15-251: Great Theoretical Ideas in Computer Science

Fall 2018, Lecture 22

Group Theory

Group Theory

Study of **symmetries** and **transformations** of mathematical objects.

Also, the study of abstract algebraic objects called '**groups**'.

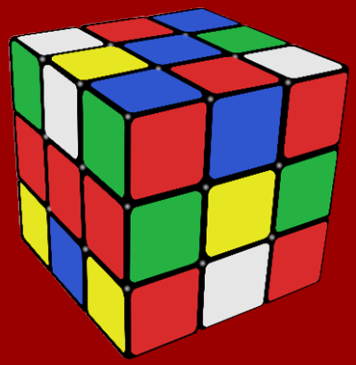
What is group theory good for?

In theoretical computer science:

- Cryptography: Fully homomorphic encryption, obfuscation…
- Quantum algorithms
- Mulmuley's approach to **P** vs. **NP**
- Checksums, error-correction schemes
- Minimizing space usage of algorithms
- Derandomization

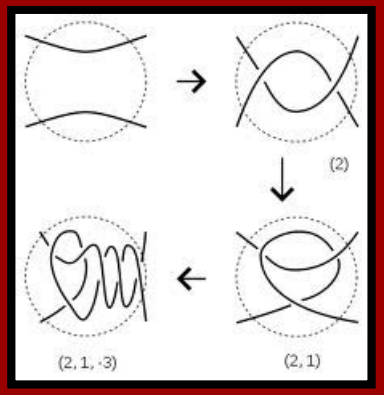
What is group theory good for?

In puzzles and games:



“15 Puzzle”

Rubik’s Cube



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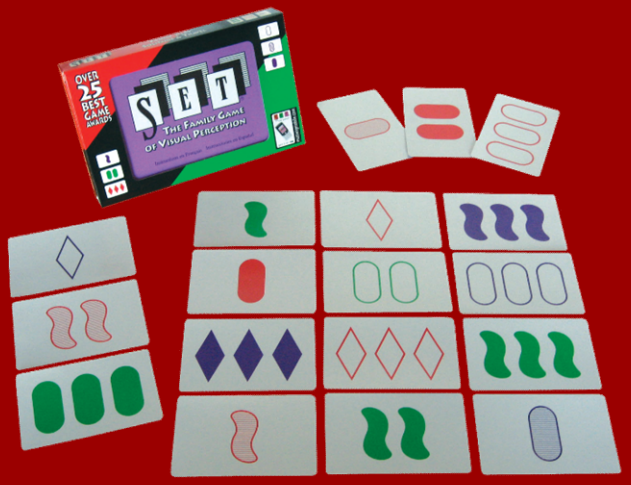


Tangles

What is group theory good for?

In math:

There's a quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What is group theory good for?

In math:

There's a cubic formula:

$$x_1 = -\frac{b}{3a} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}\right]} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}\right]}$$

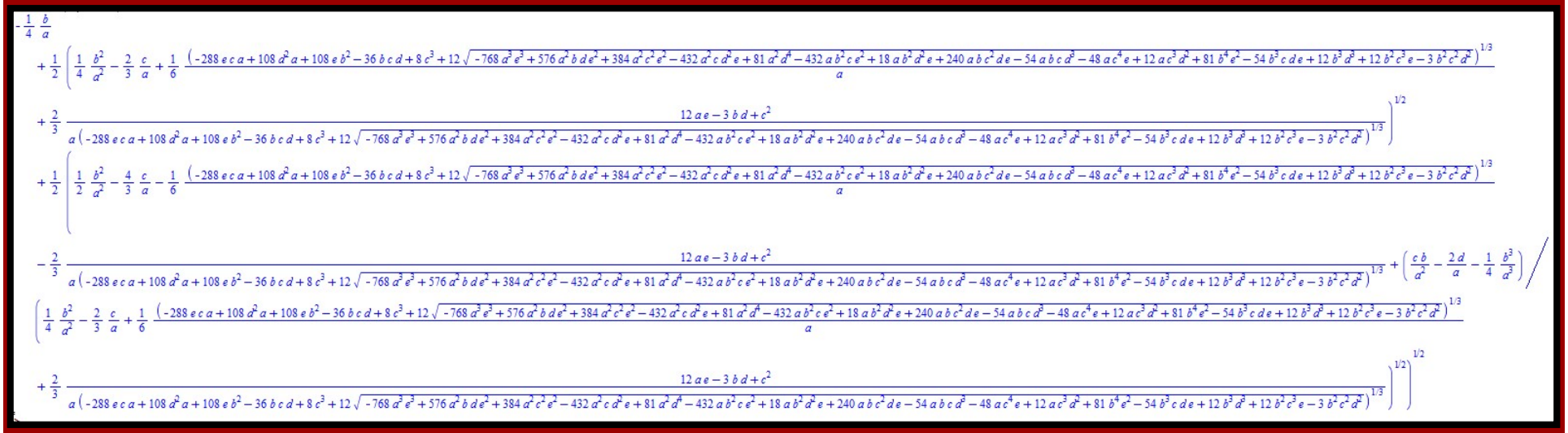
$$x_2 = -\frac{b}{3a} + \frac{1+i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}\right]} + \frac{1-i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}\right]}$$

$$x_3 = -\frac{b}{3a} + \frac{1-i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}\right]} + \frac{1+i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}\right]}$$

What is group theory good for?

In math:

There's a quartic formula:



$$
\begin{aligned}
&-\frac{1}{4}\frac{b}{a}\\
&+\frac{1}{2}\left(\frac{1}{4}\frac{b^2}{a^2}-\frac{2}{3}\frac{c}{a}+\frac{1}{6}\frac{\left(-288\,e\,c\,a+108\,d^2\,a+108\,e\,b^2-36\,b\,c\,d+8\,c^3+12\sqrt{-768\,a^3e^3+576\,a^2b\,d\,e^2+384\,a^2c^2e^2-432\,a^2c\,d^2e+81\,a^2d^4-432\,a\,b^2c\,e^2+18\,a\,b^2d^2e+240\,a\,b\,c^2d\,e-54\,a\,b\,c\,d^3-48\,a\,c^4e+12\,a\,c^3d^2+81\,b^4e^2-54\,b^3c\,d\,e+12\,b^3d^3+12\,b^2c^3e-3\,b^2c^2d^2}\right)^{1/3}}{a}\right.\\
&\left.+\frac{2}{3}\frac{12\,a\,e-3\,b\,d+c^2}{a\left(-288\,e\,c\,a+108\,d^2\,a+108\,e\,b^2-36\,b\,c\,d+8\,c^3+12\sqrt{-768\,a^3e^3+576\,a^2b\,d\,e^2+384\,a^2c^2e^2-432\,a^2c\,d^2e+81\,a^2d^4-432\,a\,b^2c\,e^2+18\,a\,b^2d^2e+240\,a\,b\,c^2d\,e-54\,a\,b\,c\,d^3-48\,a\,c^4e+12\,a\,c^3d^2+81\,b^4e^2-54\,b^3c\,d\,e+12\,b^3d^3+12\,b^2c^3e-3\,b^2c^2d^2}\right)^{1/3}}\right)^{1/2}\\
&+\frac{1}{2}\left(\frac{1}{2}\frac{b^2}{a^2}-\frac{4}{3}\frac{c}{a}-\frac{1}{6}\frac{\left(-288\,e\,c\,a+108\,d^2\,a+108\,e\,b^2-36\,b\,c\,d+8\,c^3+12\sqrt{-768\,a^3e^3+576\,a^2b\,d\,e^2+384\,a^2c^2e^2-432\,a^2c\,d^2e+81\,a^2d^4-432\,a\,b^2c\,e^2+18\,a\,b^2d^2e+240\,a\,b\,c^2d\,e-54\,a\,b\,c\,d^3-48\,a\,c^4e+12\,a\,c^3d^2+81\,b^4e^2-54\,b^3c\,d\,e+12\,b^3d^3+12\,b^2c^3e-3\,b^2c^2d^2}\right)^{1/3}}{a}\right.\\
&\left.-\frac{2}{3}\frac{12\,a\,e-3\,b\,d+c^2}{a\left(-288\,e\,c\,a+108\,d^2\,a+108\,e\,b^2-36\,b\,c\,d+8\,c^3+12\sqrt{-768\,a^3e^3+576\,a^2b\,d\,e^2+384\,a^2c^2e^2-432\,a^2c\,d^2e+81\,a^2d^4-432\,a\,b^2c\,e^2+18\,a\,b^2d^2e+240\,a\,b\,c^2d\,e-54\,a\,b\,c\,d^3-48\,a\,c^4e+12\,a\,c^3d^2+81\,b^4e^2-54\,b^3c\,d\,e+12\,b^3d^3+12\,b^2c^3e-3\,b^2c^2d^2}\right)^{1/3}}+\left(\frac{c\,b}{a^2}-\frac{2\,d}{a}-\frac{1}{4}\frac{b^3}{a^3}\right)\Big/\right.\\
&\left.\left(\frac{1}{4}\frac{b^2}{a^2}-\frac{2}{3}\frac{c}{a}+\frac{1}{6}\frac{\left(-288\,e\,c\,a+108\,d^2\,a+108\,e\,b^2-36\,b\,c\,d+8\,c^3+12\sqrt{-768\,a^3e^3+576\,a^2b\,d\,e^2+384\,a^2c^2e^2-432\,a^2c\,d^2e+81\,a^2d^4-432\,a\,b^2c\,e^2+18\,a\,b^2d^2e+240\,a\,b\,c^2d\,e-54\,a\,b\,c\,d^3-48\,a\,c^4e+12\,a\,c^3d^2+81\,b^4e^2-54\,b^3c\,d\,e+12\,b^3d^3+12\,b^2c^3e-3\,b^2c^2d^2}\right)^{1/3}}{a}\right.\right.\\
&\left.\left.+\frac{2}{3}\frac{12\,a\,e-3\,b\,d+c^2}{a\left(-288\,e\,c\,a+108\,d^2\,a+108\,e\,b^2-36\,b\,c\,d+8\,c^3+12\sqrt{-768\,a^3e^3+576\,a^2b\,d\,e^2+384\,a^2c^2e^2-432\,a^2c\,d^2e+81\,a^2d^4-432\,a\,b^2c\,e^2+18\,a\,b^2d^2e+240\,a\,b\,c^2d\,e-54\,a\,b\,c\,d^3-48\,a\,c^4e+12\,a\,c^3d^2+81\,b^4e^2-54\,b^3c\,d\,e+12\,b^3d^3+12\,b^2c^3e-3\,b^2c^2d^2}\right)^{1/3}}\right)^{1/2}\right)^{1/2}
\end{aligned}
$$

