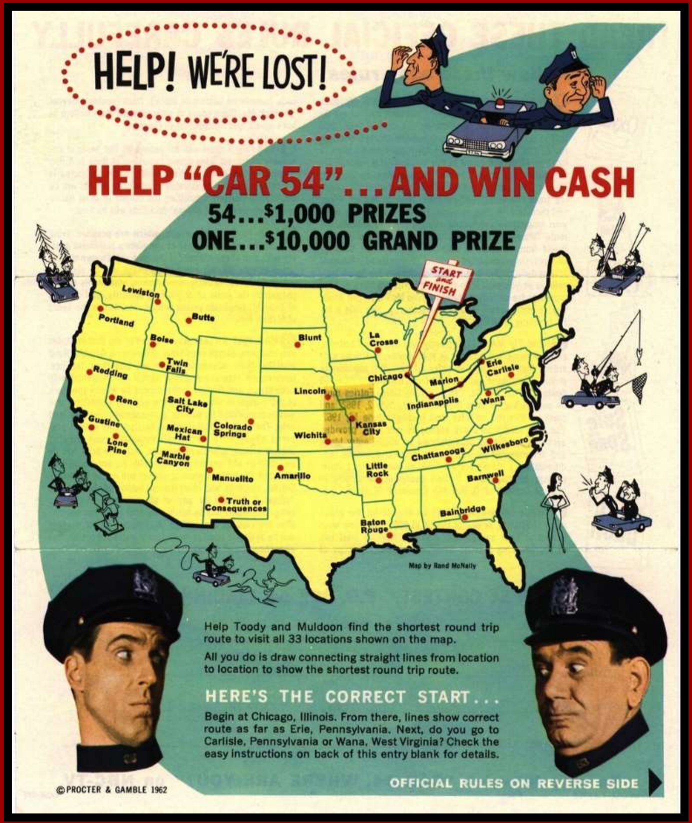
15-251: Great Theoretical Ideas in Computer Science
Fall 2018, Lecture 17

Approximation Algorithms



SAT	given a Boolean formula F, is it satisfiable?
3SAT	same, but F is a 3-CNF
Vertex-Cover	given G and k, are there k vertices which touch all edges?
Clique	are there k vertices all connected?
Max-Cut	is there a vertex 2-coloring with at least k “cut” edges?
Hamiltonian-Cycle	is there a cycle touching each vertex exactly once?

SAT	… is **NP-complete**
3SAT	… is **NP-complete**
Vertex-Cover	… is **NP-complete**
Clique	… is **NP-complete**
Max-Cut	… is **NP-complete**
Hamiltonian-Cycle	… is **NP-complete**

Decision vs. Optimization/Search

NP defined to be a class of **decision problems**.

Usually there is a natural 'optimization' version.

3SAT	Given a 3-CNF formula, is it satisfiable?
Vertex-Cover	Given G and k, are there k vertices which touch all edges?
Clique	Given G and k, are there k vertices which are all mutually connected?
Max-Cut	Is there a vertex 2-coloring with at least k "cut" edges?
Hamiltonian-Cycle	Is there a cycle touching each vertex exactly once?

Decision vs. Optimization/Search

NP defined to be a class of **decision problems**.

Usually there is a natural ‘optimization’ version.

3SAT	
Vertex-Cover	Given G, find the size of the smallest S ⊆ V touching all edges.
Clique	Given G, find the size of the largest clique (set of mutually connected vertices).
Max-Cut	Given G, find the largest number of edges ‘cut’ by some vertex 2-coloring.
Hamiltonian-Cycle	

Decision vs. Optimization/Search

NP defined to be a class of **decision problems**.

Usually there is a natural ‘optimization’ version.

3SAT	Given a 3-CNF formula, find the largest number of clauses satisfiable by a truth assignment.
Vertex-Cover	Given G, find the size of the smallest S ⊆ V touching all edges.
Clique	Given G, find the size of the largest clique (set of mutually connected vertices).
Max-Cut	Given G, find the largest number of edges ‘cut’ by some vertex 2-coloring.
Hamiltonian-Cycle	

Decision vs. Optimization/Search

NP defined to be a class of **decision problems**.

Usually there is a natural ‘optimization’ version.

3SAT	Given a 3-CNF formula, find the largest number of clauses satisfiable by a truth assignment.
Vertex-Cover	Given G, find the size of the smallest S ⊆ V touching all edges.
Clique	Given G, find the size of the largest clique (set of mutually connected vertices).
Max-Cut	Given G, find the largest number of edges ‘cut’ by some vertex 2-coloring.
TSP	Given G with edge costs, find the cost of the cheapest cycle touching each vertex once.

Decision vs. Optimization/Search

NP defined to be a class of **decision problems**.

Usually there is a natural 'optimization' version and a natural 'search' version.

3SAT	Given a 3-CNF formula, find a truth assignment with the largest number of satisfied clauses.
Vertex-Cover	Given G, find the smallest S ⊆ V touching all edges.
Clique	Given G, find the largest clique (set of mutually connected vertices).
Max-Cut	Given G, find the vertex 2-coloring which 'cuts' the largest number of edges.
TSP	Given G with edge costs, find the cheapest cycle touching each vertex once.

Decision vs. Optimization/Search

NP defined to be a class of **decision problems**.

Usually there is a natural ‘optimization’ version and a natural ‘search’ version.

Technically, the ‘optimization’ or ‘search’ versions cannot be in NP, since they’re not languages.

We often still say they are NP-hard.

This means: if you could solve them in poly-time, then you could solve any NP problem in poly-time.

Why???

Decision vs. Optimization/Search

More interestingly the opposite is usually true too:

Given an efficient solution to the decision problem we can solve the ‘optimization’ and ‘search’ versions efficiently, too.

Find the number (e.g., of satisfiable clauses) via

binary search.

Find a solution (e.g., satisfying assignment) by

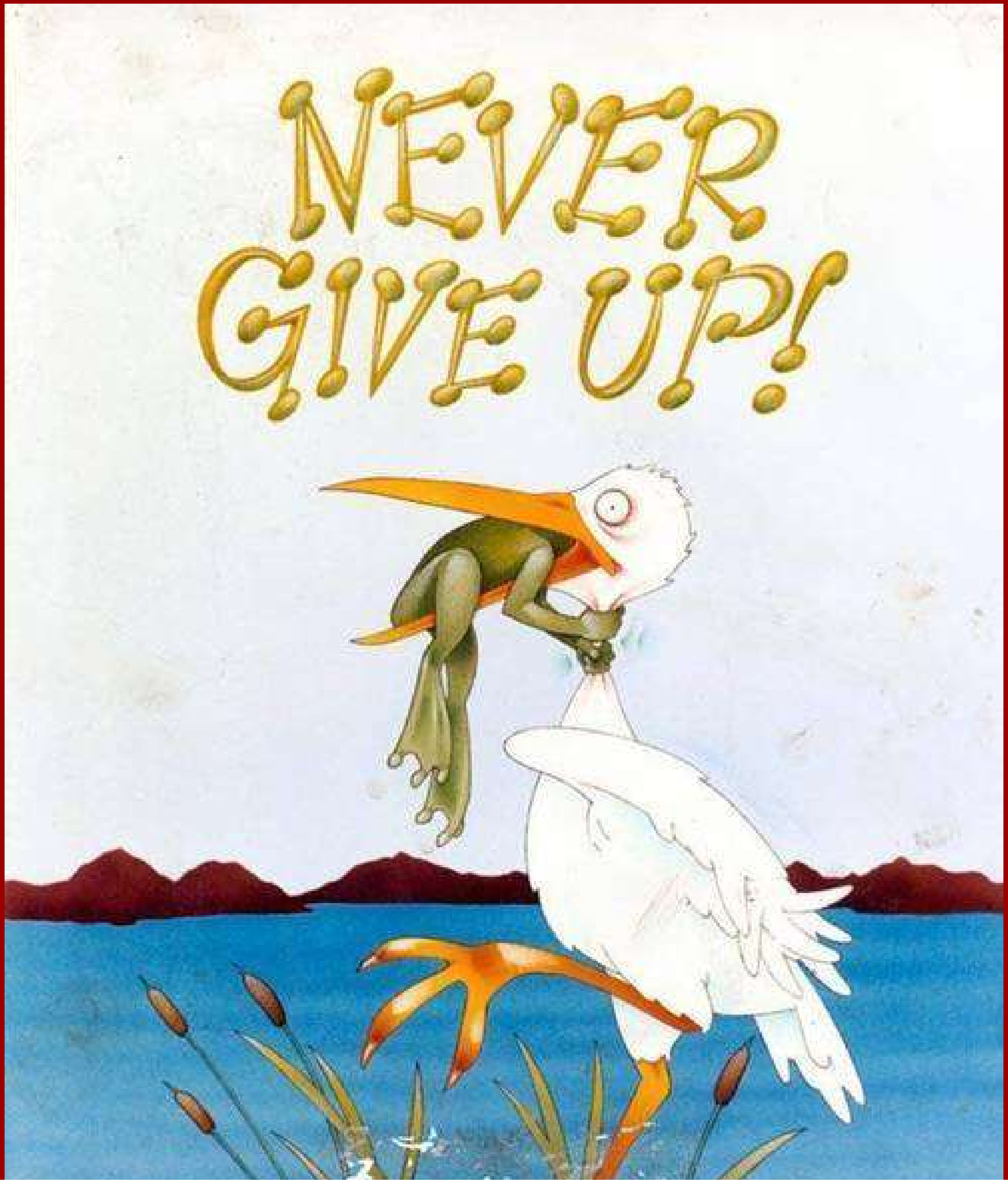
setting variables one by one an, testing each time if there is still a good assignment.

SAT	… is **NP-complete**
3SAT	… is **NP-complete**
Vertex-Cover	… is **NP-complete**
Clique	… is **NP-complete**
Max-Cut	… is **NP-complete**
Hamiltonian-Cycle	… is **NP-complete**

INVENTS BEAUTIFUL THEORY
OF ALGORITHMIC COMPLEXITY

EVERYTHING IS NP-COMPLETE

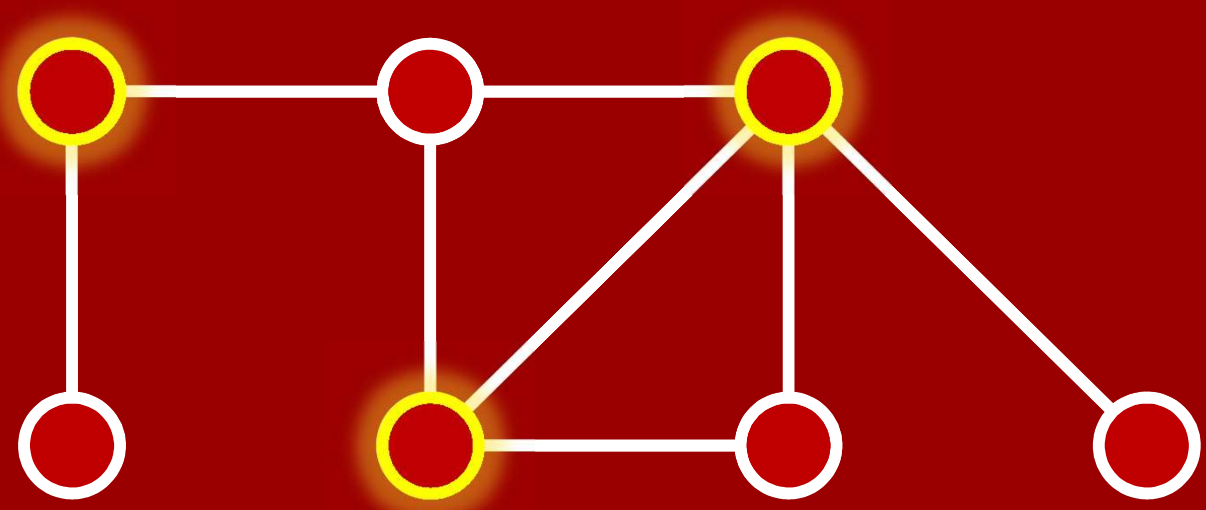
There is only one idea in this lecture:



Vertex-Cover

Given graph G = (V,E) and number k,
is there a size-k “vertex-cover” for G?

