages in **NP**

Proof (continued):

Need to show:

1.

2.

3.

Examples of languages in **NP**

Proof (continued):

Need to show:

1. if $x \in$ CLIQUE, there is some proof u (of poly-length) that makes V ACCEPT.

Examples of languages in **NP**

Proof (continued):

Need to show:

2. if $x \notin$ CLIQUE, no matter what u is, V REJECTS.

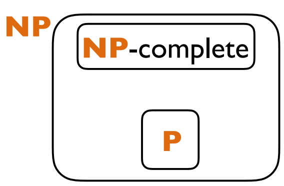
□

The complexity class **NP**

2 Observations:

1. Every decision problem in **NP** can be solved using BFS.

2. This is a big class!
Contains everything in **P**.



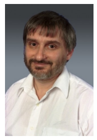
People expect **NP** contains much more than **P**.

Coming back to our main goal

Could it be true that one of
SAT, Theorem Proving, TSP, Sudoku, etc.
is **NP**-complete?

Is there **any** language that is **NP**-complete??

The Cook-Levin Theorem



Theorem (Cook 1971 - Levin 1973):

Karp's 21 **NP**-complete problems

1972: "Reducibility Among Combinatorial Problems"

0-1 Integer Programming	Partition
Clique	Clique Cover
Set Packing	Exact Cover
Vertex Cover	Hitting Set
Set Covering	Knapsack
Feedback Node Set	Steiner Tree
Feedback Arc Set	3-Dimensional Matching
Directed Hamiltonian Cycle	Job Sequencing
Undirected Hamiltonian Cycle	Max Cut
3SAT	Chromatic Number

Some other "interesting" examples

Super Mario Bros

Given a Super Mario Bros level, is it completable?

Classic Nintendo Games are (NP-)Hard

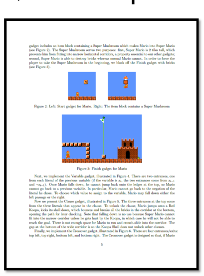
Greg Aloupis* Erik D. Demaine† Alan Guo†‡

March 9, 2012

arXiv:1203.1895v1 [cs.CC] 8 Mar 2012

Abstract

We prove NP-hardness results for five of Nintendo's largest video game franchises: Mario, Donkey Kong, Legend of Zelda, Metroid, and Pokémon. Our results apply to Super Mario Bros. 1, 3, Lost Levels, and Super Mario World; Donkey Kong Country 1–3; all Legend of Zelda games except Zelda II: The Adventure of Link; all Metroid games; and all Pokémon role-playing games. For Mario and Donkey Kong, we show NP-completeness. In addition, we observe that several games in the Zelda series are PSPACE-complete.

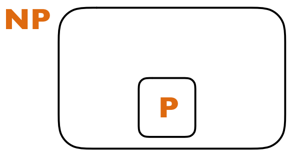


Tetris

Given a sequence of Tetris pieces, and a number k, can you clear more than k lines?

How do you show a language is **NP**-complete?

How did Cook and Levin do it ?!?



$\leq_T^P$ SAT

How did Karp do it ?!?

IMPORTANT NOTE:

How do you show a language is **NP**-complete?

It is similar to showing undecidability.

- need an initial direct proof that a language is **NP**-hard. (Cook-Levin Theorem)

- to show other languages are **NP**-hard, use poly-time reductions.

These are the topics of next 2 lectures.

The P vs NP Question

Good evidence for intractability?

If A is **NP**-hard,
that seems to be good evidence that $A \notin$ **P**…

if you believe **P** ≠ **NP**

But is **P** ≠ **NP**???

The two possible worlds

What do experts think?

Two polls from **2002** and **2012**

respondents in **2002**: 100

respondents in **2012**: 152

	$P \neq NP$	$P = NP$	Ind	DC	DK
2002	61(61%)	9(9%)	4(4%)	1(1%)	22(22%)
2012	126 (83%)	12 (9%)	5 (3%)	5 (3%)	1(0.6%)

What does **NP** stand for anyway?