(That's just the first of four roots, actually.)

What is group theory good for?

In math:

There is **NO** quintic formula.

What is group theory good for?

In physics:

Predicting the existence of elementary particles **before** they are discovered.

What is group theory good for?

In entertainment:

Driving the plot of S06E10 of Futurama,

"The Prisoner of Benda"

So: What **is** group theory?

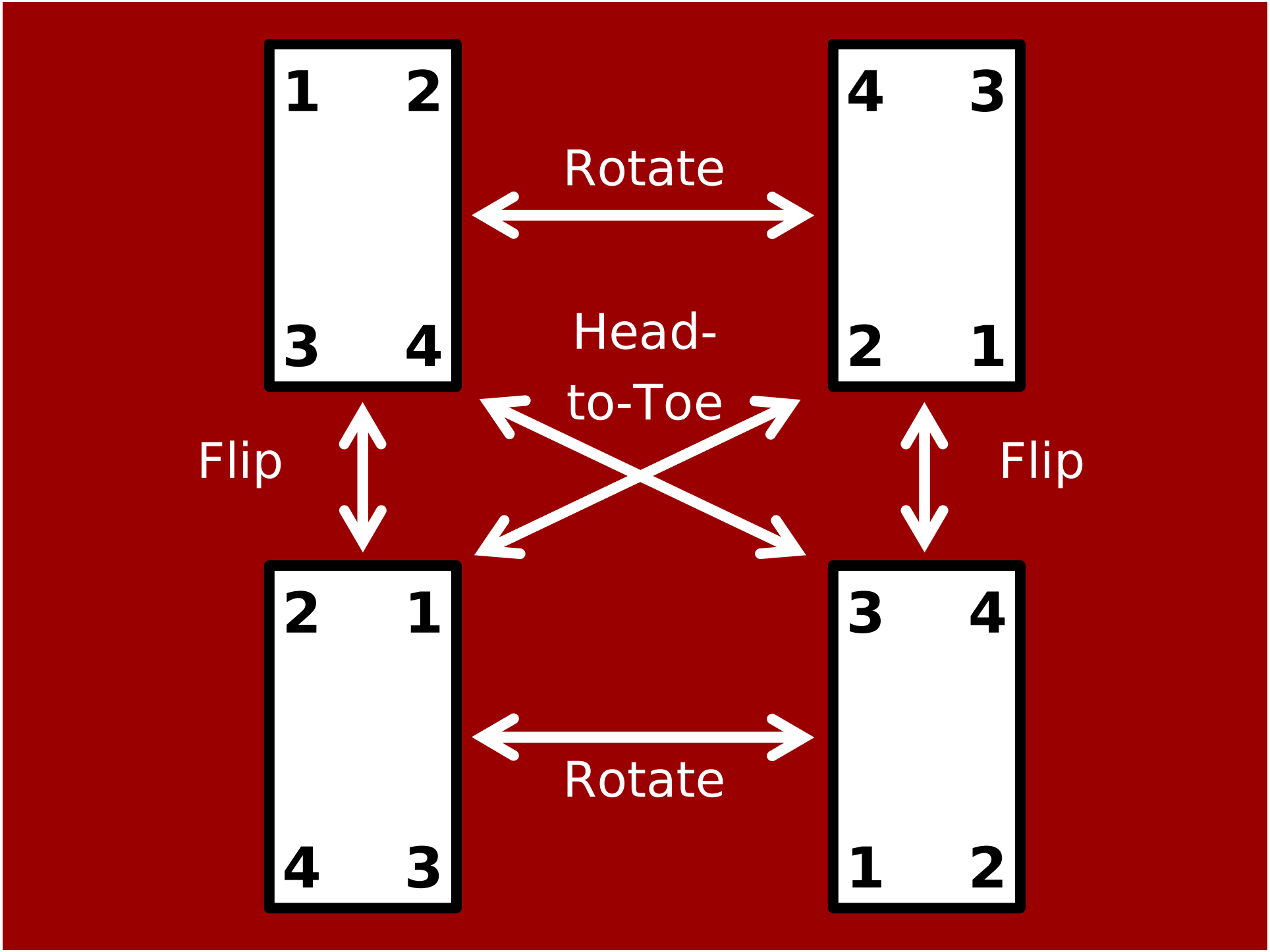
Rotate

Flip

Head-to-Toe flip

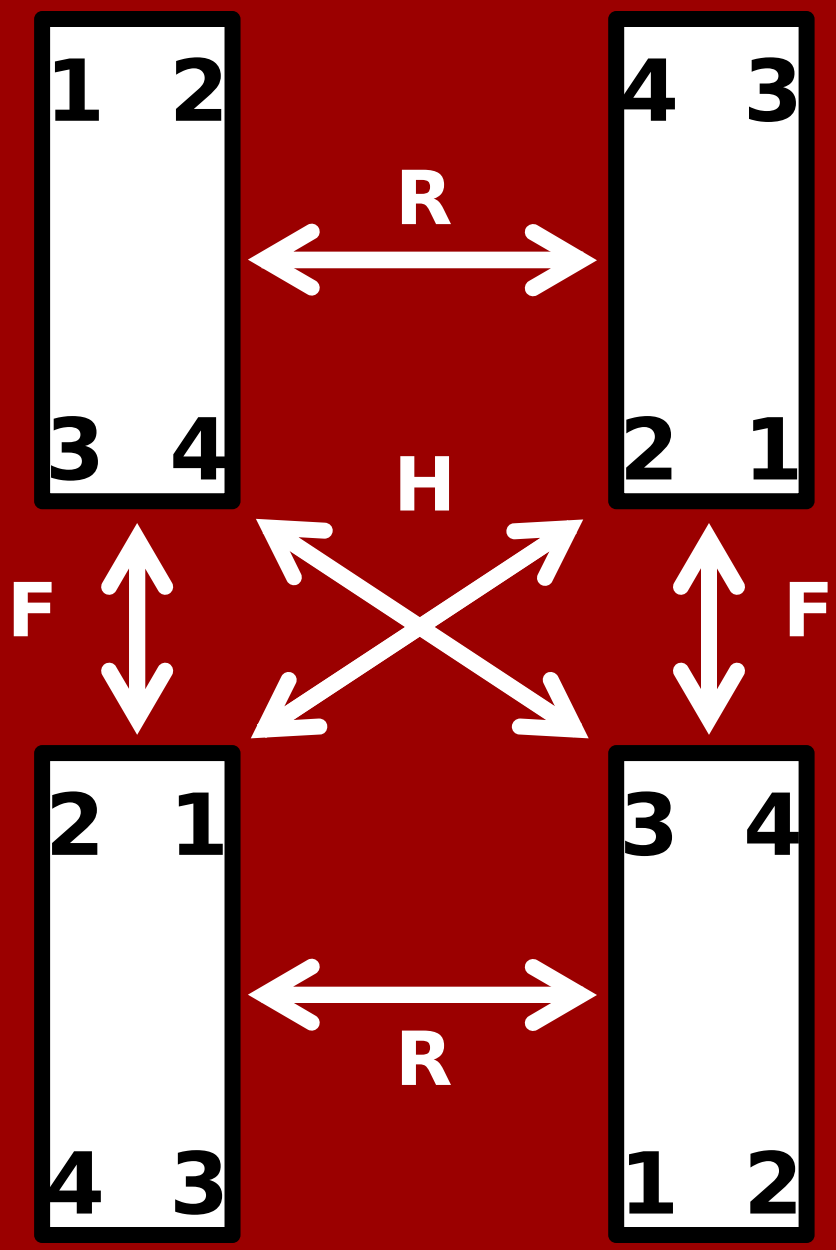
Q: How many positions can it be in?