(S ⊆ V is a “vertex-cover” if it touches all edges.)

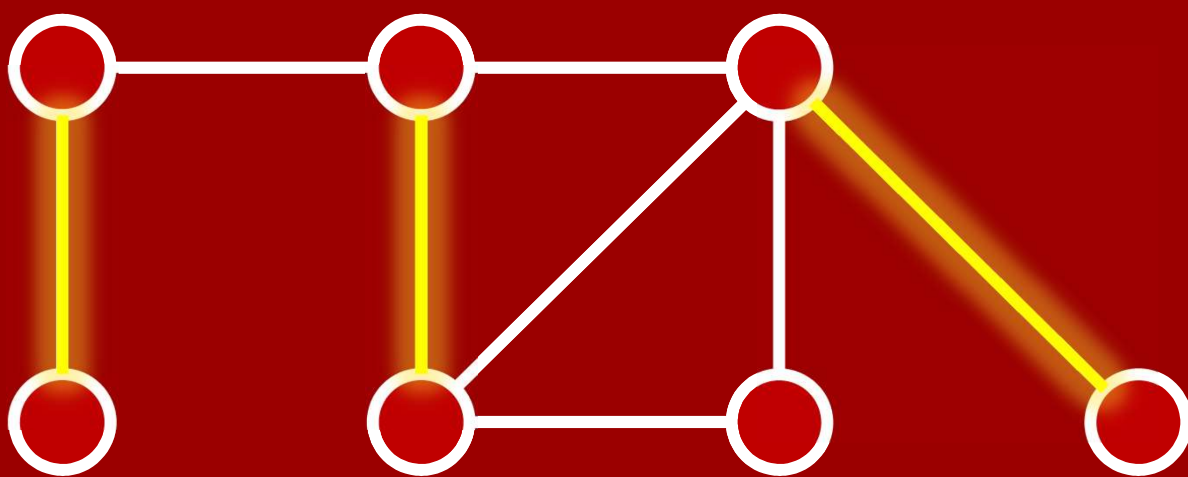


G has a vertex-cover of size 3.

Vertex-Cover

Given graph G = (V,E) and number k,
is there a size-k “vertex-cover” for G?

(S ⊆ V is a “vertex-cover” if it touches all edges.)



G has no vertex-cover of size 2.

(Because you need ≥ 1 vertex per yellow edge.)

Vertex-Cover

Given graph $G = (V,E)$ and number k,
is there a size-k “vertex-cover” for G?

($S \subseteq V$ is a “vertex-cover” if it touches all edges.)

The Vertex-Cover problem is NP-complete. ☹

➔ assuming “P ≠ NP”, there is **no** algorithm
running in **polynomial time**
which, for **all graphs** G,
finds the **minimum**-size vertex-cover.

Never Give Up

Subexponential-time algorithms:

Brute-force tries all 2^n subsets of n vertices.

Maybe there's an $O(1.5^n)$-time algorithm.

Or $O(1.1^n)$ time, or $O(2^{n^{.1}})$ time, or…

Could be quite okay if $n = 100$, say.

As of 2010: there **is** an $O(1.28^n)$-time algorithm.

→ assuming "P ≠ NP", there is **no** algorithm
running in **polynomial time**
which, for **all graphs** G,
finds the **minimum**-size vertex-cover.

Never Give Up

Special cases:

Solvable in poly-time for…

tree graphs,

bipartite graphs,

“**series-parallel**” graphs…

Perhaps for “graphs encountered in practice”?

→ assuming “P ≠ NP”, there is **no** algorithm

running in **polynomial time**

which, for **all graphs** G,

finds the **minimum**-size vertex-cover.

Never Give Up

Approximation algorithms:

Try to find pretty small vertex-covers.

Still want polynomial time, and for **all** graphs.

→ assuming "P ≠ NP", there is **no** algorithm
running in **polynomial time**
which, for **all graphs** G,
finds the **minimum**-size vertex-cover.

Gavril's Approximation Algorithm

Easy Theorem (from 1976):

There is a **polynomial-time** algorithm that,

given **any** graph $G = (V,E)$,

outputs a vertex-cover $S \subseteq V$ such that

$$|S| \leq 2|S^*|$$

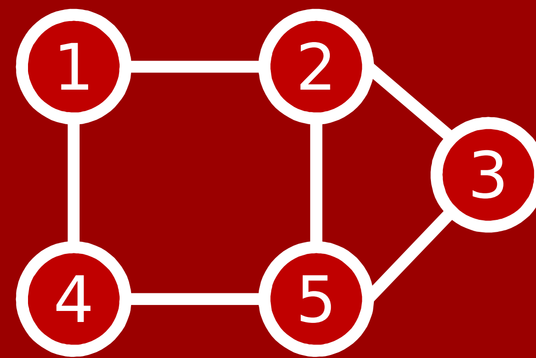
where S^* is the **smallest** vertex-cover.

"A factor 2-approximation for Vertex-Cover."

Let’s recall a similar situation from Lecture 10:

My favorite problem, **Max-Cut**.

Max-Cut

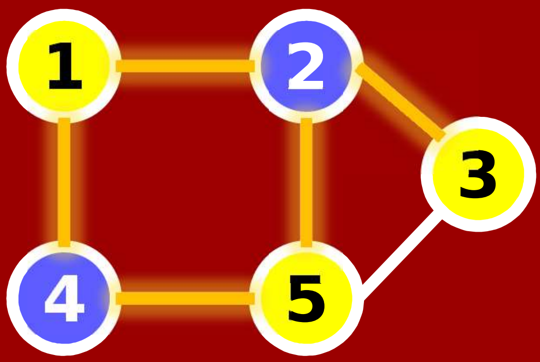


Input: A graph G=(V,E).

Output: A "**2-coloring**" of V: each vertex designated yellow or blue.

Goal: Have as many **cut** edges as possible. An edge is cut if its endpoints have different colors.

Max-Cut



Input: A graph G=(V,E).

Output: A "**2-coloring**" of V: each vertex designated yellow or blue.

Goal: Have as many **cut** edges as possible. An edge is cut if its endpoints have different colors.

Max-Cut

On one hand:

Finding the **MAX**-Cut is NP-hard.

On the other hand:

Polynomial-time “Local Search” algorithm guarantees cutting $\geq \frac{1}{2}|E|$ edges.

In particular:

(# cut by Local Search) $\geq \frac{1}{2}$ (max # cuttable)

“A factor ½-approximation for Max-Cut.”

Max-Cut

By the way:

Goemans and Williamson (1994) gave a polynomial-time

0.87856-approximation

for Max-Cut.

It is very beautiful, but pretty difficult!

Not all NP-hard problems created equal!

3SAT, Vertex-Cover, Clique, Max-Cut, TSP, …

All of these problems are equally NP-hard.

(There's no poly-time algorithm to find
the optimal solution unless P = NP.)

But from the point of view of finding
approximately optimal solutions,
there is an intricate, fascinating, and wide
range of possibilities…

Today: A case study of approximation algorithms

1. A somewhat good approximation algorithm
   for Vertex-Cover.

2. A pretty good approximation algorithm
   for the "k-Coverage Problem".

3. Some very good approximation algorithms
   for TSP.

Today: A case study of approximation algorithms

1. A somewhat good approximation algorithm for Vertex-Cover.

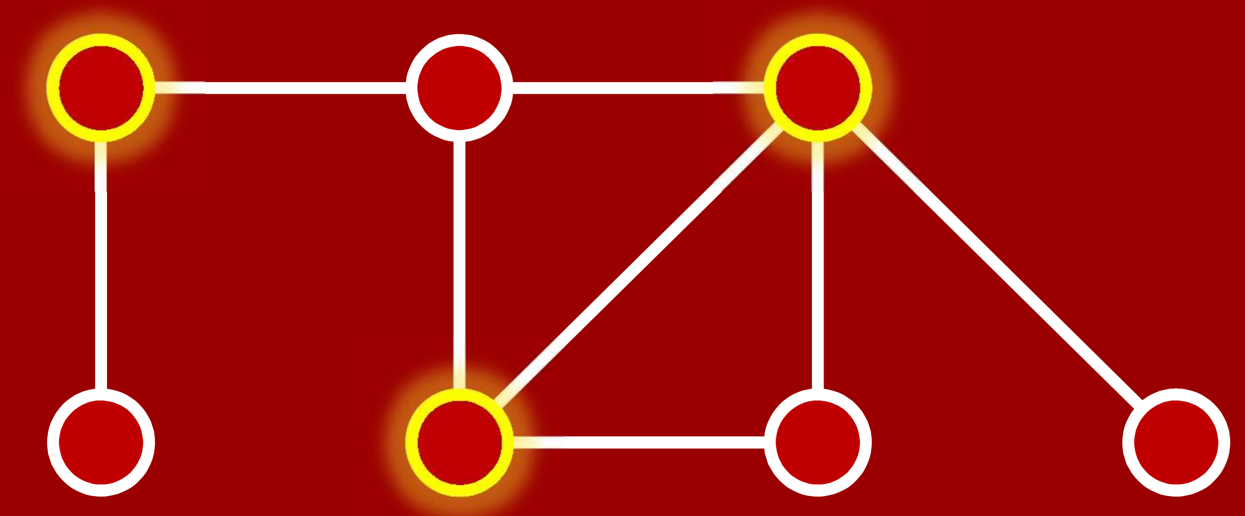
2. A pretty good approximation algorithm for the “k-Coverage Problem”.

3. Some very good approximation algorithms for TSP.

Vertex-Cover

Given graph G = (V,E) try to find the smallest “vertex-cover” for G.

(S ⊆ V is a “vertex-cover” if it touches all edges.)

