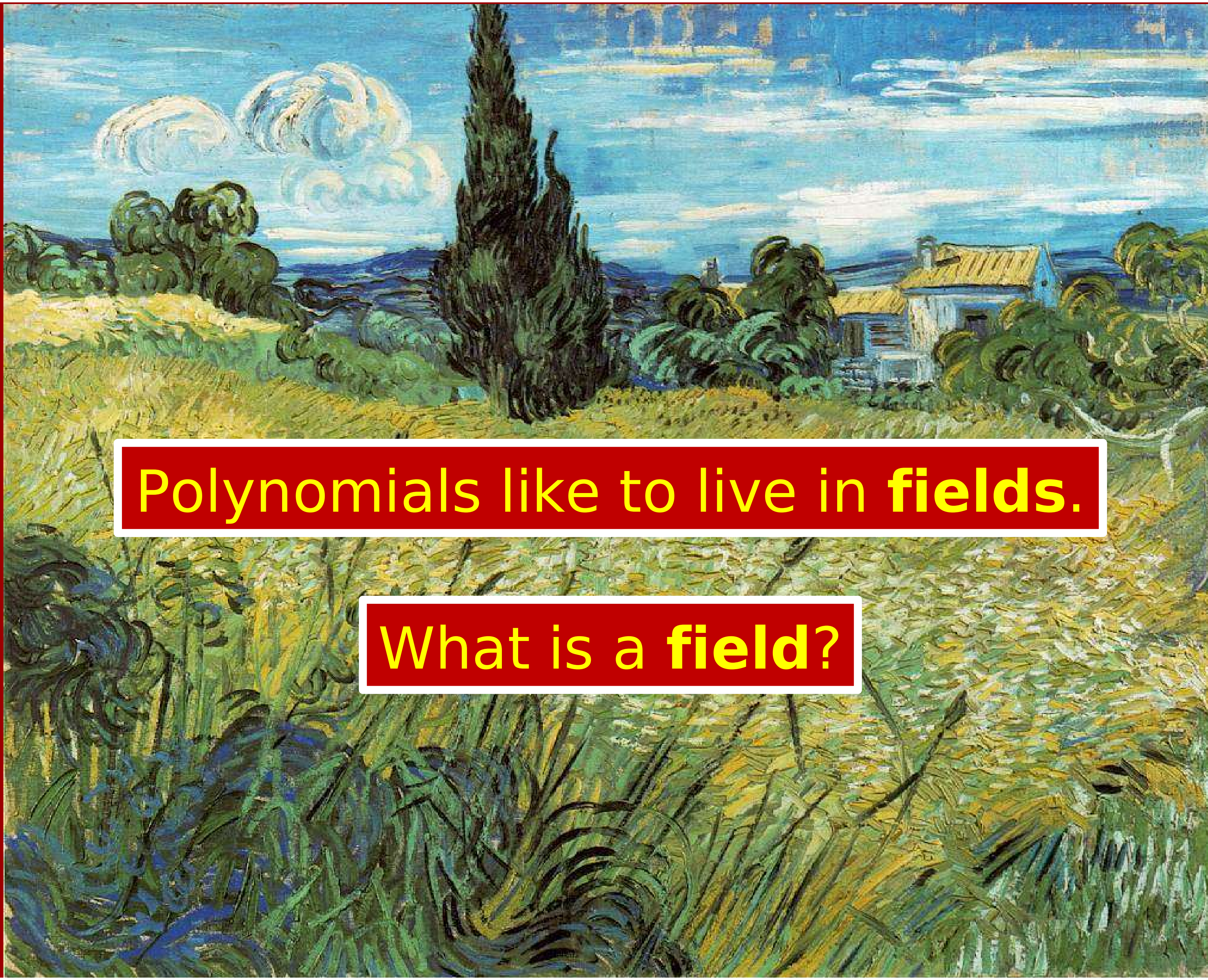
15-251: Great Theoretical Ideas in Computer Science

Fall 2018, Lecture 23

Fields and Polynomials


Polynomials like to live in **fields**.
What is a **field**?

Find out about the wonderful world of $\mathbb{F}_{2^k}$
where two equals zero, plus is minus,
and squaring is a linear operator!