A: Four.


1 2
3 4
Rotate
4 3
2 1
Head-
to-Toe
Flip
Flip
2 1
4 3
Rotate
3 4
1 2

Group theory is not so much
about **objects** (like mattresses).

It’s about the **transformations**
on objects and how they (inter)act.



F(**R**(mattress)) =

H(mattress)

H(**F**(mattress)) =

R(mattress)

R(**F**(**H**(mattress))) =

Id(mattress)

F∘**R**=**H**

H∘**F**=**R**

R∘**F**∘**H**=**Id**

R∘**Id**∘**H**∘**F**∘**H** = **H**

The kinds of questions asked:

What is **R**∘**Id**∘**H**∘**F**∘**H** ?

Do transformations **A** and **B** “commute”?
I.e., does **A**∘**B** = **B**∘**A** ?

What is the “order” of transformation **A**?
I.e., how many times do you have to
apply **A** before you get to **Id** ?

Definition of a **group of transformations**

Let X be a set.

Let G be a set of **bijections** $p : X \to X$.

We say G is a **group of transformations** if:

1. If **p** and **q** are in G then so is $\mathbf{p} \circ \mathbf{q}$.

   G is "**closed**" under composition.

2. The 'do-nothing' bijection **Id** is in G.

3. If **p** is in G then so is its inverse, $\mathbf{p}^{-1}$.

   G is "closed" under inverses.

Example: Rotations of a rectangular mattress

X = set of all physical points of the mattress

G = { **Id**, **R**otate, **F**lip, **H**ead-to-toe }

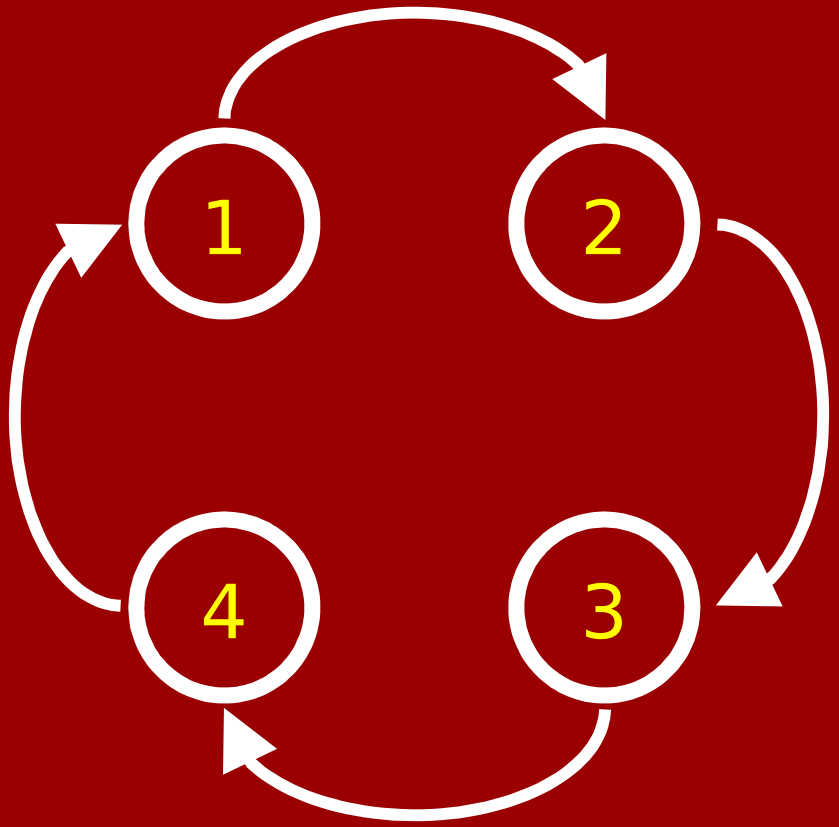
Check the 3 conditions:

1. If **p** and **q** are in G then so is **p**∘**q**. ✔

2. The ‘do-nothing’ bijection **Id** is in G. ✔

3. If **p** is in G then so is its inverse, $\mathbf{p}^{-1}$. ✔

Example: Symmetries of a directed cycle



X = labelings of the vertices by 1,2,3,4

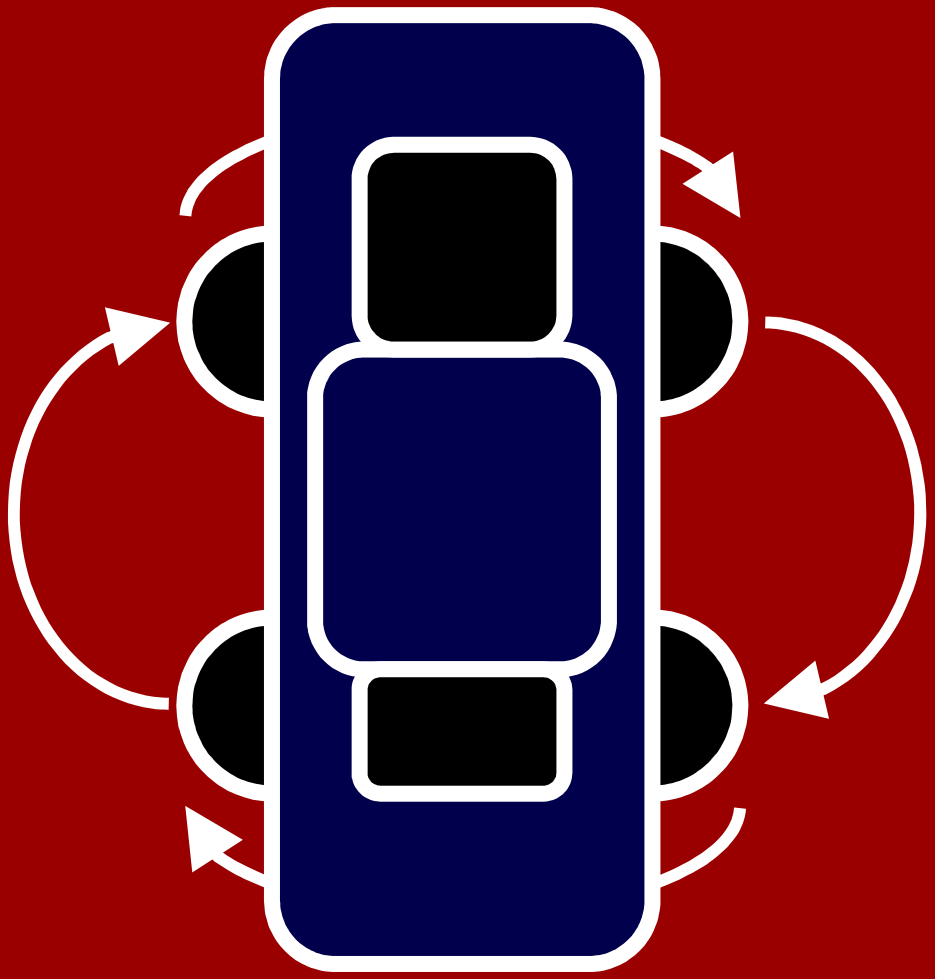
|X| = 24

G = permutations of the labels which don't change the graph

|G| = 4

G = { **Id**, $\mathbf{Rot_{90}}$, $\mathbf{Rot_{180}}$, $\mathbf{Rot_{270}}$ }

Example: Symmetries of a directed cycle



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Example: Symmetries of a directed cycle