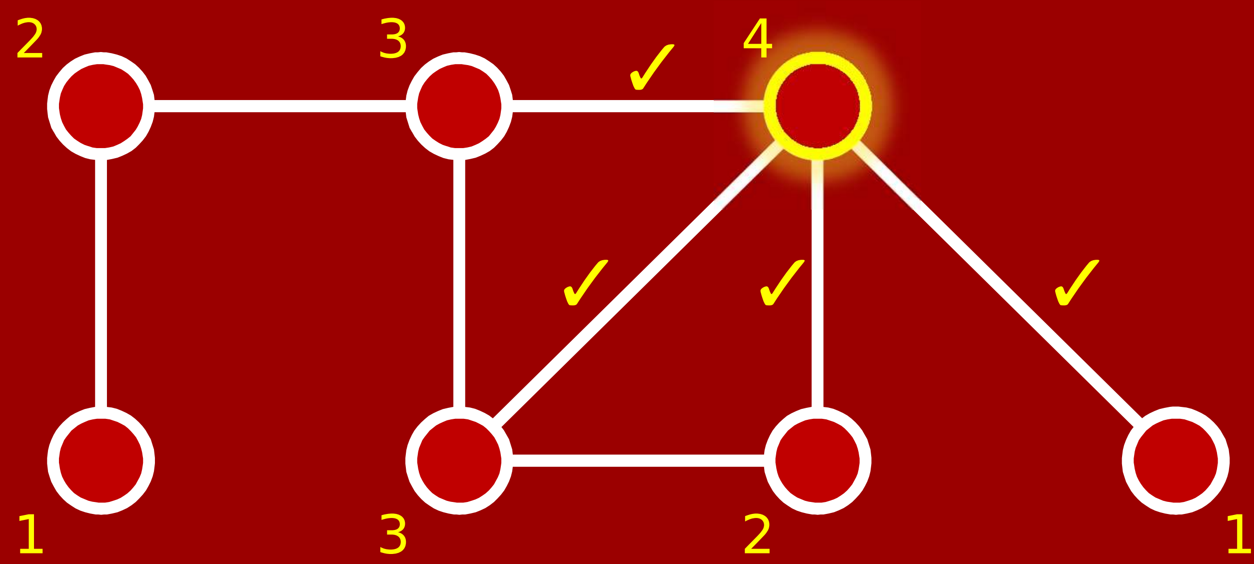


A possible Vertex-Cover algorithm

Simplest heuristic you might think of:

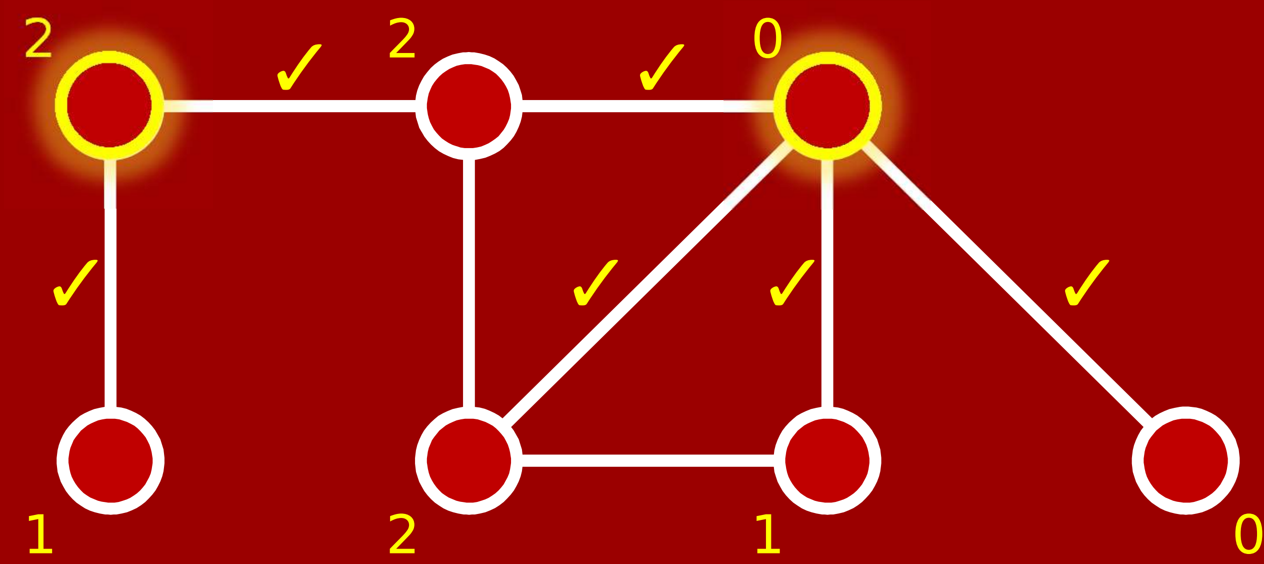
```
GreedyVC(G)
  S ← ∅
  while not all edges marked as "covered"
    find v∈V touching most unmarked edges
    S ← S ∪ {v}
    mark all edges v touches
```

GreedyVC example

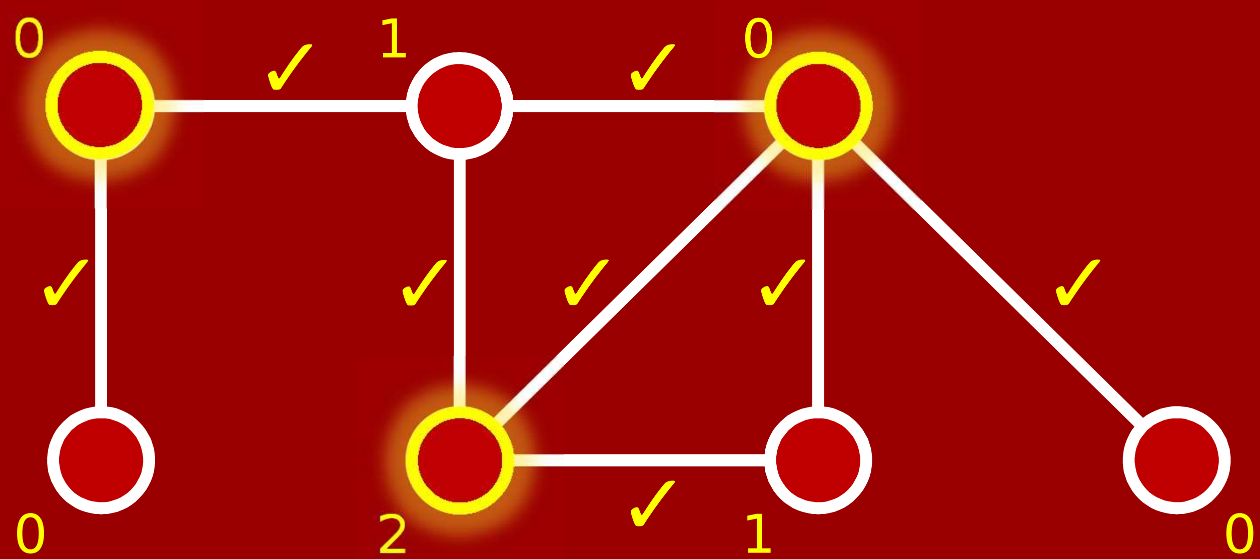


GreedyVC example

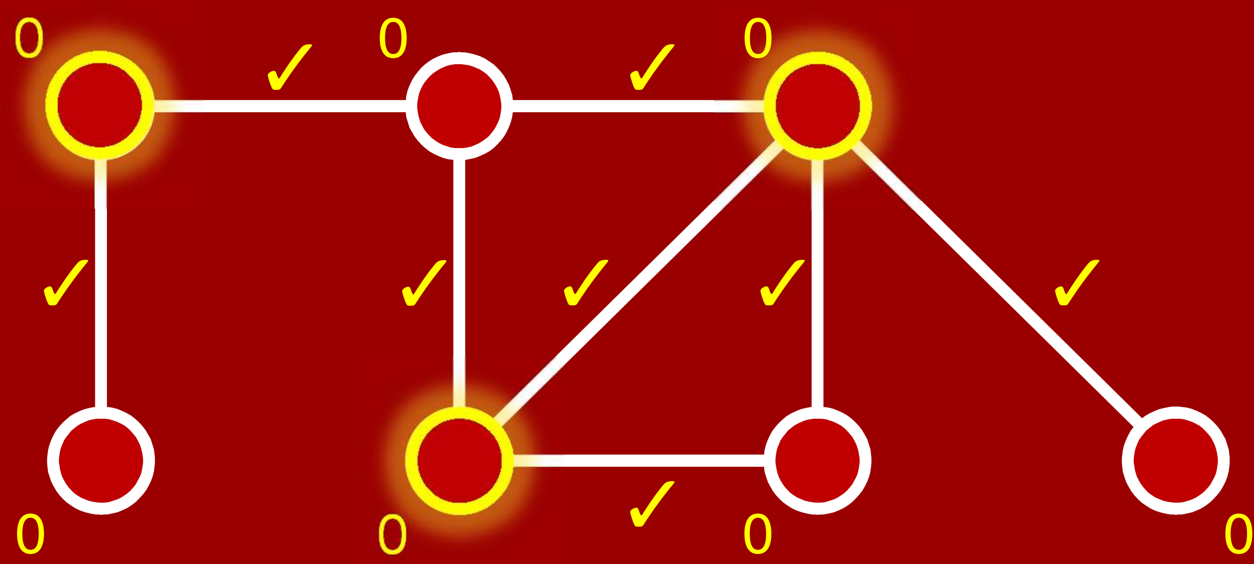
(Break ties arbitrarily.)



GreedyVC example



GreedyVC example



Done. Vertex-cover size 3 (optimal) ☺.

GreedyVC analysis

Correctness:

✓ Always outputs a **valid** vertex-cover.

Running time:

✓ Polynomial time.

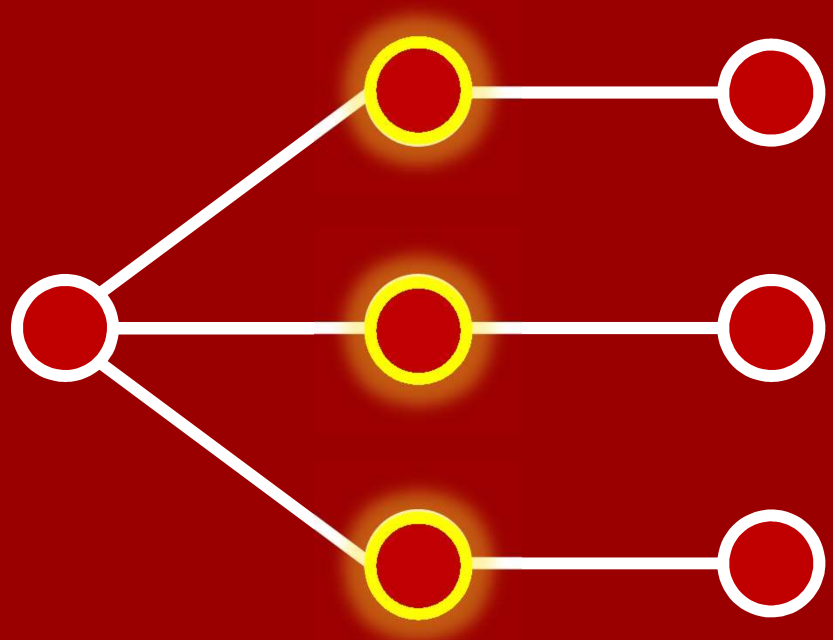
Solution quality:

This is the interesting question.

There must be some graph G where it doesn’t find the **smallest** vertex-cover.