– Rich Schroeppel

Fields

Informally, it's a number system where you can add, subtract, multiply, and divide (by nonzero).

Examples:	Real numbers	$\mathbb{R}$
	Rational numbers	$\mathbb{Q}$
	Complex numbers	$\mathbb{C}$
	Integers mod **prime**	Z_p
NON-examples:	Integers $\mathbb{Z}$	division??
	Positive reals $\mathbb{R}^+$	subtraction??

Field – formal definition

A **field** is a set F with two binary operations, called + and •.

(F,+) an abelian group, with identity element called 0

(F \ {0},•) an abelian group, identity element called 1

Distributive Law holds:

a•(b+c) = a•b + a•c

Example:

$\mathbb{F}_3 = Z_3$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

•	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Finite fields

Some familiar infinite fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

Finite fields we know: Z_p aka $\mathbb{F}_p$, for p a prime

Is there a field with 2 elements? Yes

Is there a field with 3 elements? Yes

Is there a field with 4 elements? Yes

$\mathbb{F}_4$

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

•	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Finite fields

Is there a field with 2 elements? Yes

Is there a field with 3 elements? Yes

Is there a field with 4 elements? Yes

Is there a field with 5 elements? Yes

Is there a field with 6 elements? No

Is there a field with 7 elements? Yes

Is there a field with 8 elements? Yes

Is there a field with 9 elements? Yes

Is there a field with 10 elements? No

Finite fields

Theorem:

There is a field with q elements
if and only if q is a power of a prime.

Up to isomorphism, it is unique.

I.e., all fields with q elements have the
same addition and multiplication tables,
after renaming elements.

This field is denoted $\mathbb{F}_q$.

Finite fields

Question:

If q is a prime power but not just a prime, what **are** the addition and multiplication tables of $\mathbb{F}_q$?

Answer:

It's a bit hard to describe.

We'll see it later, but for 251's purposes, you only need to know about prime q.

Polynomials

Polynomials

Informally, a polynomial is an expression that looks like this:

$$6x^3 - 2.3x^2 + 5x + 4.1$$

x is a symbol, called the variable

the 'numbers' standing next to powers of x are called the coefficients

Polynomials

Informally, a polynomial is an expression that looks like this:

$$6x^3 - 2.3x^2 + 5x + 4.1 \in \mathbb{R}[\mathbf{x}]$$

Actually, coefficients can come from any field.

Can allow multiple variables; we won't in this lecture.

The set of polynomials with variable x and coefficients from field F is denoted with $\mathbf{F[x]}$.

Polynomials – formal definition

Let F be a field and let x be a variable symbol.

$F[x]$ is the set of polynomials over F,

defined to be expressions of the form

$$c_d\, x^d + c_{d-1}\, x^{d-1} + \cdots + c_2\, x^2 + c_1\, x + c_0$$

where each c_i is in F, and $c_d \neq 0$.

We call d the degree of the polynomial.

Also, the expression 0 is a polynomial.

(By convention, we call its degree $-\infty$.)

Adding and multiplying polynomials

You can add and multiply polynomials.

Example. Here are two polynomials in $\mathbb{F}_{11}[x]$:

$$P(x) = x^2 + 5x - 1$$
$$Q(x) = 3x^3 + 10x$$

$$\begin{aligned} P(x) + Q(x) &= 3x^3 + x^2 + 15x - 1 \\ &= 3x^3 + x^2 + 4x - 1 \\ &= 3x^3 + x^2 + 4x + 10 \end{aligned}$$

Adding and multiplying polynomials

You can add and multiply polynomials.

Example. Here are two polynomials in $\mathbb{F}_{11}[x]$:

$$P(x) = x^2 + 5x - 1$$
$$Q(x) = 3x^3 + 10x$$

$$P(x) \bullet Q(x) = (x^2 + 5x - 1)(3x^3 + 10x)$$
$$= 3x^5 + 15x^4 + 7x^3 + 50x^2 - 10x$$
$$= 3x^5 + 4x^4 + 7x^3 + 6x^2 + x$$

Adding and multiplying polynomials

Polynomial addition is associative and commutative.