X = labelings of directed 4-cycle

G = { **Id**, $\mathbf{Rot_{90}}$, $\mathbf{Rot_{180}}$, $\mathbf{Rot_{270}}$ }

Check the 3 conditions:

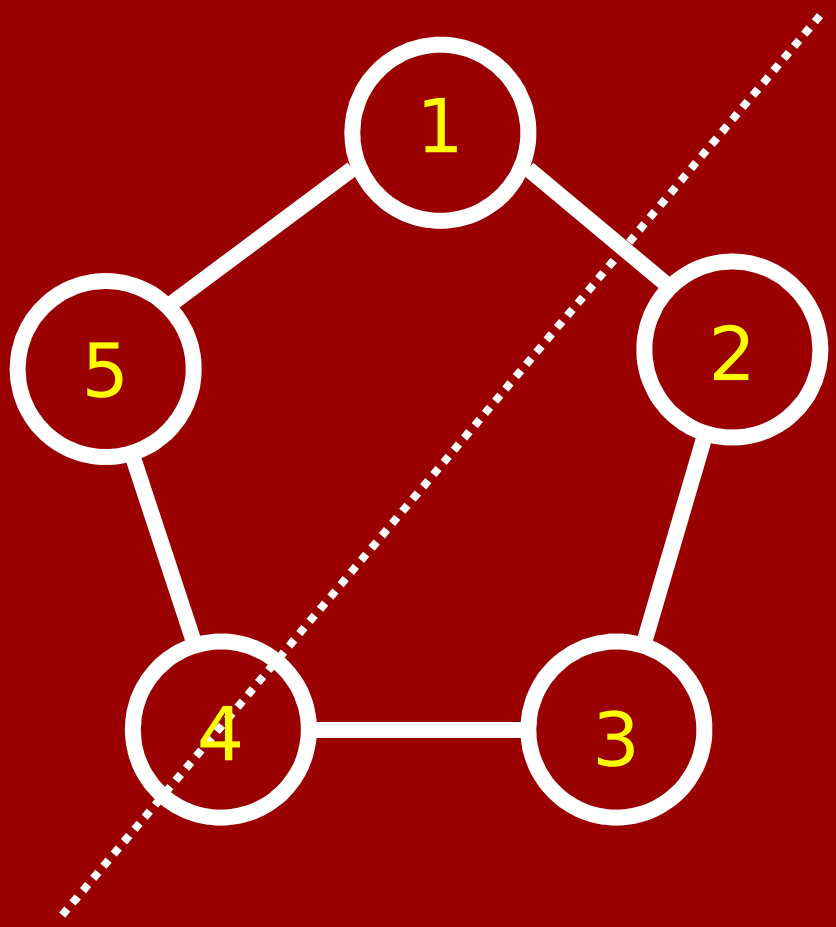
1. If **p** and **q** are in G then so is **p**∘**q**. ✔

2. The ‘do-nothing’ bijection **Id** is in G. ✔

3. If **p** is in G then so is its inverse, $\mathbf{p^{-1}}$. ✔

“Cyclic group of size 4”

Example: Symmetries of **undirected** n-cycle



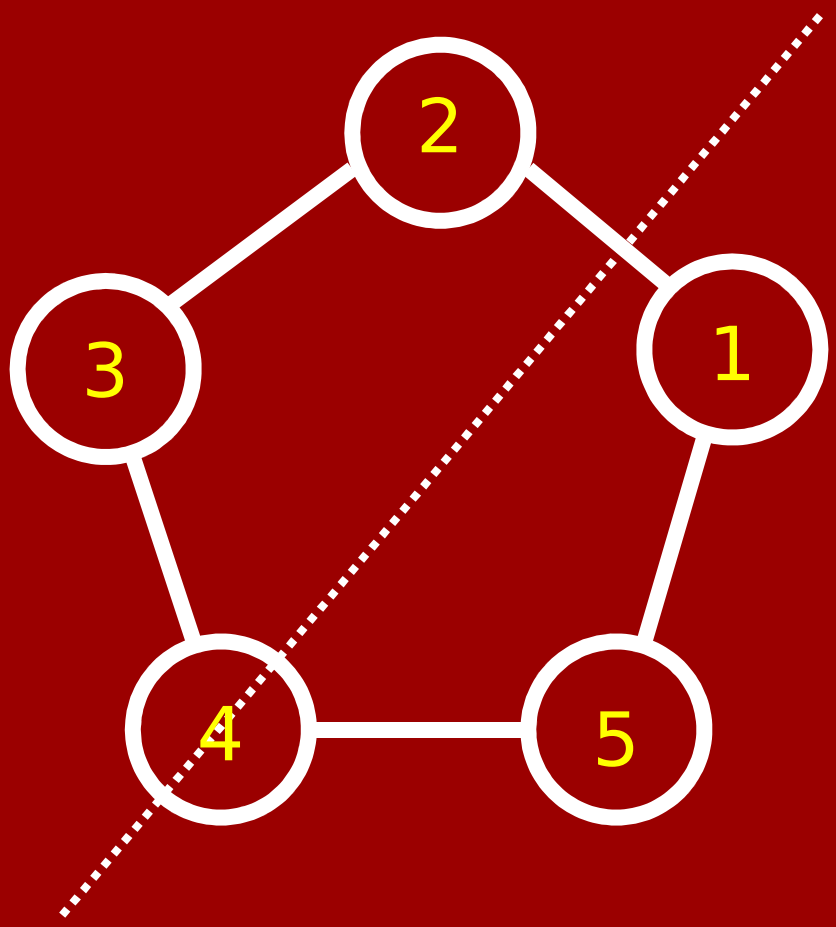
X = labelings of the vertices by 1,2, …, n