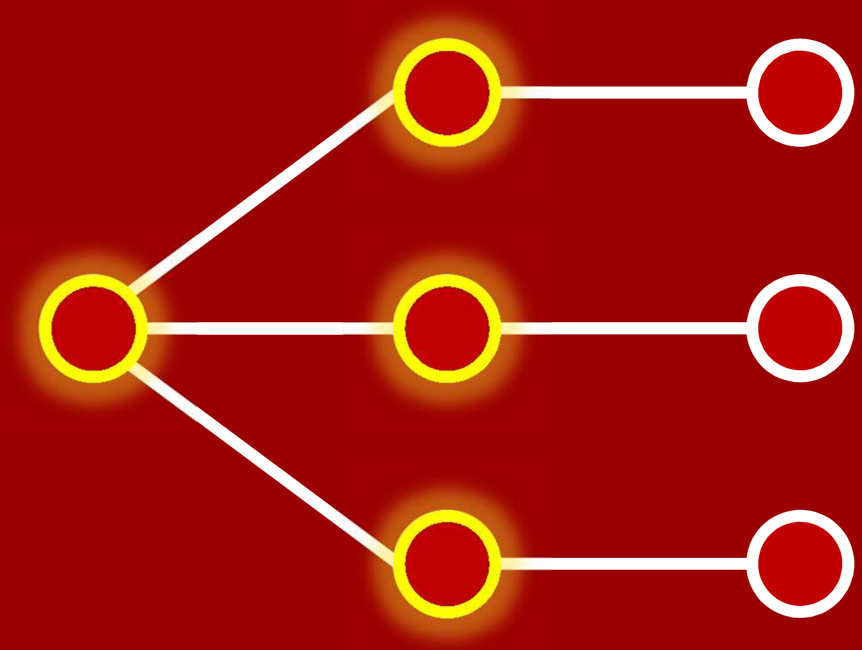
Because otherwise… P = NP!

A bad graph for GreedyVC



Smallest? 3

A bad graph for GreedyVC

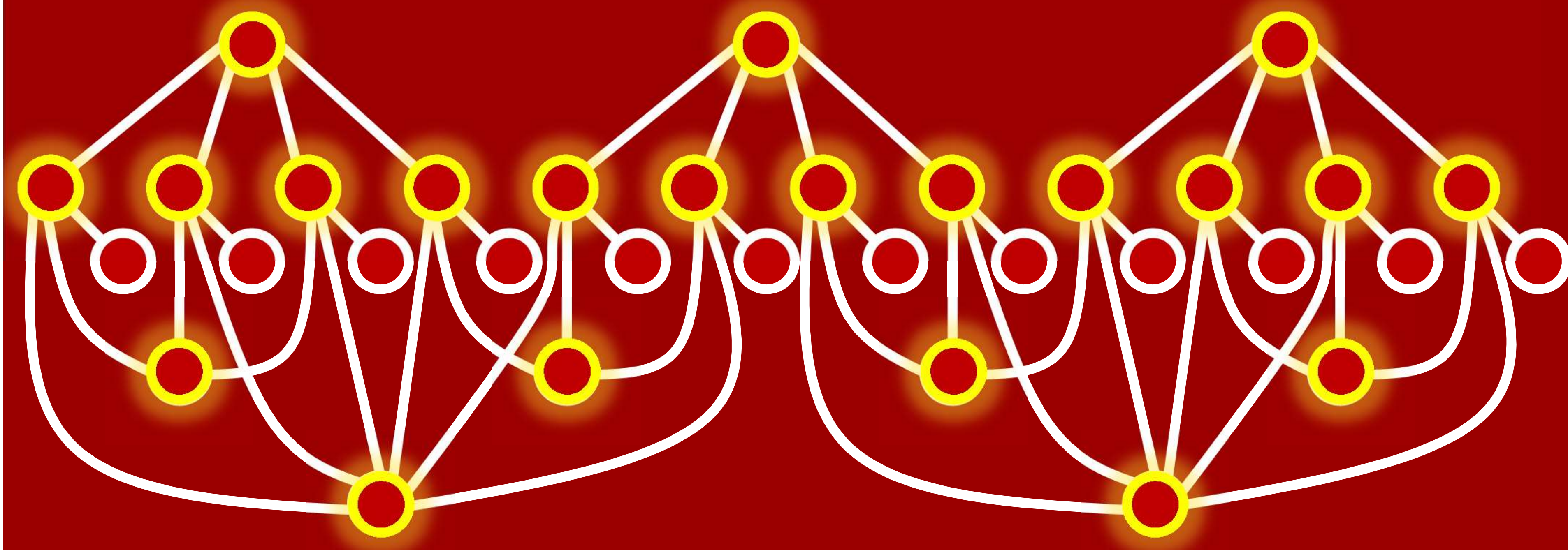


Smallest? 3

GreedyVC? 4

So GreedyVC is **not** a 1.33-approximation.
(Because 1.33 < 4/3.)

A worse graph for GreedyVC



Smallest?	???
GreedyVC?	21

So GreedyVC is **not** a 1.74-approximation. (Because 1.74 < 21/12.)

Even worse graph for GreedyVC

Well… it's a good homework problem.

We know GreedyVC is **not** a 1.74-approximation.

Fact: GreedyVC is **not** a 2.08-approximation.

Fact: GreedyVC is **not** a 3.14-approximation.

Fact: GreedyVC is **not** a 42-approximation.

Fact: GreedyVC is **not** a 999-approximation.

Greed is Bad (for Vertex-Cover)

Theorem: $\forall C$, GreedyVC is **not** a C-approximation.

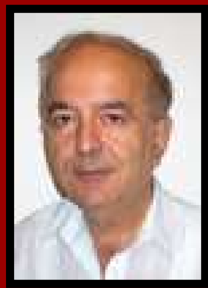
In other words:

For any constant C,

there is a graph G such that

$$|\text{GreedyVC}(G)| > C \cdot |\text{Min-Vertex-Cover}(G)|.$$

Gavril to the rescue



GavrilVC(G)

S ← ∅

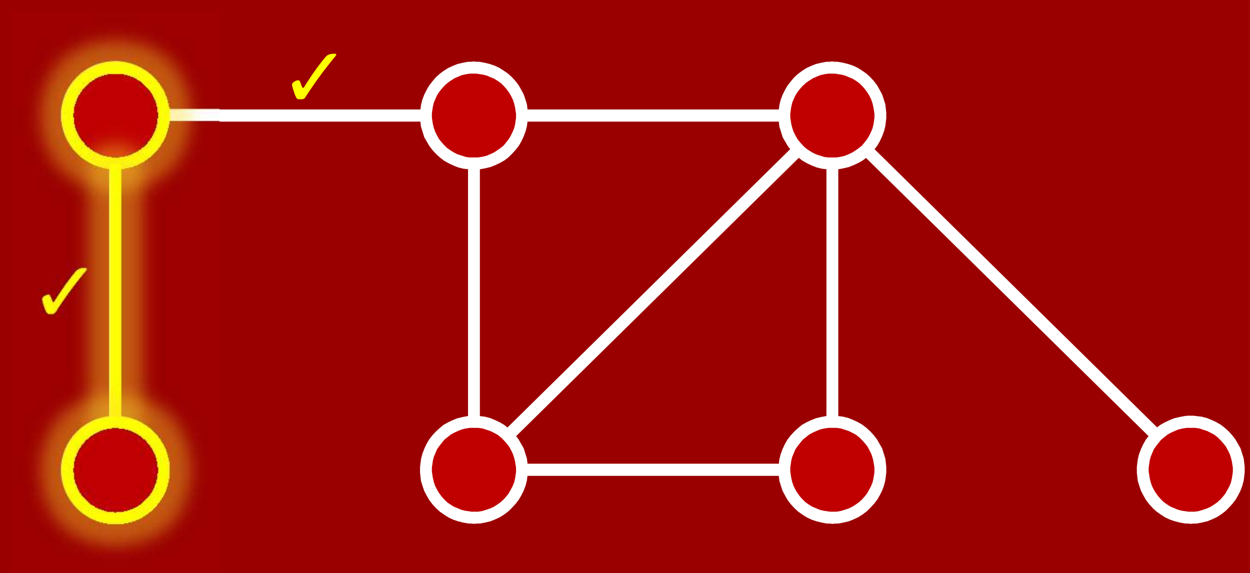
while **not** all edges marked as “covered”

let {v,w} be any unmarked edge

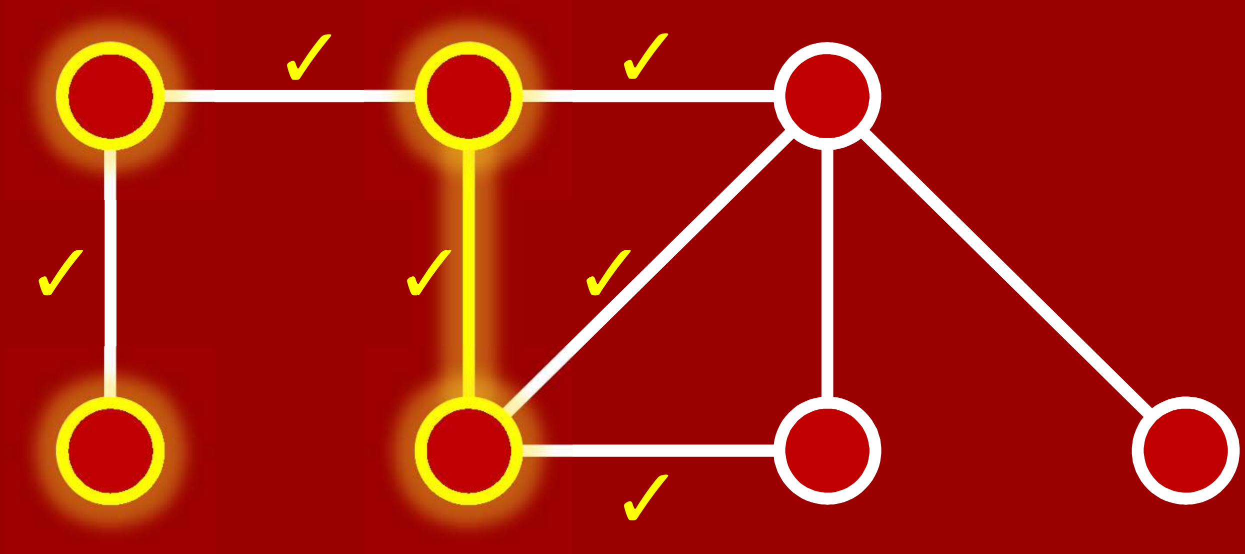
S ← S ∪ {v,w} ?