$0 + P(x) = P(x) + 0 = P(x)$.

$P(x) + (-P(x)) = 0$.

So $(F[x], +)$ is an abelian group!

Polynomial multiplication is associative and commutative.

$1 \bullet P(x) = P(x) \bullet 1 = P(x)$.

Multiplication distributes over addition:

$$P(x) \bullet (Q(x) + R(x)) = P(x) \bullet Q(x) + P(x) \bullet R(x)$$

If $P(x) / Q(x)$ were always a polynomial,

then $F[x]$ would be a field! Alas…

Dividing polynomials?

$P(x) / Q(x)$ is not necessarily a polynomial.

So $F[x]$ is not quite a field.
(It's just a "commutative ring with identity".)

Same with $\mathbb{Z}$, the integers:
it has everything except division.

Actually, there are many analogies between
$F[x]$ and $\mathbb{Z}$.

Dividing polynomials?

$\mathbb{Z}$ has the concept of "division with remainder":

Given $a,b \in \mathbb{Z}$, $b \neq 0$, can write

$$a = q \cdot b + r,$$

where r is "smaller than" b.

F[x] has the same concept:

Given $A(x), B(x) \in F[x]$, $B(x) \neq 0$, can write

$$A(x) = Q(x) \cdot B(x) + R(x),$$

where $\deg(R(x)) < \deg(B(x))$.

"Division with remainder" for polynomials

Example: Divide $6x^4+8x+1$ by $2x^2+4$ in $\mathbb{F}_{11}[x]$

$$
\begin{array}{r|l}
 & 3x^2+5 \\
\hline
2x^2+4 & 6x^4+8x+1 \\
- & 6x^4+x^2 \\
\hline
 & \quad -x^2+8x+1 \\
- & \quad -x^2+9 \\
\hline
 & \qquad 8x+3
\end{array}
$$

Check:

$$6x^4+8x+1$$
$$= (3x^2+5)(2x^2+4)+(8x+3)$$

(in $\mathbb{F}_{11}[x]$)

Integers $\mathbb{Z}$

"size" = abs. value

"division":

$a = qb+r, \quad |r| < |b|$

can use Euclid's Algorithm to find GCDs

p is "prime": no nontrivial divisors

$\mathbb{Z}$ mod p: a field if p is prime

Polynomials F[x]

"size" = degree

"division":

$A(x) = Q(x)B(x)+R(x),$

$\deg(R) < \deg(B)$

can use Euclid's Algorithm to find GCDs

P(x) is "irreducible": no nontrivial divisors

F[x] mod P(x): a field if P(x) is irreducible (with $|F|^{\deg(P)}$ elements)

Enough algebraic theory.

Let’s play with polynomials!

Evaluating polynomials

Given a polynomial $P(x) \in F[x]$,

$P(a)$ means its evaluation at element a.

E.g., if $P(x) = x^2+3x+5$ in $\mathbb{F}_{11}[x]$,

$P(6) = 6^2+3\cdot 6+5 = 36+18+5 = 59 = 4$

$P(4) = 4^2+3\cdot 4+5 = 16+12+5 = 33 = 0$

Definition: r is a **root** of P(x) if P(r) = 0.

Polynomial roots

Theorem:

Let $P(x) \in F[x]$ have degree 1.

Then $P(x)$ has exactly 1 root.

Proof:

Write $P(x) = cx + d$ (where $c \neq 0$).

$$
\begin{aligned}
\text{Then } P(r) = 0 \quad &\Leftrightarrow \quad cr + d = 0 \\
&\Leftrightarrow \quad cr = -d \\
&\Leftrightarrow \quad r = -d/c.
\end{aligned}
$$

Polynomial roots

Theorem:

Let $P(x) \in F[x]$ have degree 2.

Then $P(x)$ has… how many roots??

E.g.: x^2+1…

# of roots over $\mathbb{F}_2[x]$: 1 (namely, 1)

# of roots over $\mathbb{F}_3[x]$: 0

# of roots over $\mathbb{F}_5[x]$: 2 (namely, 2 and 3)

# of roots over $\mathbb{R}[x]$: 0

# of roots over $\mathbb{C}[x]$: 2 (namely, i and −i)

The single most important theorem about polynomials over fields:

A nonzero degree-d polynomial has at most d roots.

