15-251: Great Theoretical Ideas in Computer Science

Lecture 5

Turing's Legacy

What is **computation**?

What is an **algorithm**?

How can we mathematically define them?

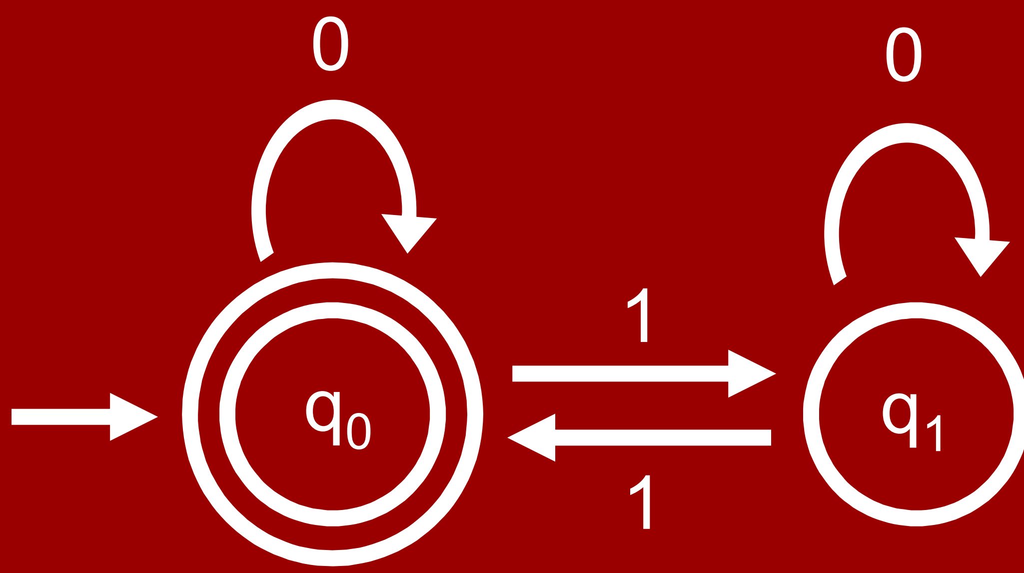
Quick Recap

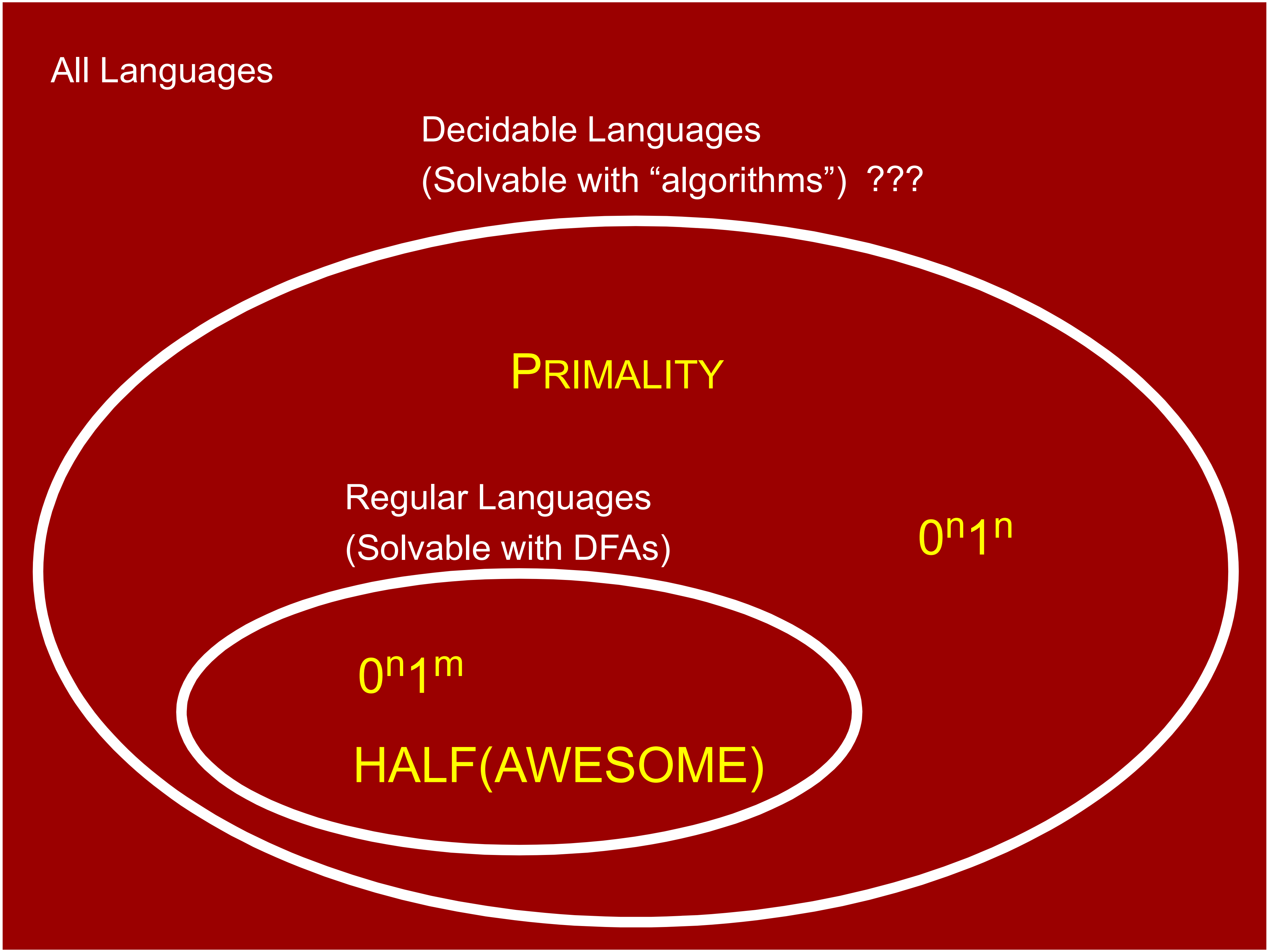
Mathematical definition of a (computational) problem:

Input / output function: $f: \Sigma^* \to \Sigma^*$

Language: $L = \{x \in \Sigma^* | f(x) = 1\} \subseteq \Sigma^*$

A simple mathematical model for algorithms: DFAs



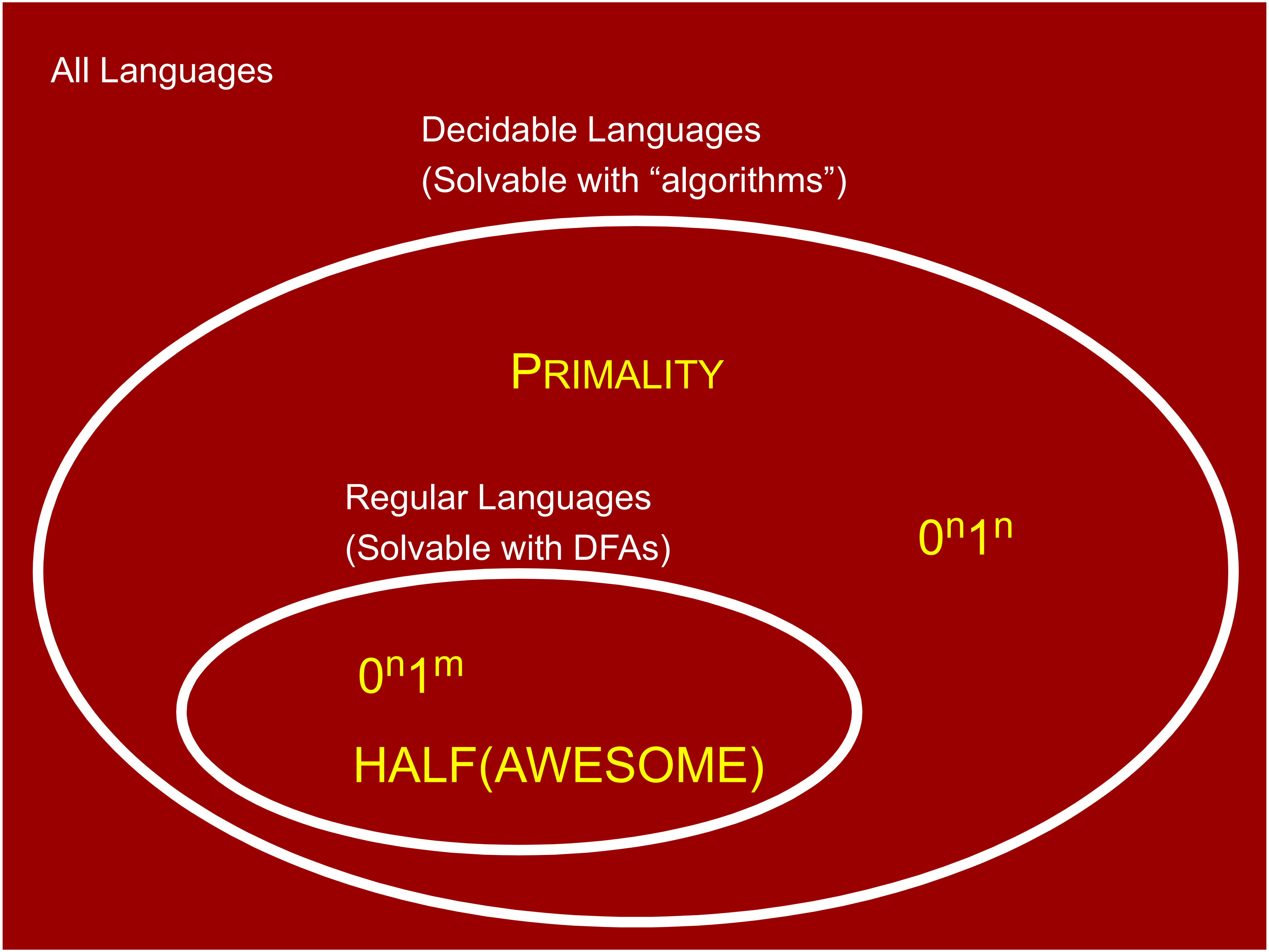

All Languages
Decidable Languages
(Solvable with "algorithms") ???
PRIMALITY
Regular Languages
(Solvable with DFAs)
0^n1^n
0^n1^m
HALF(AWESOME)

Solving 0^n1^n with Python

```
# Determines if string S is of form 0^n 1^n
def Solution( S ):
    i = 0
    j = len(S)-1
    while j >= i:
        if S[i] != '0' or S[j] != '1':
            return False
        i = i + 1
        j = j - 1
    return True
```