$|X| = n!$

G = permutations of the labels which don't change the graph

$|G| = 2n$

Example: Symmetries of **undirected** n-cycle



X = labelings of the vertices by 1,2, ..., n

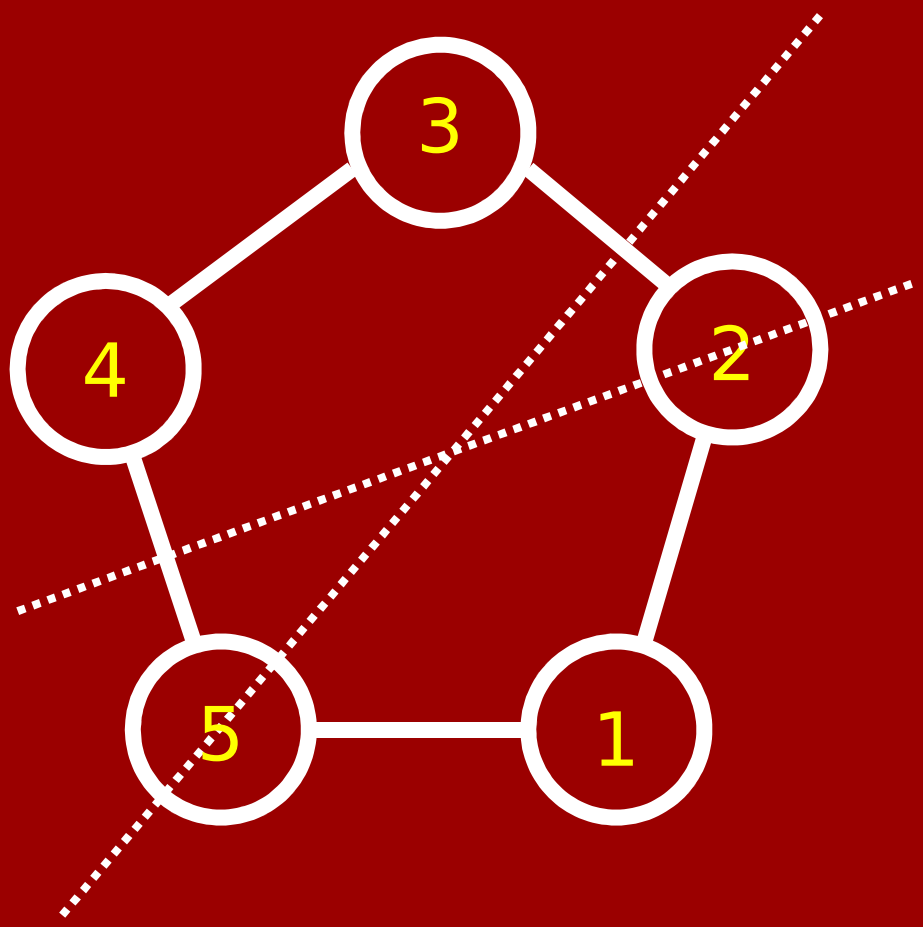
$|X| = n!$

G = permutations of the labels which don't change the graph

$|G| = 2n$

+ one clockwise twist

Example: Symmetries of **undirected** n-cycle



X = labelings of the vertices by 1,2, ..., n

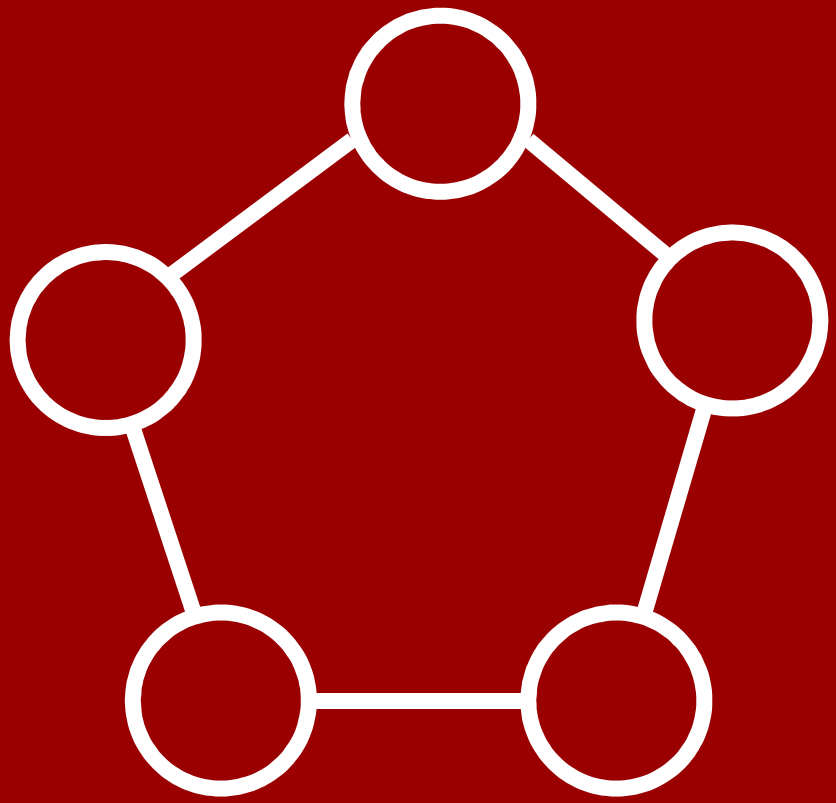
|X| = n!

G = permutations of the labels which don't change the graph

|G| = 2n

+ one clockwise twist =

Example: Symmetries of **undirected** n-cycle



X = labelings of the vertices by 1,2, …, n

$|X| = n!$

G = permutations of the labels which don't change the graph

$|G| = 2n$

G = { **Id**, n−1 'rotations', n 'reflections' }

"Dihedral group of size 2n"

Example: “All permutations”

$X = \{1, 2, \ldots, n\}$

G = all permutations of X

e.g., for n = 4, a typical element of G is:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

“Symmetric group, Sym(n)”

More groups of transformations

Motions of 3D space: translations + rotations
(preserve laws of Newtonian mechanics)

Translations of 2D space by an integer amount horizontally and an integer amount vertically

Rotations which preserve an old-school soccer ball.

$|G| = 60$

Group theory is not so much
about **objects** (like mattresses).

It’s about the **transformations**
on objects and how they (inter)act.

There is no mattress.

$$F \circ R = H$$

$$H \circ F = R$$

$$R \circ F \circ H = Id$$

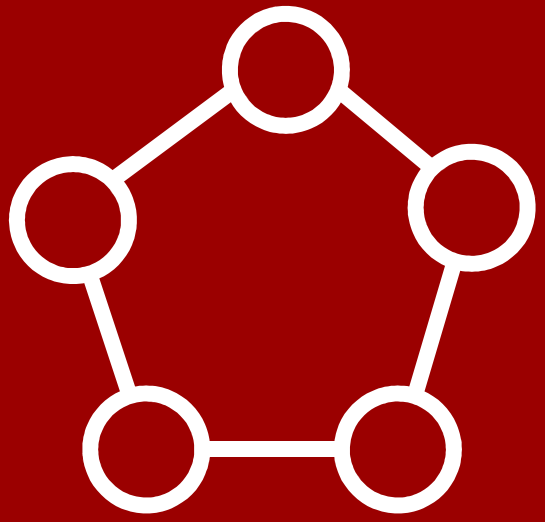
$$R \circ Id \circ H \circ F \circ H = H$$

The laws of mattress rotation

$$G = \{ Id, R, F, H \}$$

$Id \circ Id = Id$	$F \circ Id = F$
$Id \circ R = R$	$F \circ R = H$
$Id \circ F = F$	$F \circ F = Id$
$Id \circ H = H$	$F \circ H = R$
$R \circ Id = R$	$H \circ Id = H$
$R \circ R = Id$	$H \circ R = F$
$R \circ F = H$	$H \circ F = R$
$R \circ H = F$	$H \circ H = Id$

The laws of the dihedral group of size 10



$G =$

$\{ Id, r_1, r_2, r_3, r_4, f_1, f_2, f_3, f_4, f_5 \}$

$\circ$	Id	r_1	r_2	r_3	r_4	f_1	f_2	f_3	f_4	f_5
Id	Id	r_1	r_2	r_3	r_4	f_1	f_2	f_3	f_4	f_5
r_1	r_1	r_2	r_3	r_4	Id	f_4	f_5	f_1	f_2	f_3
r_2	r_2	r_3	r_4	Id	r_1	f_2	f_3	f_4	f_5	f_1
r_3	r_3	r_4	Id	r_1	r_2	f_5	f_1	f_2	f_3	f_4
r_4	r_4	Id	r_1	r_2	r_3	f_3	f_4	f_5	f_1	f_2
f_1	f_1	f_3	f_5	f_2	f_4	Id	r_3	r_1	r_4	r_2
f_2	f_2	f_4	f_1	f_3	f_5	r_2	Id	r_3	r_1	r_4
f_3	f_3	f_5	f_2	f_4	f_1	r_4	r_2	Id	r_3	r_1
f_4	f_4	f_1	f_3	f_5	f_2	r_1	r_4	r_2	Id	r_3
f_5	f_5	f_2	f_4	f_1	f_3	r_3	r_1	r_4	r_2	Id

Let's define an abstract **group**.

Let G be a set.

Let ∘ be a "**binary operation**" on G;

think of it as defining a "multiplication table".

E.g., if G = { a, b, c } then...

... is a binary operation.

This means that c∘a = b.

∘	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

Definition of an (abstract) **group**

We say G is a "**group** under operation $\circ$" if:

1. Operation $\circ$ is **associative**:

   i.e., $a \circ (b \circ c) = (a \circ b) \circ c \quad \forall\, a,b,c \in G$

2. There exists an element $e \in G$

   (called the "identity element") such that

   $$a \circ e = a, \;\; e \circ a = a \quad \forall\, a \in G$$

3. For each $a \in G$ there is an element $a^{-1} \in G$

   (called the "inverse of a") such that

   $$a \circ a^{-1} = e, \;\; a^{-1} \circ a = e$$

Examples of (abstract) groups

Any group of transformations is a group.

(Only need to check that composition of functions is associative.)

E.g., the ‘mattress group’ (AKA Klein 4-group)

∘	Id	R	F	H
Id	Id	R	F	H
R	R	Id	H	F
F	F	H	Id	R
H	H	F	R	Id

identity element is Id

$R^{-1} = R$

$F^{-1} = F$

$H^{-1} = H$

Examples of (abstract) groups

Any group of transformations is a group.