Theorem: Over any field, a nonzero degree-d polynomial has at most d roots.

Proof by induction on $d \in \mathbb{N}$:

Base case: If $P(x)$ is degree-0 then $P(x) = a$ for some $a \neq 0$. This has 0 roots.

Induction:

Assume true for $d \geq 0$. Let $P(x)$ have degree $d+1$.

If $P(x)$ has 0 roots: we're done! Else let b be a root.

Divide with remainder: $P(x) = Q(x)(x-b) + R(x)$. $(*)$

$\deg(R) < \deg(x-b) = 1$, so $R(x)$ is a constant. Say $R(x)=r$.

Plug $x = b$ into $(*)$: $0 = P(b) = Q(b)(b-b)+r = 0+r = r$.

So $P(x) = Q(x)(x-b)$. Hence $\deg(Q) = d$. $\therefore$ Q has $\leq d$ roots.

$\therefore$ $P(x)$ has $\leq d+1$ roots, completing the induction.

Theorem: Over any field, a nonzero degree-d polynomial has at most d roots.

Reminder:

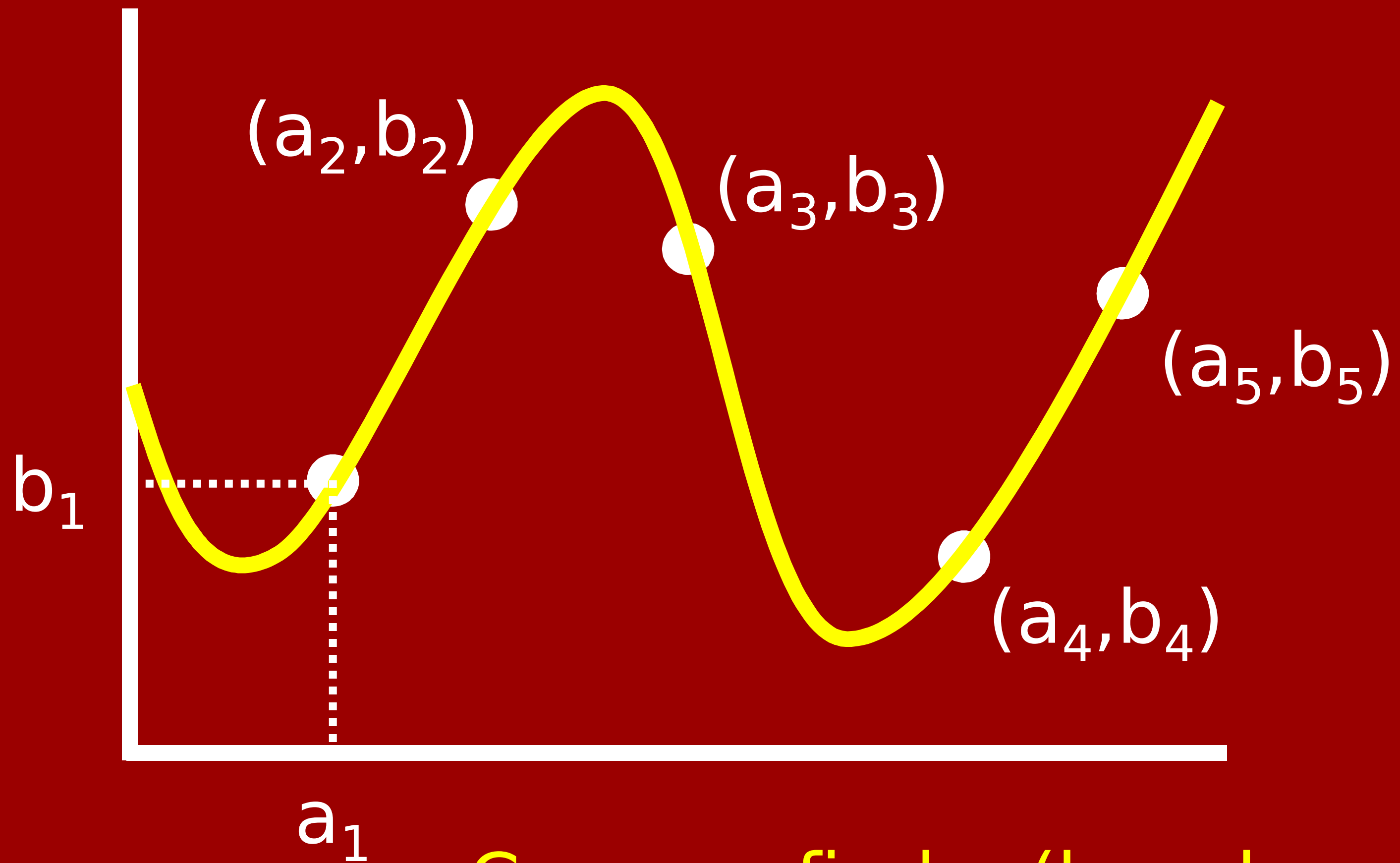
This is only true over a field.