Solving 0^n1^n with C

```
/* Determines if string S is of form 0^n 1^n */
int Solution(char S[])
{
    int i = 0, j;
    while (S[j] != NULL)  /* NULL is end-of-string char */
        j++;
    j--;
    while (j >= i)
    {
        if (S[i] != '0' || S[j] != '1')
            return 0;  /* Reject */
        i++;
        j--;
    }
    return 1;  /* Accept */
}
```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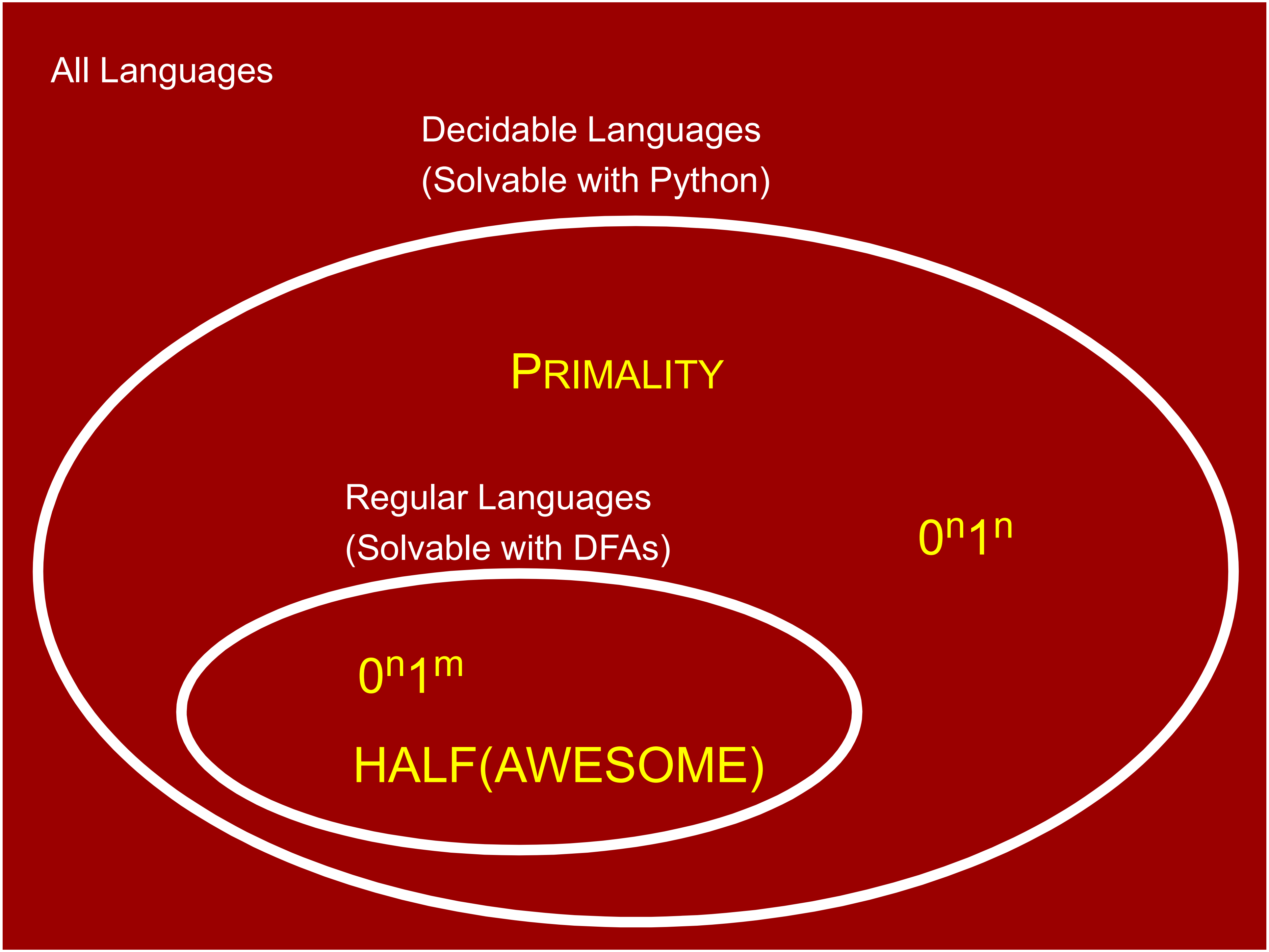
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0^n1^n
0^n1^m
HALF(AWESOME)

Question:

Should we just define “algorithm” to mean a function written in Python?

(allowed access to unlimited memory)

Answer:

Actually, we’ll see that this would be OK!

Downsides as a formal definition:

- Why choose Python?
  Why not C, or Java, or SML, or…?

- Extremely complicated to rigorously define.
  E.g., official 2011 ISO definition of C
  requires a 701-page PDF file!