mark all edges v,w touch

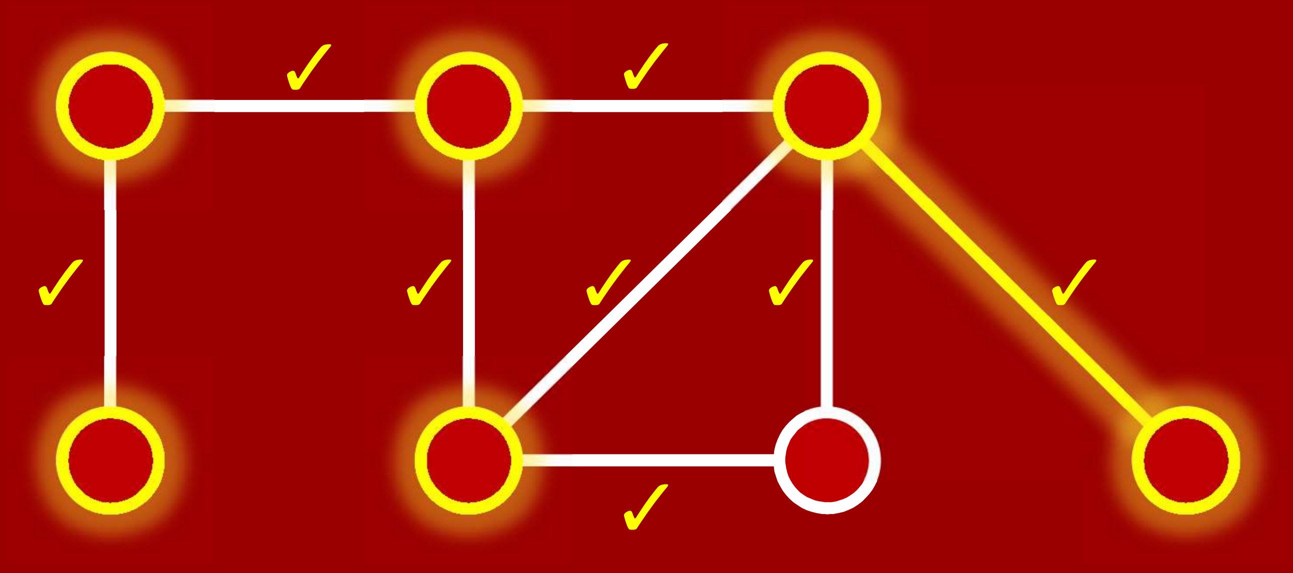
GavrilVC example



GavrilVC example



GavrilVC example



Smallest: 3

GavrilVC: 6

So GavrilVC is **at best** a 2-approximation.

Theorem:

GavrilVC is a **2**-approximation for Vertex-Cover.

Proof:

Say GavrilVC(G) does T iterations. So its $|S| = \underline{2T}$.

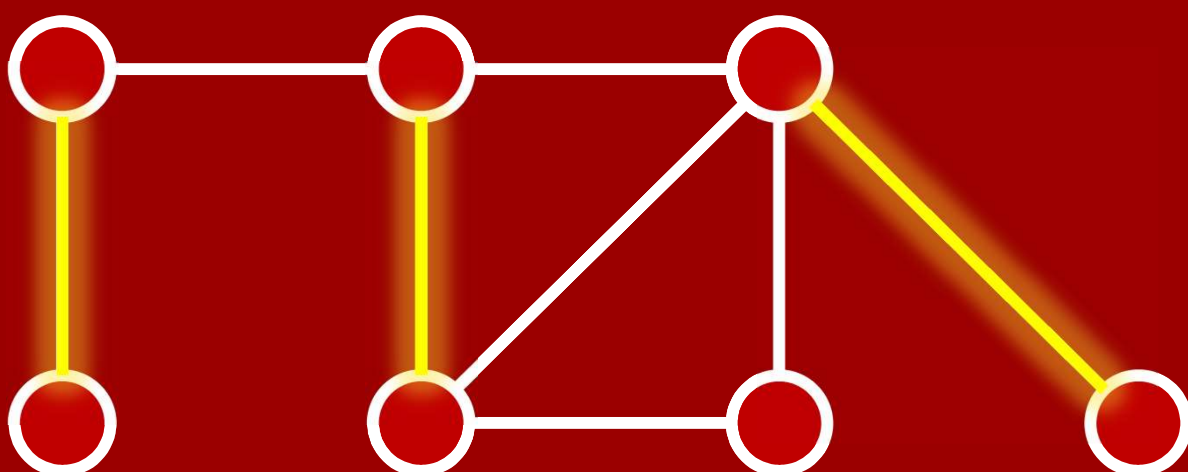
Say it picked edges $e_1, e_2, \ldots, e_T \in E$.

Key claim: $\{e_1, e_2, \ldots, e_T\}$ is a matching.

Because… when e_j is picked, it's unmarked,

so its endpoints are not among $e_1, \ldots, e_{j-1}$.

So **any** vertex-cover must have ≥ 1 vertex from each e_j.



Theorem:

GavrilVC is a **2**-approximation for Vertex-Cover.

Proof:

Say GavrilVC(G) does T iterations. So its $|S| = \underline{2T}$.

Say it picked edges $e_1, e_2, \ldots, e_T \in E$.

Key claim: $\{e_1, e_2, \ldots, e_T\}$ is a matching.

Because… when e_j is picked, it's unmarked,

so its endpoints are not among $e_1, \ldots, e_{j-1}$.

So **any** vertex-cover must have ≥ 1 vertex from each e_j.

Including the **minimum** vertex-cover S^*, whatever it is.

Thus $|S^*| \geq T$.

So for Gavril's final vertex-cover S,

$$|S| = 2T \leq 2|S^*|. \quad \square$$

Today: A case study of approximation algorithms

1. A 2-approximation algorithm for Vertex-Cover.

2. A pretty good approximation algorithm
   for the "k-Coverage Problem".

3. Some very good approximation algorithms
   for TSP.

Today: A case study of approximation algorithms

1. A 2-approximation algorithm for Vertex-Cover.

2. A pretty good approximation algorithm for the “k-Coverage Problem”.

3. Some very good approximation algorithms for TSP.

“k-Coverage” problem

“Pokémon-Coverage” problem

Let’s say you have some Pokémon,

and some trainers, each having a subset of Pokémon.

Given k, choose a team of k trainers to maximize the # of distinct Pokémon.

“Pokémon-Coverage” problem

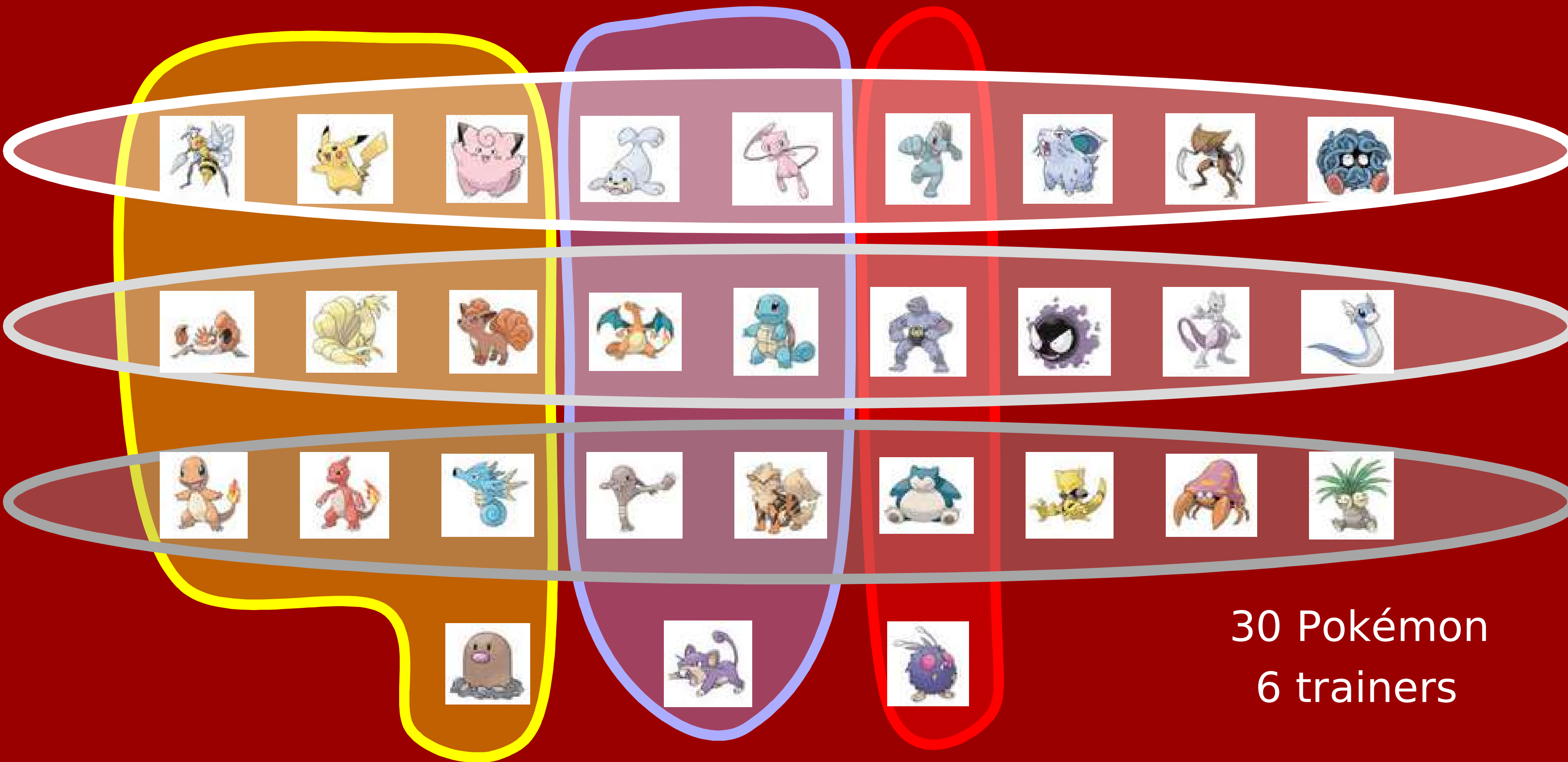
This problem is NP-hard. ☹

Approximation algorithm?

We could try to be greedy again…

```
GreedyCoverage()
    for i = 1…k
    add to the team the trainer bringing in the
       most new Pokémon, given the team so far
```

Example with k=3:



Optimum: 27

GreedyCoverage: 21

So Greedy is **at best** a 77.7%-approximation.

Greed is Pretty Good (for k-Coverage)

Theorem:

GreedyCoverage is a **63%**-approximation for k-Coverage.

More precisely, $1 - 1/e$

where $e \approx 2.718281828\ldots$

Proof: (Don't read if you don't want to.)

Let P* be the Pokémon covered by the best k trainers.

Define r_i = |P*| − # Pokémon covered after i steps of Greedy.

We'll prove by induction that $r_i \le (1-1/k)^i \cdot |P^*|$.

The base case i=0 is clear, as $r_0 = |P^*|$.

For the inductive step, suppose Greedy enters its ith step.

At this point, the number of uncovered Pokémon in P* must be $\ge r_{i-1}$.

We know there are some k trainers covering all these Pokémon.

Thus one of these trainers must cover at least r_{i-1}/k of them.

Therefore the trainer chosen in Greedy's ith step will cover $\ge r_{i-1}/k$ Pokémon.

Thus $r_i \le r_{i-1} - r_{i-1}/k = (1-1/k)\cdot r_{i-1} \le (1-1/k)\cdot(1-1/k)^i\cdot|P^*|$ by induction.

Thus we have completed the inductive proof that $r_i \le (1-1/k)^i \cdot |P^*|$.

Therefore the Greedy algorithm terminates with $r_k \le (1-1/k)^k \cdot |P^*|$.