E.g., consider $P(x) = 3x$ over Z_6.

It has degree 1, but 3 roots: 0, 2, and 4.

Interpolation

Say you're given a bunch of "data points"



Can you find a (low-degree) polynomial which "fits the data"?

Interpolation

Let pairs (a_1,b_1), (a_2,b_2), …, (a_{d+1},b_{d+1}) from a field F be given (with all a_i's distinct).

Theorem:

There is exactly one polynomial $P(x)$ of degree at most d such that $P(a_i) = b_i$ for all $i = 1\ldots d+1$.

E.g., thru 2 points there is a unique linear polynomial.

Interpolation

There are two things to prove.

1. There is at least one polynomial of degree $\leq d$ passing through all d+1 data points.

2. There is at most one polynomial of degree $\leq d$ passing through all d+1 data points.

Let's prove #2 first.

Interpolation

Theorem:

Let pairs (a_1,b_1), (a_2,b_2), …, (a_{d+1},b_{d+1}) from a field F be given (with all a_i's distinct).
Then there is at most one polynomial $P(x)$ of degree at most d with $P(a_i) = b_i$ for all i.

Proof: Suppose $P(x)$ and $Q(x)$ both do the trick.

Let $R(x) = P(x) - Q(x)$.

Since $\deg(P)$, $\deg(Q) \le d$ we must have $\deg(R) \le d$.

But $R(a_i) = b_i - b_i = 0$ for all $i = 1 \ldots d+1$.

$\therefore$ $R(x)$ is the 0 polynomial; i.e., $P(x)=Q(x)$.

Interpolation

Now let's prove the other part,
that there is at least one polynomial.

Theorem:

Let pairs (a_1,b_1), (a_2,b_2), …, (a_{d+1},b_{d+1})
from a field F be given (with all a_i's distinct).
Then there exists a polynomial $P(x)$ of
degree at most d with $P(a_i) = b_i$ for all i.

Interpolation

The method for constructing the polynomial is called Lagrange Interpolation.

Discovered in 1779 by Edward Waring.

Rediscovered in 1795 by J.-L. Lagrange.

Lagrange Interpolation

a_1	b_1
a_2	b_2
a_3	b_3
...	...
a_d	b_d
a_{d+1}	b_{d+1}

Want $P(x)$

(with degree $\leq d$)

such that $P(a_i) = b_i \;\; \forall i.$

Lagrange Interpolation

a_1	1
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	0

Can we do this special case?

Promise: once we solve this special case,
the general case is very easy.

Lagrange Interpolation

a_1	1
a_2	0
a_3	0
$\cdots$	$\cdots$
a_d	0
a_{d+1}	0

Just divide P(x) by this number.

Idea #1: $P(x) = (x-a_2)(x-a_3)\cdots(x-a_{d+1})$

Degree is d. ✓

$P(a_2) = P(a_3) = \cdot\cdot\cdot = P(a_{d+1}) = 0.$ ✓

$P(a_1) = (a_1-a_2)(a_1-a_3)\cdots(a_1-a_{d+1}).$??