$\mathbb{Z}$ (the integers) is a group under operation $+$

Check:

0. $+$ really is a binary operation on $\mathbb{Z}$
1. $+$ is associative: $a+(b+c) = (a+b)+c$
2. "e" is 0: $a+0 = a$, $0+a = a$
3. "a^{-1}" is $-a$: $a+(-a) = 0$, $(-a)+a = 0$

Examples of (abstract) groups

Any group of transformations is a group.

$\mathbb{Z}$ (the integers) is a group under operation $+$

$\mathbb{R}$ (the reals) is a group under operation $+$

$\mathbb{R}^+$ (the positive reals) is a group under $\times$

$\mathbb{R} \setminus \{0\}$ is a group under $\times$

$\mathbb{Z}_n$ (the integers mod n) is a group under $+$

NONEXAMPLES of groups

G = {all odd integers}, operation $+$

$+$ is not a binary operation on G!

$\mathbb{Z}$, operation $-$

$-$ is not associative!

$\mathbb{Z} \setminus \{0\}$, operation $\times$

1 is the only possible identity element;

but then most elements don't have inverses!

Abstract algebra on groups

Theorem 1:

If $(G, \circ)$ is a group, identity element is unique.

Proof:

Suppose f and g are both identity elements.

Since g is identity, $f \circ g = f$.

Since f is identity, $f \circ g = g$.

Therefore $f = g$. ■

Abstract algebra on groups

Theorem 2:

In any group $(G, \circ)$, inverses are unique.

Proof:

Given $a \in G$, suppose b, c are both inverses of a.

Let e be **the** identity element.

By assumption, $a \circ b = e$ and $c \circ a = e$.

Now: $c = c \circ e = c \circ (a \circ b)$

$= (c \circ a) \circ b = e \circ b = b$ ■

Abstract algebra on groups

Theorem 3:

For all a in group G we have $(a^{-1})^{-1} = a$.

Theorem 4:

For $a,b \in G$ we have $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$.

Theorem 5:

In group $(G, \circ)$, it doesn't matter how you put parentheses in an expression like

$$a_1 \circ a_2 \circ a_3 \circ \cdots \circ a_k$$

("generalized associativity").

Notation

In abstract groups, it's tiring to always write $\circ$.
So we often write ab rather than $a \circ b$.

Sometimes write 1 instead of e for the identity.

For $n \in \mathbb{N}^+$, write a^n instead of $aaa\cdots a$ (n times).
Also a^{-n} instead of $a^{-1}a^{-1}\cdots a^{-1}$, and a^0 means 1.

Then $a^j a^k = a^{j+k}$ holds for all $j,k \in \mathbb{Z}$.

Algebra practice

Problem: In the mattress group {1, R, F, H}, simplify the element $R^2 (H^3 R^{-1})^{-1}$

One (slightly roundabout) solution:

$H^3 = H\ H^2 = H\ 1 = H$, so we reach $R^2 (H R^{-1})^{-1}$.

$(H R^{-1})^{-1} = (R^{-1})^{-1} H^{-1} = R H$, so we get $R^2 R H$.

But $R^2 = 1$, so we get $1 R H = R H = F$.

Moral: the usual rules of multiplication, **except**...

Commutativity?

In a group we do NOT NECESSARILY have

$$a \circ b = b \circ a$$

Actually, in the mattress group we do have this for all elements. E.g., RF = FR (=H).

Definition:

"a,b∈G commute" means ab = ba.

"G is commutative" means **all** pairs commute.

In group theory, “commutative groups” are usually called **abelian** groups.

Niels Henrik Abel (1802−1829)

Norwegian

Died at 26 of tuberculosis ☹

Age 22: proved there is no quintic formula.

Evariste Galois (1811−1832)

French

Died at 20 in a duel ☹

One of the main inventors of group theory.

Some abelian groups:

"Mattress group" ("Klein 4-group")

Symms of a **directed** cycle ("cyclic group")

$(\mathbb{R}, +)$

Some nonabelian groups:

Symms of an **undirected** cycle ("dihedral group")

Motions of 3D space

Sym(n) ("symmetric group on n elements")

Another fun group:

Quaternion group

$$Q_8 = \{ 1, -1, i, -i, j, -j, k, -k \}$$

Multiplication defined by:

1 is the identity

$(-1)^2 = 1, \quad (-1)a = a(-1) = -a$

$i^2 = j^2 = k^2 = -1$

$ij = k, \; ji = -k$

$jk = i, \; kj = -i$

$ki = j, \; ik = -j$

Exercise: valid def. of a (nonabelian) group.

Application to computer graphics

"Quaternions": expressions like

$$3.2 + 1.4i - .5j + 1.1k$$

which generalize complex numbers ($\mathbb{C}$).

Suppose we store points (x,y,z) in 3D space as quaternions $xi + yj + zk$.

To rotate point p an angle of θ around an axis defined by unit vector (u,v,w), let

$$q = \cos(\theta/2) + \sin(\theta/2)u\,i + \sin(\theta/2)v\,j + \sin(\theta/2)w\,k.$$

Then the rotated point is qpq^{-1}.

Isomorphism

Here's a group: V = { 00, 01, 10, 11 }

⊕ (bitwise XOR) is the operation

There's something familiar about this group…

V

⊕	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

same

after renaming:

00↔Id

01↔R

10↔F

11↔H

The mattress

∘	Id	R	F	H
Id	Id	R	F	H
R	R	Id	H	F
F	F	H	Id	R
H	H	F	R	Id

Isomorphism

Groups $(G,\circ)$ and $(H,\bullet)$ are "**isomorphic**" if there is a way to rename elements so that they have the same multiplication table.

Fundamentally,
they're the "same" abstract group.

Isomorphism and orders

Obviously, if G and H are isomorphic
we must have $|G| = |H|$.

$|G|$ is called the **order** of G.

E.g.: Let C_4 be the group of transformations
preserving the directed 4-cycle.

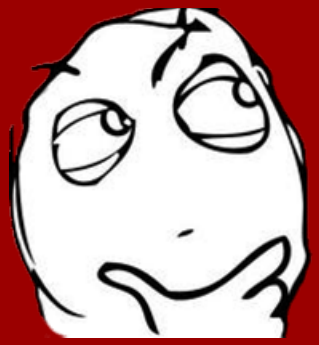
$$|C_4| = 4$$

Q: Is C_4 isomorphic to the mattress group V ?

Isomorphism and orders

Q: Is C_4 isomorphic to the mattress group V ?

A: No!



$a^2 = 1$ for every element $a \in V$.

But in C_4, $\mathrm{Rot}_{90}^2 = \mathrm{Rot}_{270}^2 \neq \mathrm{Rot}_{180}^2 = \mathrm{Id}^2$

Motivates studying powers of elements.

Order of a group element

Let G be a finite group. Let $a \in G$.

Look at $1, a, a^2, a^3, \ldots$ till you get some repeat.

Say $a^k = a^j$ for some $k > j$.

Multiply this equation by a^{-j} to get $a^{k-j} = 1$.

So the first repeat is always 1.

Definition: The order of a, denoted $|a|$, is the smallest $m \geq 1$ such that $a^m = 1$.

Note that $a, a^2, a^3, \ldots, a^{m-1}, a^m=1$ all distinct.