Since $1-1/k \le e^{-1/k}$ (Taylor expansion), we get $r_k \le e^{-1} \cdot |P^*|$.

Thus Greedy covers at least $|P^*| - e^{-1} \cdot |P^*| = (1-1/e) \cdot |P^*|$ Pokémon.

This completes the proof that Greedy is a (1−1/e)-approximation algorithm.

∎

Today: A case study of approximation algorithms

1. A 2-approximation algorithm for Vertex-Cover.

2. A 63% ($1-1/e$) approximation algorithm for the "k-Coverage Problem".

3. Some very good approximation algorithms for TSP.

Today: A case study of approximation algorithms

1. A 2-approximation algorithm for Vertex-Cover.

2. A 63% $(1-1/e)$ approximation algorithm for the "k-Coverage Problem".

3. Some very good approximation algorithms for TSP.

TSP

(Traveling Salesperson Problem)

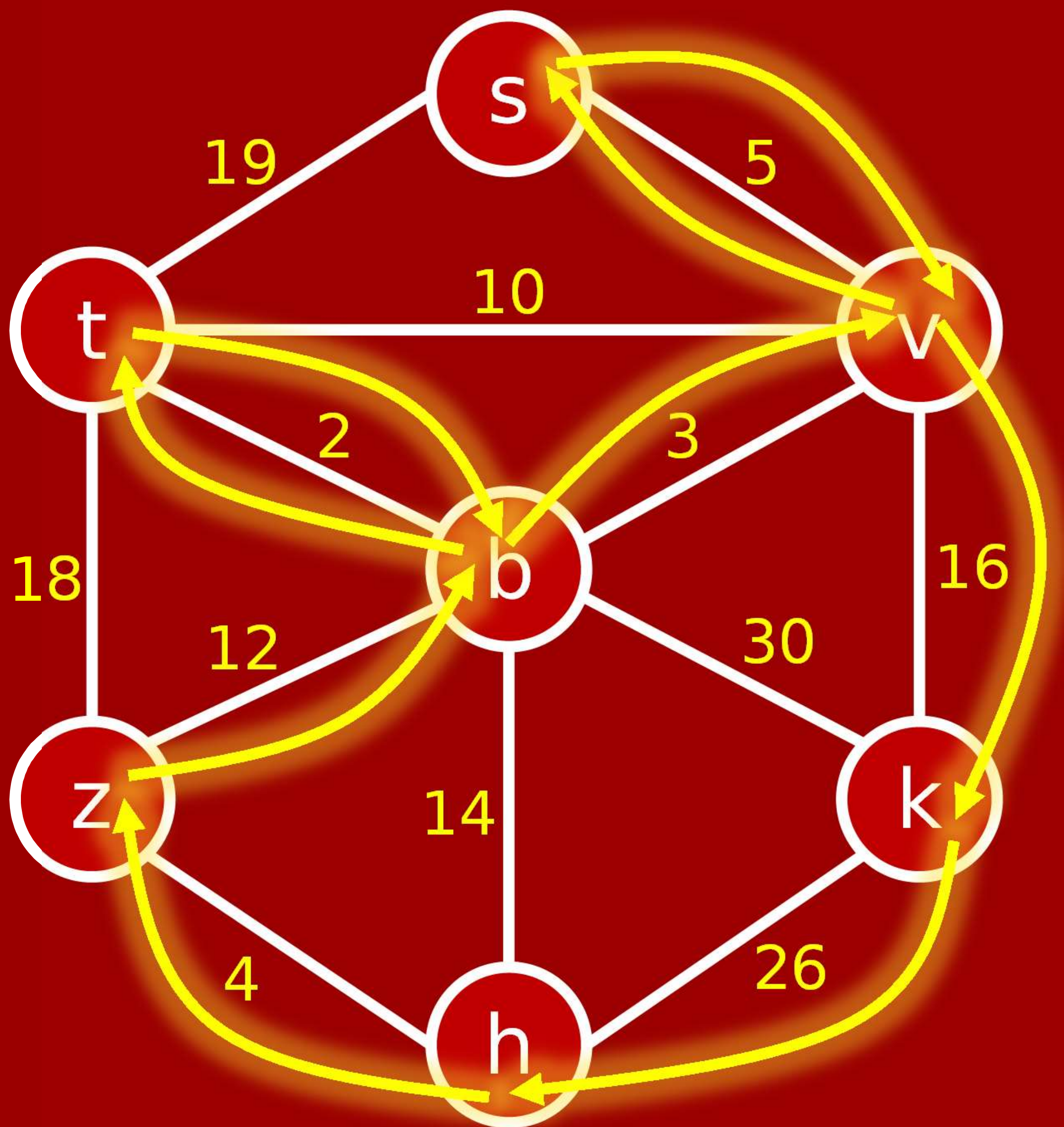
Many variants. Most common is “Metric-TSP”:

Input: A graph G=(V,E) with edge costs.

Output: A “tour”: i.e., a walk that visits each vertex **at least** once, and starts and ends at the same vertex.

Goal: Minimize total cost of tour.

TSP example



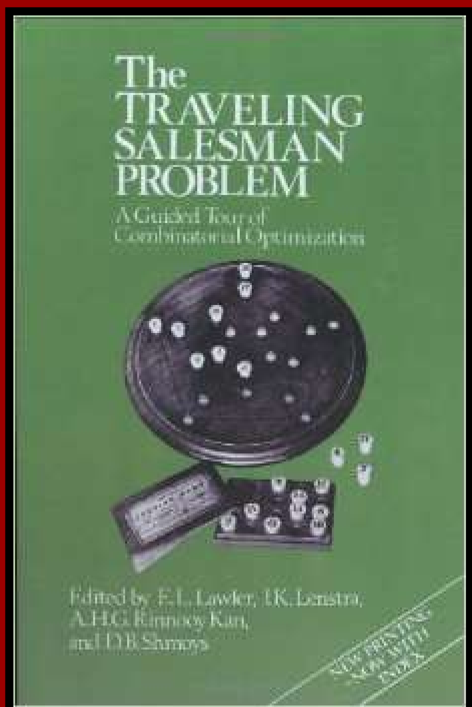
Cheapest tour:

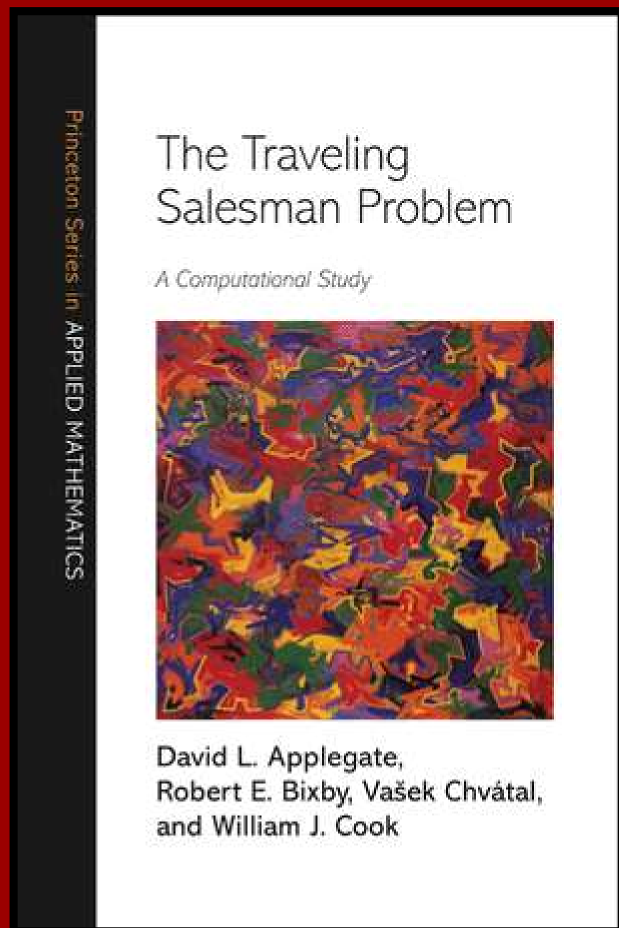
3
+ 5
+ 5
+ 16
+ 26
+ 4
+ 12
+ 2
+ 2
= 71

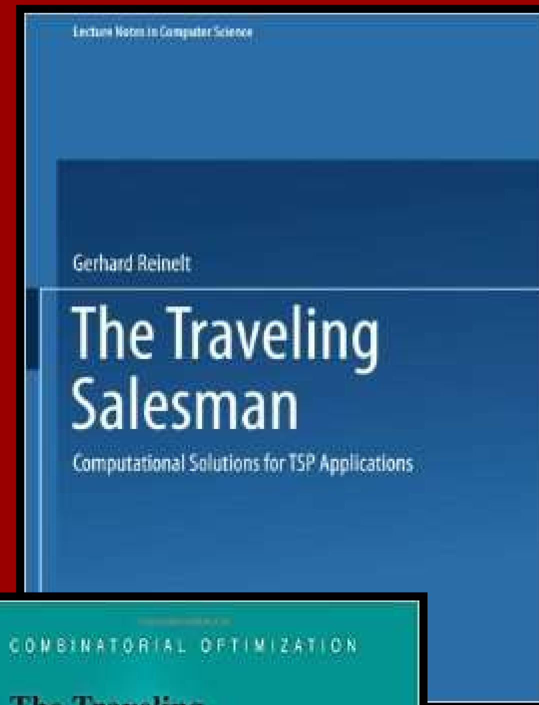
TSP is probably the most famous NP-complete problem.

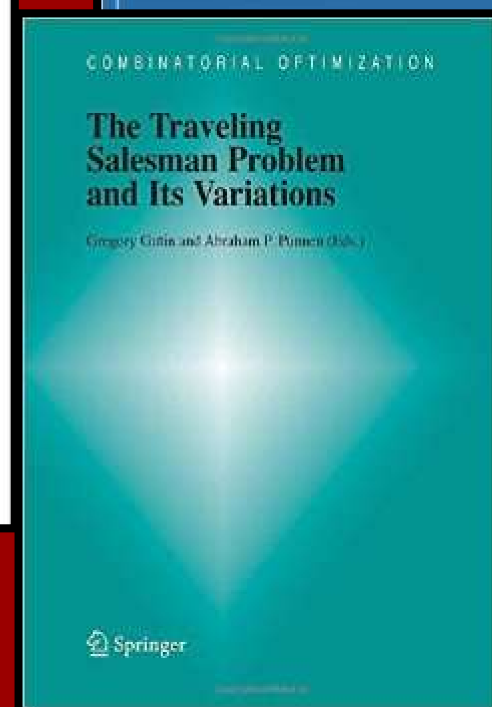
It has inspired many things...