Lagrange Interpolation

Numerator is a deg. d polynomial

a_1	1
a_2	0
a_3	0
...	...
a_d	0
a_{d+1}	0

Denominator is a nonzero field element

Idea #2:

$$S_1(x) = \frac{(x-a_2)(x-a_3)\cdots(x-a_{d+1})}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_{d+1})}$$

Call this the selector polynomial for a_1.

Lagrange Interpolation

a_1	0
a_2	1
a_3	0
...	...
a_d	0
a_{d+1}	0

Great! But what about this data?

$$S_2(x) = \frac{(x - a_1)(x - a_3)\cdots(x - a_{d+1})}{(a_2 - a_1)(a_2 - a_3)\cdots(a_2 - a_{d+1})}$$

Lagrange Interpolation

a_1	0
a_2	0
a_3	0
…	…
a_d	0
a_{d+1}	1

Great! But what about this data?

$$S_{d+1}(x) = \frac{(x-a_1)(x-a_2)\cdots(x-a_d)}{(a_{d+1}-a_1)(a_{d+1}-a_2)\cdots(a_{d+1}-a_d)}$$

Lagrange Interpolation

a_1	b_1
a_2	b_2
a_3	b_3
...	...
a_d	b_d
a_{d+1}	b_{d+1}

Great! But what about this data?

$$P(x) = b_1 \cdot S_1(x) + b_2 \cdot S_2(x) + \cdots + b_{d+1} \cdot S_{d+1}(x)$$

Lagrange Interpolation – example

Over $\mathbb{F}_{11}$, find a polynomial P of degree ≤ 2 such that P(5) = 1, P(6) = 2, P(7) = 9.

$$S_5(x) = 6\,(x-6)(x-7) \qquad \frac{1}{(5-6)(5-7)}$$

$$S_6(x) = -\,(x-5)(x-7)$$

$$S_7(x) = 6\,(x-5)(x-6)$$

$$P(x) = 1\,S_5(x) + 2\,S_6(x) + 9\,S_7(x)$$

$$= 6(x^2-13x+42) - 2(x^2-12x+35) + 54(x^2-11x+30)$$

$$= 3x^2+x+9$$

Recall: Interpolation

Let pairs (a_1,b_1), (a_2,b_2), …, (a_{d+1},b_{d+1}) from a field F be given (with all a_i's distinct).

Theorem:

There is exactly one polynomial $P(x)$ of degree at most d such that $P(a_i) = b_i$ for all $i = 1…d+1$.

Representing Polynomials

Let $P(x) \in F[x]$ be a degree-d polynomial.

Representing $P(x)$ using $d+1$ field elements:

1. List the $d+1$ coefficients.
2. Give P's value at $d+1$ different elements.

Rep 1 to Rep 2: Evaluate at $d+1$ elements

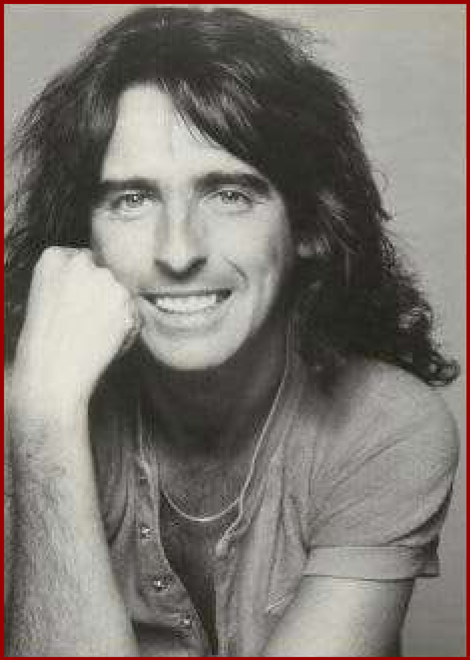
Rep 2 to Rep 1: Lagrange Interpolation

Application:

Error-correcting codes

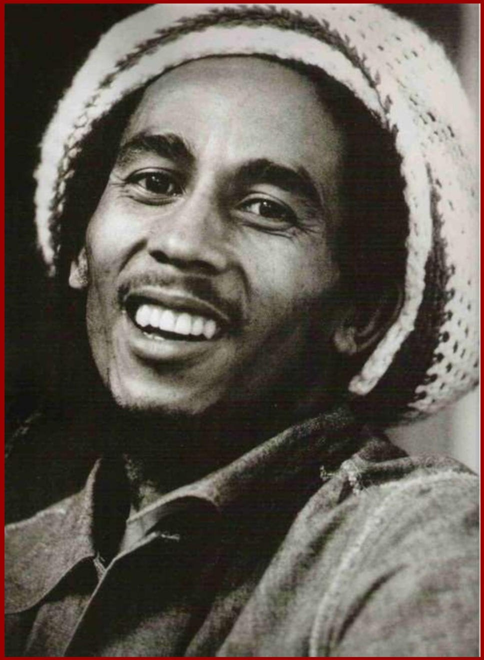
Sending messages on a noisy channel

Alice



“ bit.ly/vrxUBN ”

Bob



The channel may corrupt up to k symbols.

How can Alice still get the message across?

Sending messages on a noisy channel

Let's say messages are sequences from $\mathbb{F}_{257}$

vrxUBN $\leftrightarrow$ 118 114 120 85 66 78

$$\downarrow \text{noisy channel}$$

118 114 104 85 35 78

The channel may corrupt up to k symbols.

How can Alice still get the message across?

Sending messages on a noisy channel

Let's say messages are sequences from $\mathbb{F}_{257}$

vrxUBN $\leftrightarrow$ 118 114 120 85 66 78

noisy channel

118 114 104 85 35 78

How to correct the errors?

How to even detect that there are errors?

Simpler case: "Erasures"

118 114 120 85 66 78

erasure channel

118 114 ?? 85 ?? 78

What can you do to handle up to k erasures?

Repetition code

Have Alice repeat each symbol k+1 times.

118 114 120 85 66 78

becomes

118 118 118 114 114 114 120 120 120 85 85 85 66 66 66 78 78 78

erasure channel ↓

118 118 118 ?? ?? 114 120 120 120 85 85 85 66 66 66 78 78 78

If at most k erasures, Bob can figure out each symbol.

Repetition code – noisy channel

Have Alice repeat each symbol 2k+1 times.

118 114 120 85 66 78

becomes

118 118 118 114 114 114 120 120 120 85 85 85 66 66 66 78 78 78

noisy channel

118 118 118 114 223 114 120 120 120 85 85 85 66 66 66 78 78 78

At most k corruptions: Bob can take maj. of each block.

This is pretty wasteful!

To send message of $d+1$ symbols and guard against k erasures, we had to send $(d+1)(k+1)$ total symbols.

Can we do better?

This is pretty wasteful!

To send message of d+1 symbols and

guard against k erasures, we had

to send (d+1)(k+1) total symbols.

To send even 1 message symbol with
k erasures, **need** to send k+1 total symbols.

Maybe for d+1 message symbols with k erasures,
d+k+1 total symbols can suffice??

Enter polynomials

Say Alice's message is d+1 elements from $\mathbb{F}_{257}$

118 114 120 85 66 78

Alice thinks of it as the coefficients of a degree-d polynomial $P(x) \in \mathbb{F}_{257}[x]$

$$P(x) = 118x^5 + 114x^4 + 120x^3 + 85x^2 + 66x + 78$$

Now trying to send the degree-d polynomial P(x).

Send it in the Values Representation!

$P(x) = 118x^5 + 114x^4 + 120x^3 + 85x^2 + 66x + 78$

Alice sends P(x)'s values on d+k+1 inputs:

P(1), P(2), P(3), …, P(d+k+1)

This is called the **Reed–Solomon encoding**.

Send it in the Values Representation!

$P(x) = 118x^5 + 114x^4 + 120x^3 + 85x^2 + 66x + 78$

Alice sends P(x)'s values on d+k+1 inputs:

P(1), P(2), P(3), …, P(d+k+1)

If there are at most k erasures, then

Bob still knows P's value on d+1 points.

Bob recovers P(x) using Lagrange Interpolation!

Example

What about corruptions, not erasures?

Trickier. So let Alice now send P(x)’s value on
d + 2k + 1 inputs.