- A "philosophical" justification would be nice…

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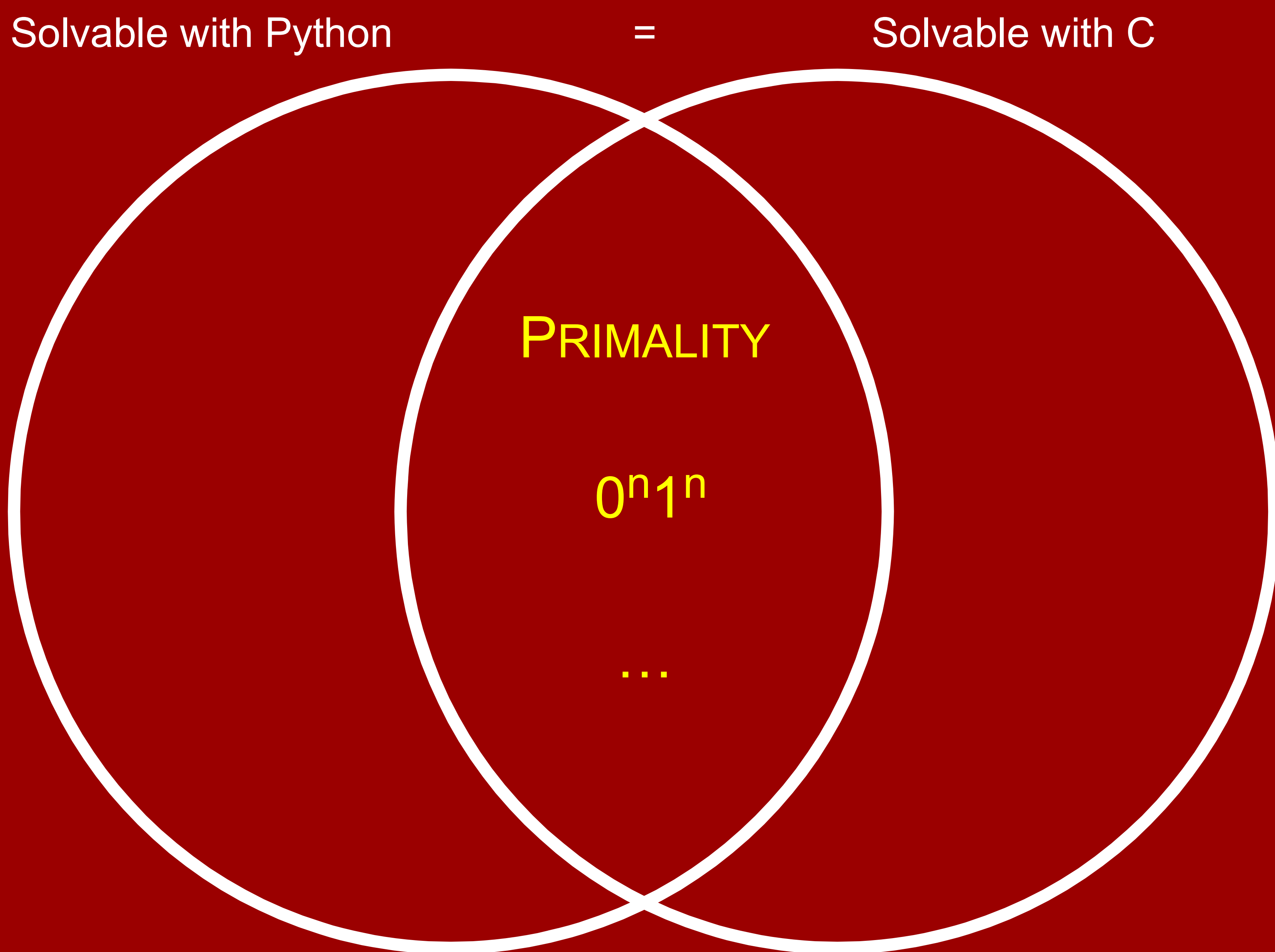
Claim:



Claim:

Solvable with Python = Solvable with C

Proof intuition:

Our shared experience with programming.

"Proof:"

1. Solvable with Python $\subseteq$ Solvable with C.
   The standard Python interpreter is written in C.

2. Solvable with C $\subseteq$ Solvable with Python.
   It's pretty clear one can write a C interpreter in Python.

Interpreters

A Python function is (representable by) a string.

A **Python interpreter** is an algorithm M that

takes two inputs: `P`, a Python function;

`x`, a string;

and step-by-step simulates `P(x)`.

In particular, M(P,x) accepts if and only if P(x) accepts.

Interpreters

You can write a Python interpreter in C.

You can write a C interpreter in Python.

You can write a Python interpreter in Java.

You can write a Java interpreter in Python.