Textbooks

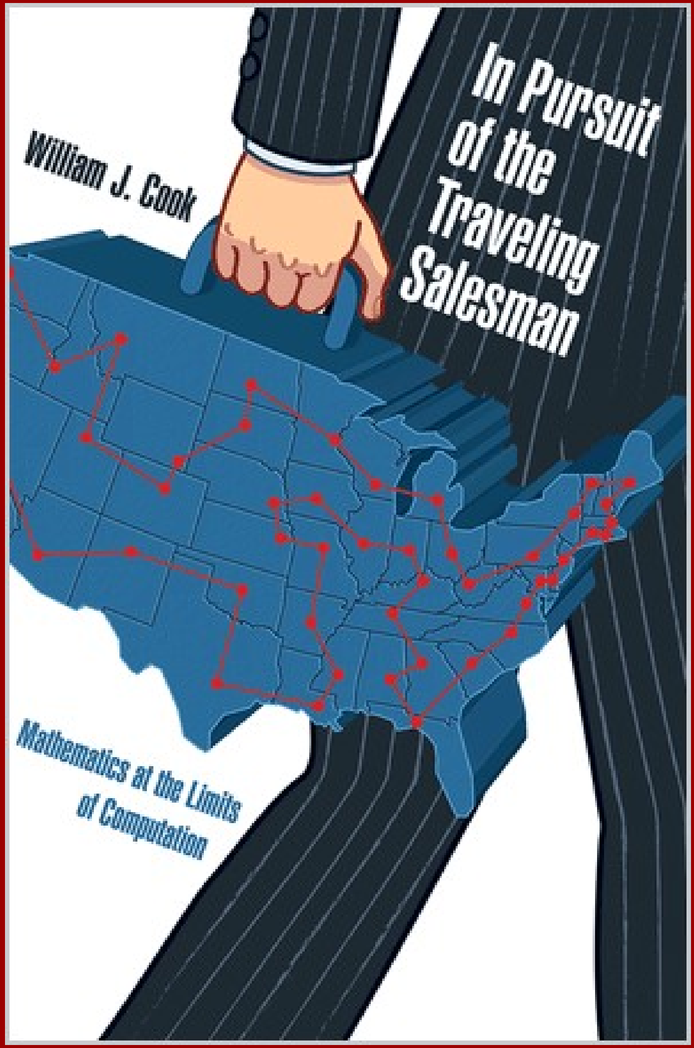

The TRAVELING SALESMAN PROBLEM
A Guided Tour of Combinatorial Optimization
Edited by E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, and D.B. Shmoys
NEW PRINTING NOW WITH INDEX


Princeton Series in APPLIED MATHEMATICS
The Traveling Salesman Problem
A Computational Study
David L. Applegate, Robert E. Bixby, Vašek Chvátal, and William J. Cook


Lecture Notes in Computer Science
Gerhard Reinelt
The Traveling Salesman
Computational Solutions for TSP Applications

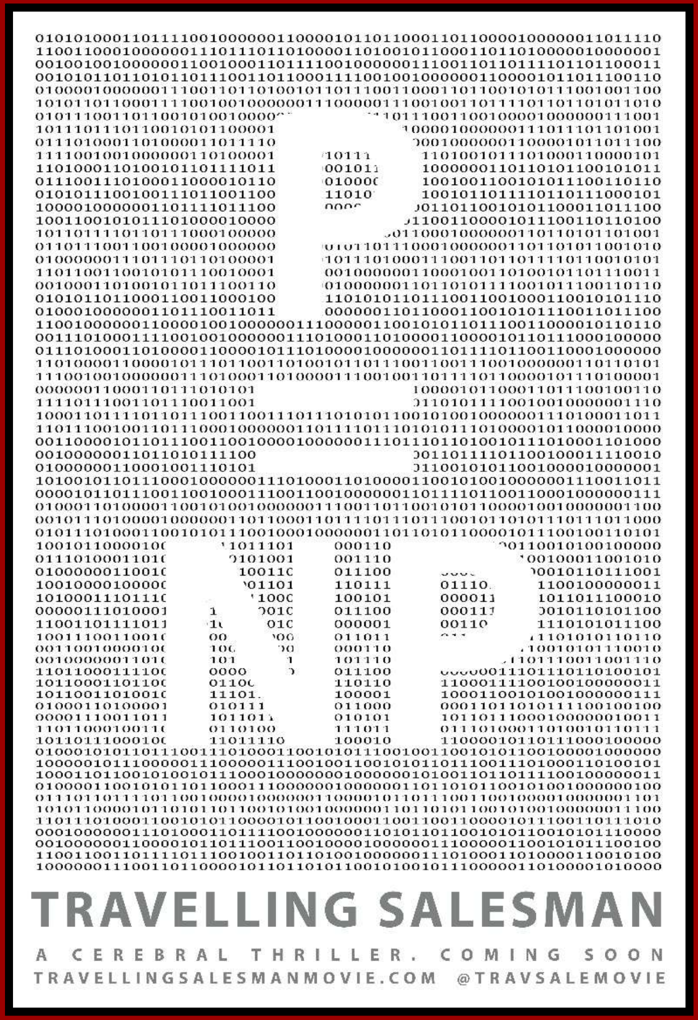

COMBINATORIAL OPTIMIZATION
The Traveling Salesman Problem and Its Variations
Gregory Gutin and Abraham P. Punnen (Eds.)
Springer

“Popular” books

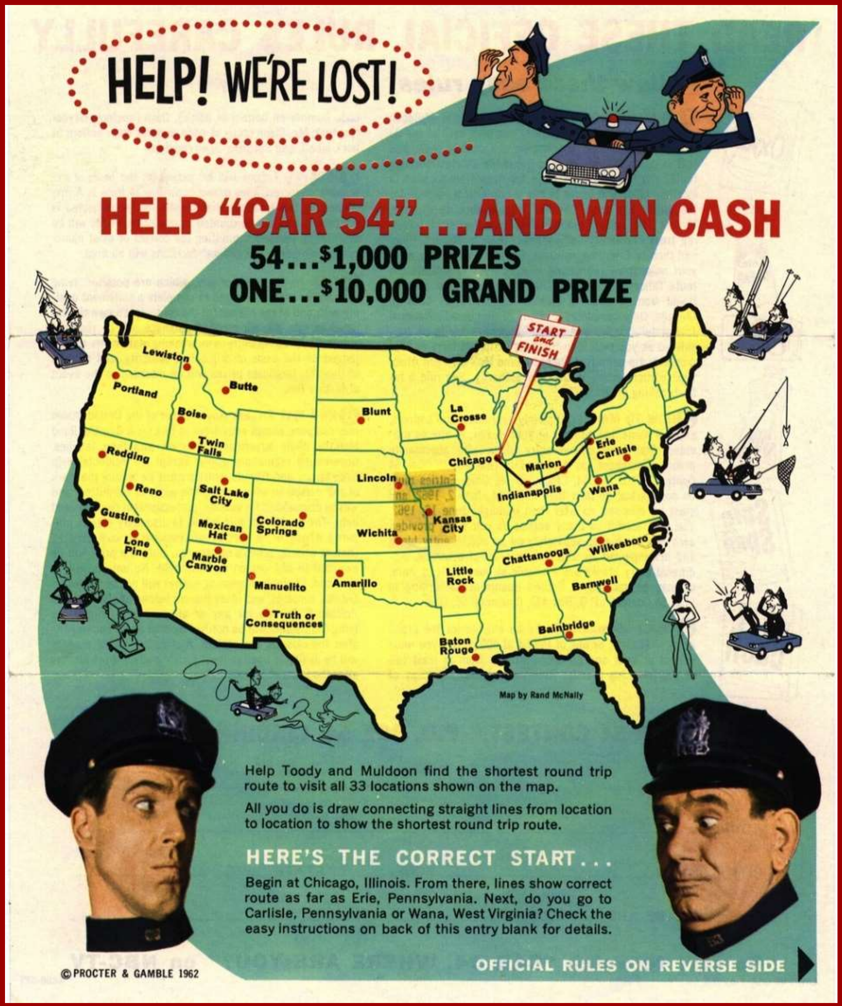


Museum exhibits

Movies


P
NP
TRAVELLING SALESMAN
A CEREBRAL THRILLER. COMING SOON
TRAVELLINGSALESMANMOVIE.COM @TRAVSALEMOVIE

’60s sitcom-themed household-goods conglomerate ad/contests



People genuinely want to solve large instances.

Applications in:

- Schoolbus routing
- Moving farm equipment
- Package delivery
- Space interferometer scheduling
- Circuit board drilling
- Genome sequencing
- ...

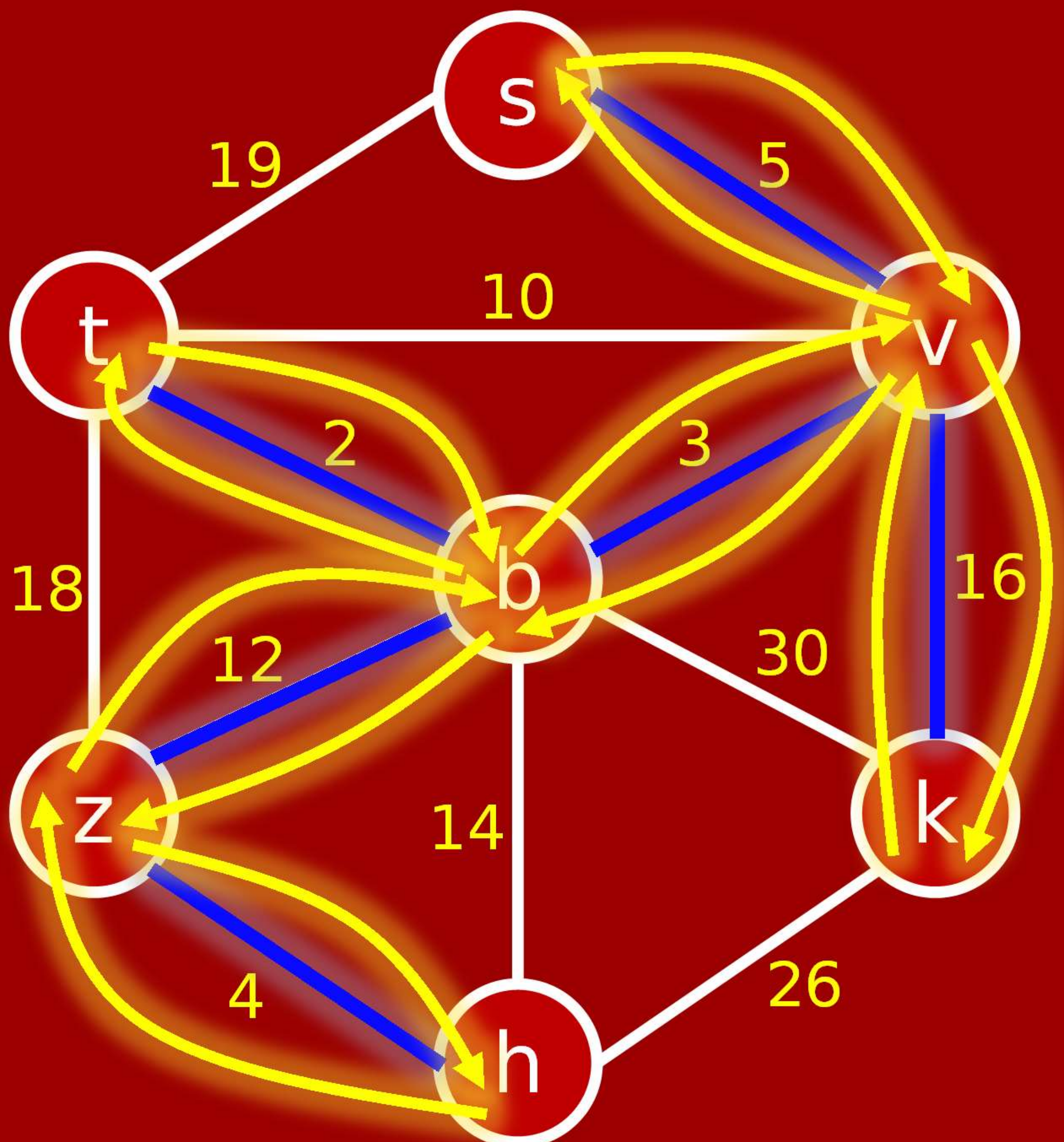
Basic Approximation Algorithm:

The MST Heuristic

Given G with edge costs…

1. Compute an **MST** T for G, rooted at any $s \in V$.
2. Visit the vertices via **DFS** from s.

MST Heuristic example



Step 1: MST

Step 2: DFS

Valid tour? ✓

Poly-time? ✓

Cost?