Examples:

In mattress group (order 4),

$|\mathrm{Id}| = 1, \quad |\mathrm{R}| = |\mathrm{F}| = |\mathrm{H}| = 2.$

In directed-4-cycle group (order 4),

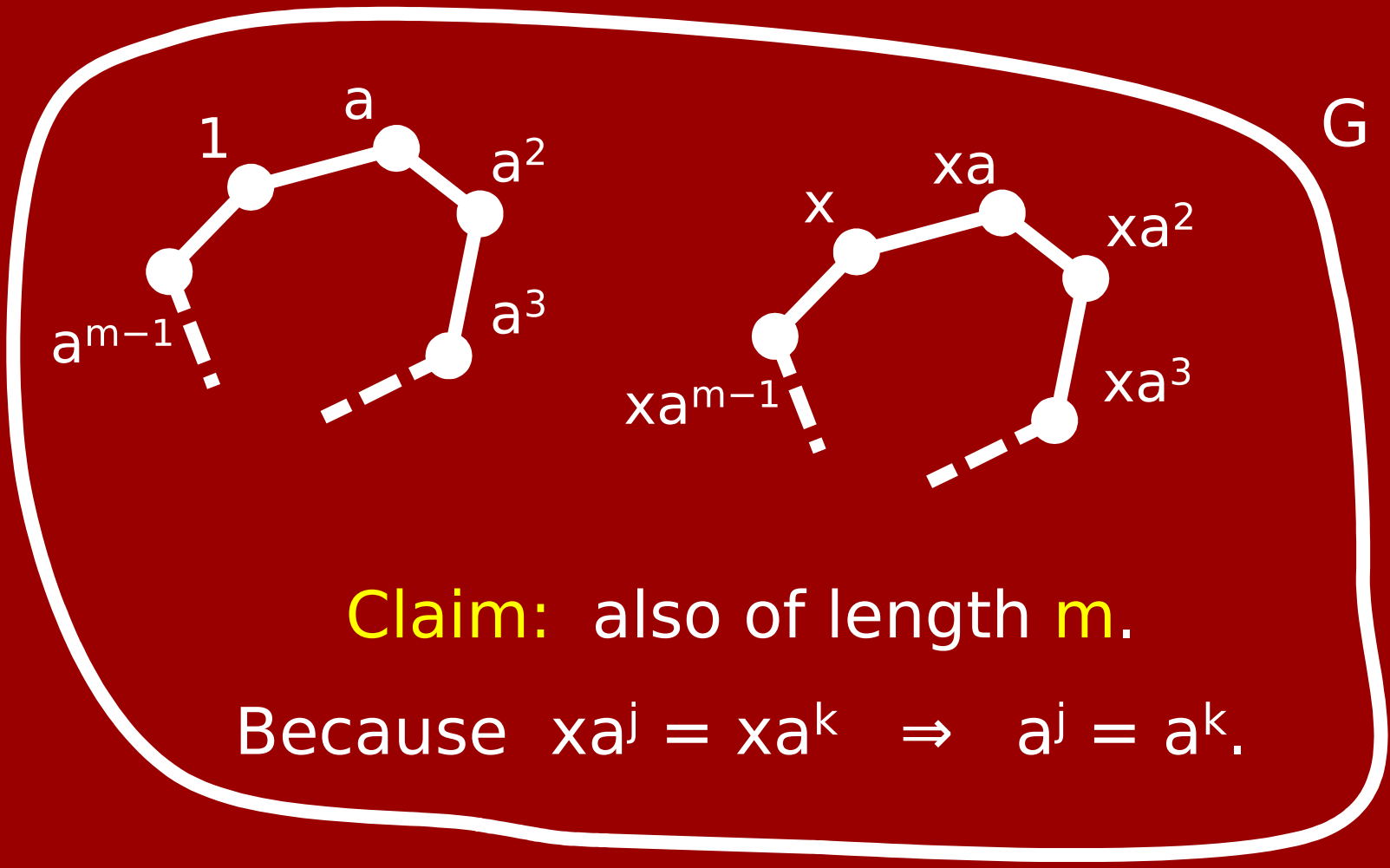
$|\mathrm{Id}| = 1, \ |\mathrm{Rot}_{180}| = 2, \ |\mathrm{Rot}_{90}| = |\mathrm{Rot}_{270}| = 4.$

In dihedral group of order 10
(symmetries of undirected 5-cycle)

$|\mathrm{Id}| = 1, \ |\text{any rotation}| = 5, \ |\text{any reflection}| = 2.$

Order Theorem:

|a| always divides evenly into |G|.

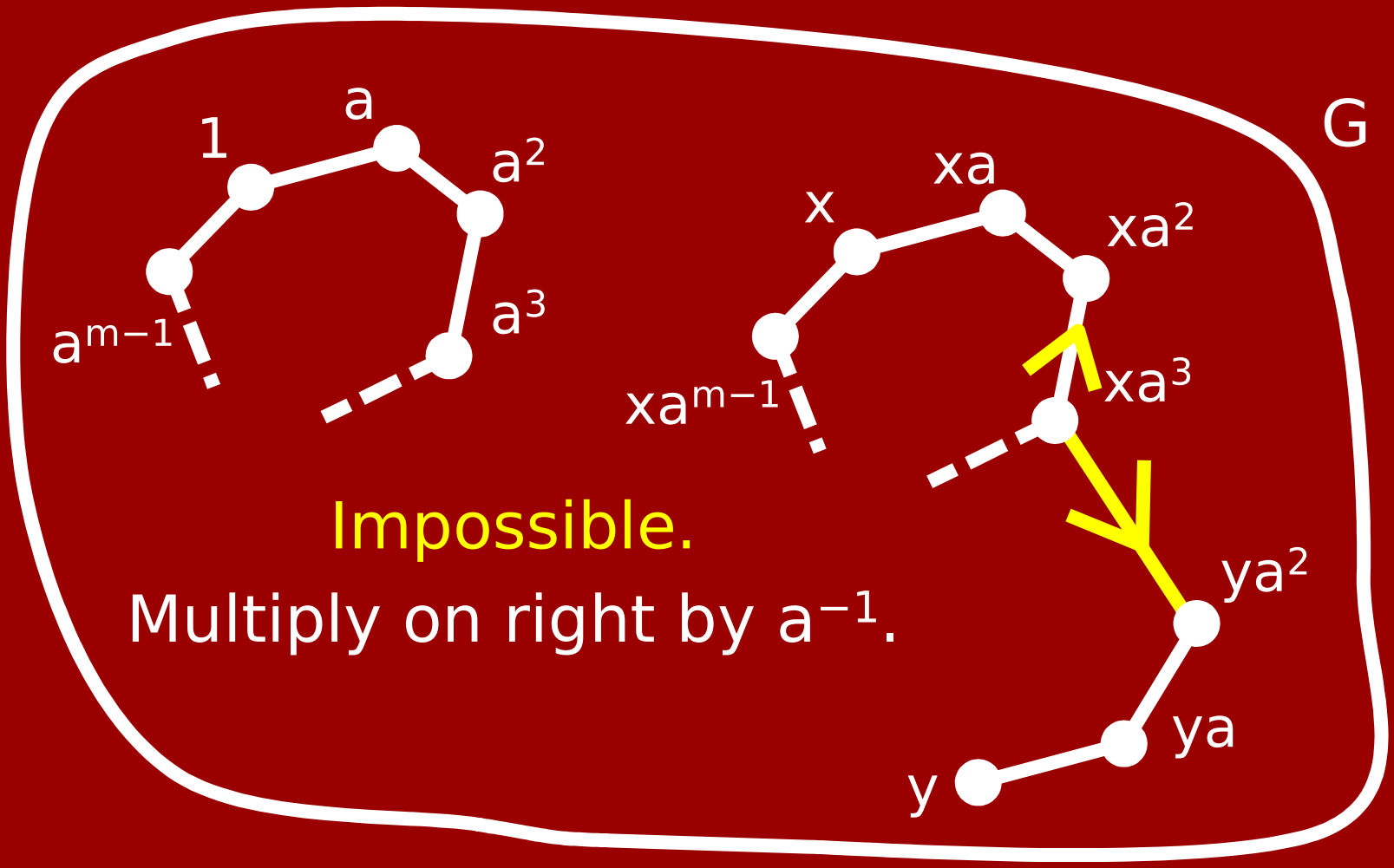


Claim: also of length m.

Because $xa^j = xa^k \Rightarrow a^j = a^k$.

Order Theorem:

|a| always divides evenly into |G|.