Assuming at most k corruptions,
Bob will have at least d+k+1 ‘correct’ values.

P(1), P(2), bogus, P(4), bogus, P(6), …, P(d+2k+1)

Trouble: Bob does not know which values are bogus.

Corruptions under Reed–Solomon

Assuming at most k corruptions,
Bob will have at least $d+k+1$ 'correct' values.

P(1), P(2), bogus, P(4), bogus, P(6), …, P(d+2k+1)

P(x) is a poly of degree ≤ d
which disagrees with the received
data on at most k positions.

Theorem: It is the **only** such polynomial.

Corruptions under Reed–Solomon

Theorem: P(x) is the **only** polynomial of degree $\leq d$ which disagrees with the data on $\leq k$ positions.

Proof:

Suppose Q(x) is another such poly.

P(x) and Q(x) disagree with each other on at most 2k positions.

∴ they agree with each other on at least $(d+2k+1)-2k = d+1$ positions.

∴ $P(x) = Q(x)$ since they are degree $\leq d$.

Corruptions under Reed–Solomon

Theorem: $P(x)$ is the **only** polynomial of degree $\leq d$ which disagrees with the data on $\leq k$ positions.

Therefore Bob can determine $P(x)$!

Brute force algorithm:

Take each set of $d+1$ out of $d+2k+1$ values.

Interpolate to get a polynomial $Q(x)$ of deg $\leq d$.

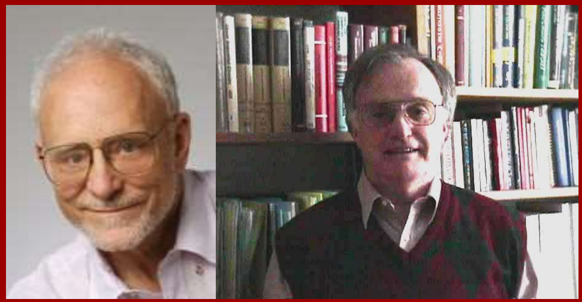
Check if it agrees with $\geq d+k+1$ values.

Efficient Reed–Solomon

Brute-force 'decoding' takes $2^{O(d)}$ time. ☹

Peterson 1960: a $O(d^3)$ decoding alg.

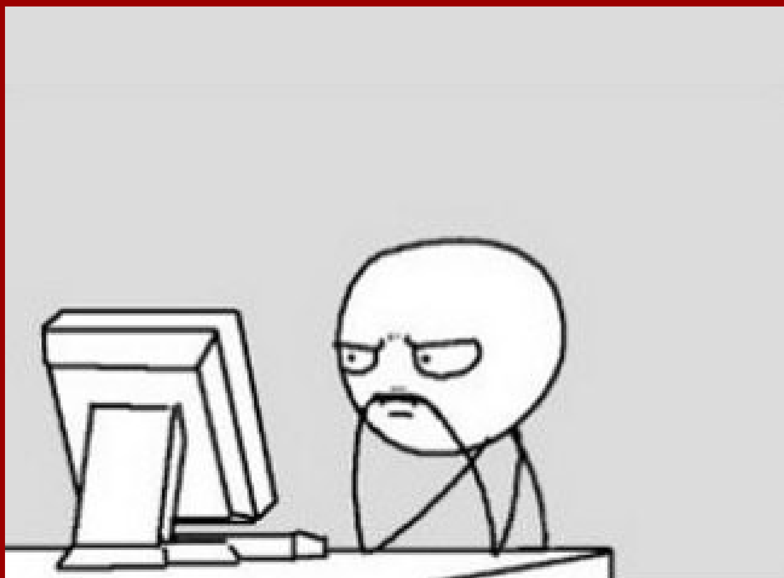
Berlekamp & Massey, late '60s:
key practical improvements



CMU's Prof. Guruswami:
efficient algorithms to meaningfully correct more than k corruptions

Reed−Solomon codes are used in practice!

These are all RS codes.



Study Guide

Definitions:

Fields, polynomials

Theorem/proof:

Degree-d polys have at most d roots.

Algorithms:

Polynomial division with remainder

Lagrange Interpolation

Error correction and detection with Reed−Solomon