2 × MST Cost

(84 in this case)

MST Heuristic

Theorem: MST Heuristic is factor-2 approximation.

Key Claim: Optimal TSP cost ≥ MST Cost always.

This implies the Theorem, since

MST Heuristic Cost = **2** × MST Cost.

Proof of Claim:

Take all edges in optimal TSP solution.

They form a connected graph on all |V| vertices.

Take any spanning tree from within these edges.

Its cost is at least the MST Cost.

Therefore the original TSP tour's cost is ≥ MST Cost. ∎

Can we do better?

Nicos Christofides, Tepper faculty, 1976:

There is a polynomial-time, factor **1.5**-approximation algorithm for (Metric) TSP.

Proof is not **too** hard. Ingredients:

- MST Heuristic
- Eulerian Tours
- Cheapest Perfect Matching algorithm

Even better in a special case

In the important special case "Euclidean-TSP",
vertices are points in $\mathbb{R}^2$,
costs are just the straight-line distances.

This special case is still NP-hard.

Theorem (Arora, Mitchell, 1998):
For Euclidean-TSP, there is a polynomial-time factor 1.3 approximation algorithm.

Even better in a special case

In the important special case "Euclidean-TSP",

vertices are points in $\mathbb{R}^2$,

costs are just the straight-line distances.

This special case is still NP-hard.

Theorem (Arora, Mitchell, 1998):

For Euclidean-TSP, there is a polynomial-time factor 1.1 approximation algorithm.

Even better in a special case

In the important special case "Euclidean-TSP",
vertices are points in $\mathbb{R}^2$,
costs are just the straight-line distances.

This special case is still NP-hard.

Theorem (Arora, Mitchell, 1998):
For Euclidean-TSP, there is a
polynomial-time factor 1.01
approximation algorithm.


Even better in a special case

In the important special case “Euclidean-TSP”,

vertices are points in $\mathbb{R}^2$,

costs are just the straight-line distances.

This special case is still NP-hard.

Theorem (Arora, Mitchell, 1998):

For Euclidean-TSP, there is a

polynomial-time factor 1.001

approximation algorithm.

Even better in a special case

In the important special case “Euclidean-TSP”,

vertices are points in $\mathbb{R}^2$,

costs are just the straight-line distances.

This special case is still NP-hard.

Theorem (Arora, Mitchell, 1998):

For Euclidean-TSP, there is a

polynomial-time factor 1.0001

approximation algorithm.

Even better in a special case

In the important special case "Euclidean-TSP",

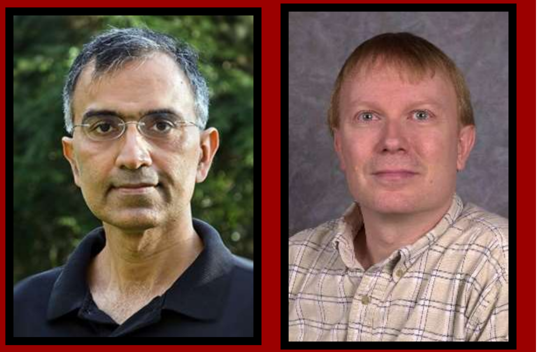
vertices are points in $\mathbb{R}^2$,

costs are just the straight-line distances.

This special case is still NP-hard.

Theorem (Arora, Mitchell, 1998):

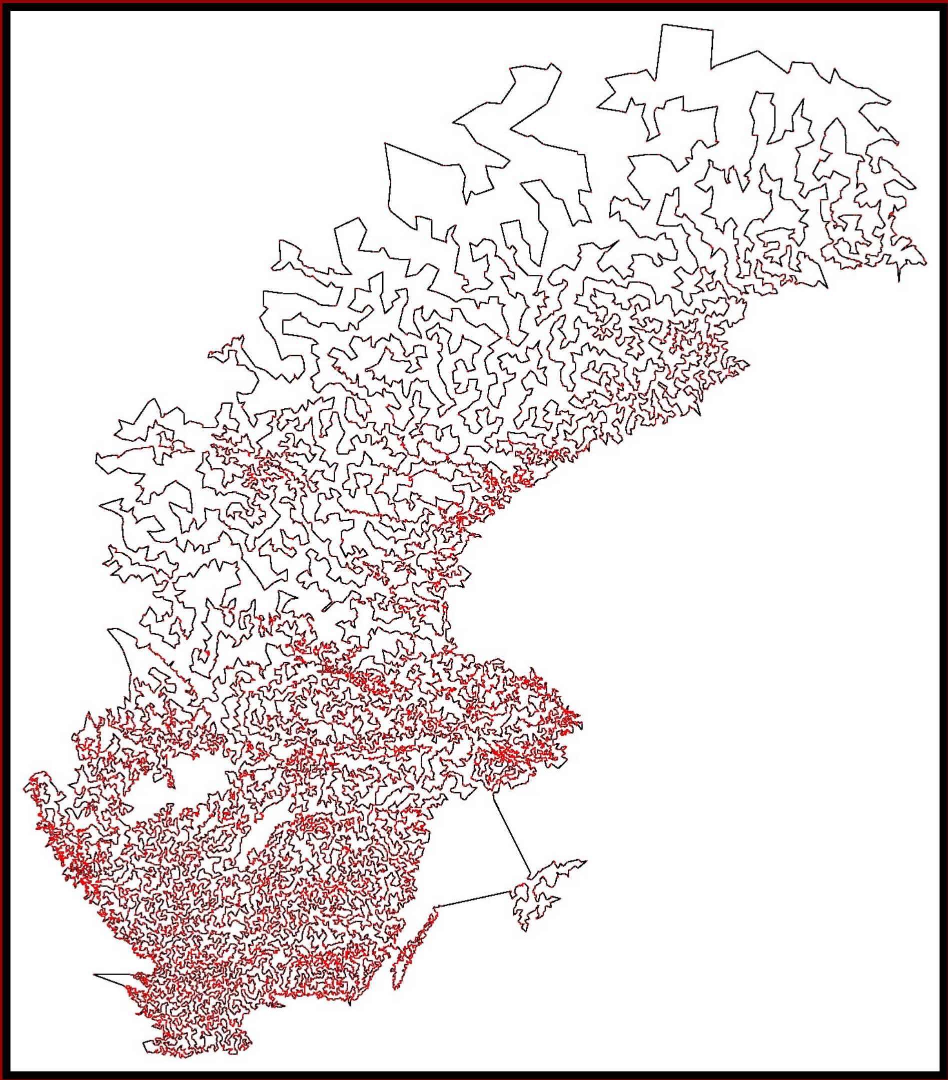
For Euclidean-TSP, there is a polynomial-time factor $1+\varepsilon$ approximation algorithm, for any $\varepsilon > 0$.



(Running time is like $O(n (\log n)^{1/\varepsilon})$.)

Euclidean-TSP:

NP-hard, but not **that** hard



n > 10,000
is feasible

Can we do better?

1. A 2-approximation algorithm for Vertex-Cover.

2. A 63% $(1-1/e)$ approximation algorithm for the "k-Coverage Problem".

3. A $(1+\varepsilon)$-approximation alg. for Euclidean-TSP.

Can we do better?

2. A 63% (1−1/e) approximation algorithm
   for the "k-Coverage Problem".

We cannot do better. (Unless P=NP.)

Theorem: For **any** $\beta > 1-1/e$, it is NP-hard
to factor β-approximate k-Coverage.

Proved in 1998 by Feige,
building on many prior works.
Proof length of reduction: ≈ 100 pages.

Can we do better?

1. A 2-approximation algorithm for Vertex-Cover.

We have no idea if we can do better.

Theorem (Dinur & Safra, 2002, Annals of Math.):
For any $\beta > 10\sqrt{5} - 21 \approx 1.36$,
it is NP-hard to β-approximate Vertex-Cover.

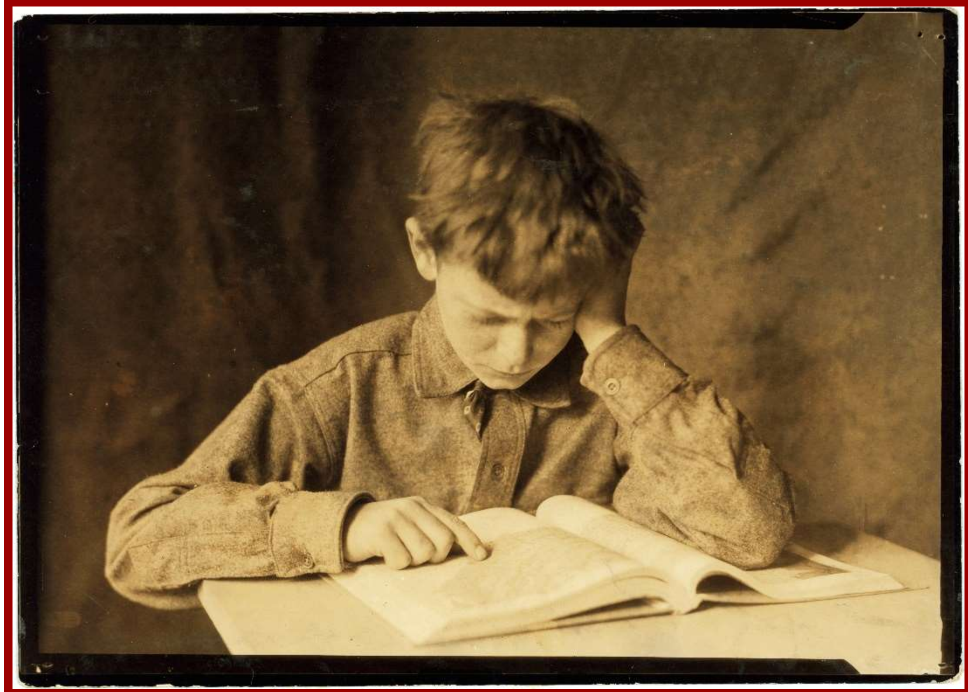

Approximating Vertex-Cover

Approximation Factor

Between 1.36 and 2: totally unknown.

Raging controversy.

Study Guide



Definitions:

Approximation algorithm.

The idea of “greedy” algorithms.

Algorithms and analysis:

Gavril algorithm for Vertex-Cover.

MST Heuristic for TSP.