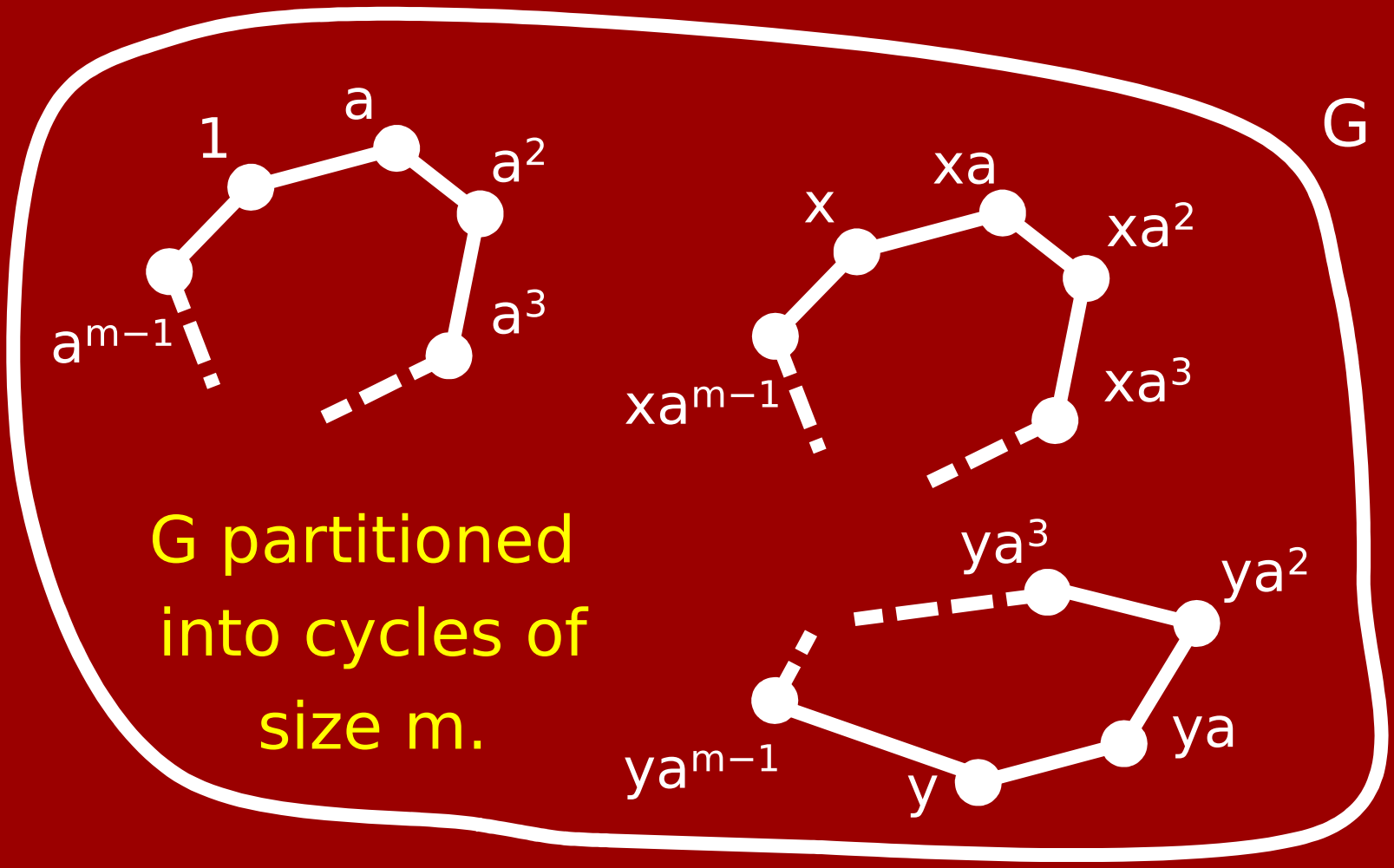


Impossible.

Multiply on right by a^{-1}.

Order Theorem:

|a| always divides evenly into |G|.



Order Theorem:

|a| always divides evenly into |G|.

Corollary: If $|G| = n$, then $a^n = 1$ for all $a \in G$.

Proof: Let $|a| = m$. Write $n = mk$.

Then $a^n = (a^m)^k = 1^k = 1$.

A Group Theory Application

Check Digits

Say you have important strings of digits:

credit card numbers
EFT routing numbers
UPC numbers
money serial numbers
book ISBNs

People screw up when transcribing them.

Check Digits

Most common human screwups:

single digit wrong (e.g., 6→8):	60-90%
omitting/adding digit:	10-20%
transposition (e.g., 35→53):	10-20%
other screwups:	≤ 5%

Instead of making them n random digits,
make them n random digits + a ‘check digit’.

Check Digits

Example: Book ISBNs before 2007.

Desired id#: 1 3 6 0 4 2 9 9 4 **7**

10 9 8 7 6 5 4 3 2

dot-prod mod 11: $1\times10+3\times9+6\times8+0\times7+4\times6+2\times5+9\times4+9\times3+8\times2 = 4$

check digit: top it off to get 0 mod 11

Pros: You can detect any single-digit or transposition error.

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check digit: top it off to get 0 mod 11

Cons: Um, check digit should be 10? “Write X”!

Doesn’t scale if you want longer id#’s.

Verhoeff Check Digit Method

Encode digits by elements of dihedral group of order 10.

Let σ be the permutation $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 5 & 7 & 6 & 2 & 8 & 3 & 0 & 9 & 4 \end{pmatrix}$

Given a desired id# $a_0\, a_1\, a_2 \cdots a_{n-1}$,
choose unique check digit a_n satisfying group equation

$$\mathrm{enc}(\sigma^0(a_0)) \circ \mathrm{enc}(\sigma^1(a_1)) \circ \mathrm{enc}(\sigma^2(a_2)) \circ \cdots \circ \mathrm{enc}(\sigma^n(a_n)) = e$$

Pros: Detects single-digit & transposition errors.
Uses just digits 0, 1, 2, ..., 9.
Scales to any length of id#.

Verhoeff Check Digit Method

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Cons: Can't really be done by a human.

Verhoeff Check Digit Method

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Is this really a con?

What human manually checksums credit cards?

We have computers, you know.

Verhoeff Check Digit Method

Nevertheless, it's like the Dvorak keyboard of check digit methods. ☹

German federal bank started using it for Deutsche Marks (with some letters?) in 1990.

Then they went and got the euro (which uses a different scheme).

The 10 is a good denomination
for mathematicians.

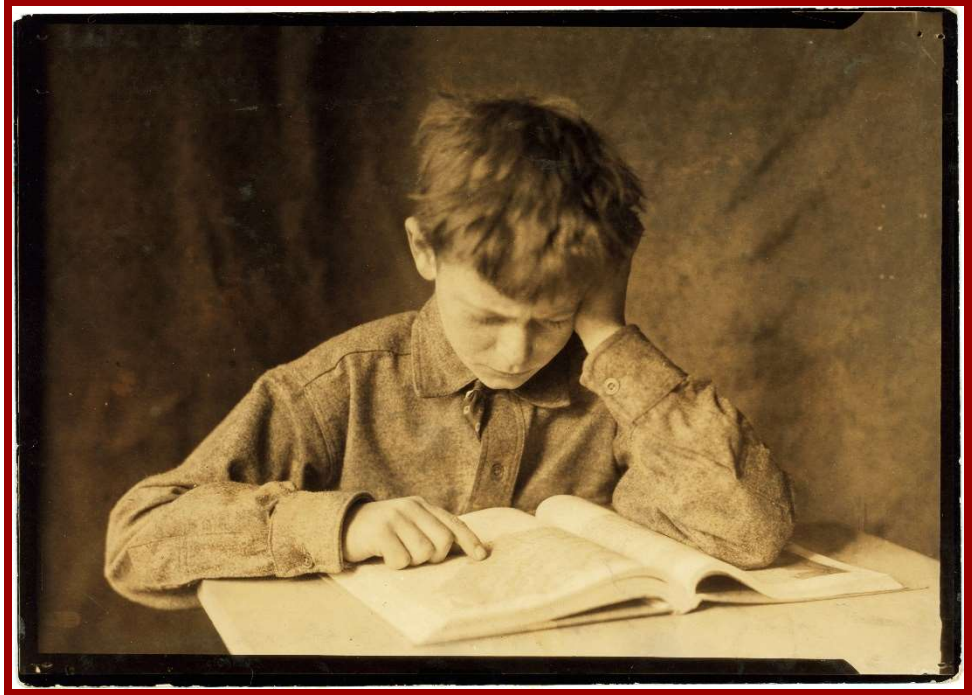
Leonhard Euler on the back of the old
10 Swiss franc note.

The 10 is a good denomination
for mathematicians.

I do not know
how this works.

Cahit Arf and an equation in the group $\mathbb{Z}_2$
starring on the back of a Turkish 10 lira.

Study Guide



Definitions:

- Groups
- Commutative/abelian
- Isomorphism
- Order

Groups:

- Klein 4-, cyclic, dihedral, symmetric, quaternions

Doing:

- Checking for groupness
- Computations in groups

Theorem/proof:

- Order Theorem