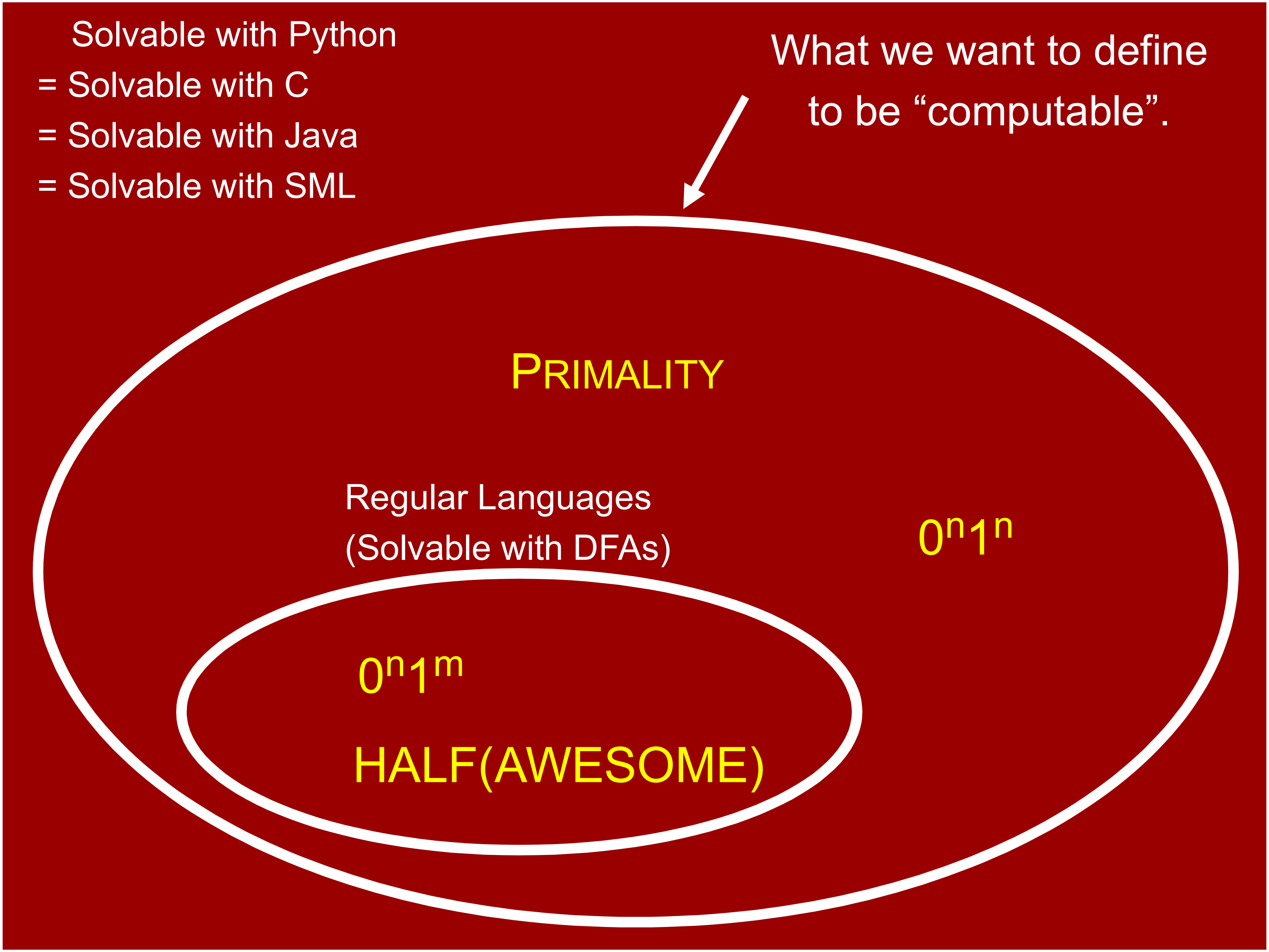
You can write a Python interpreter in SML.

You can write an SML interpreter in Python.

You can write a Python interpreter in Python!!

The last one is called a

"Universal Python Program"


Solvable with Python
= Solvable with C
= Solvable with Java
= Solvable with SML
What we want to define
to be "computable".
PRIMALITY
Regular Languages
(Solvable with DFAs)
0^n1^n
0^n1^m
HALF(AWESOME)

Downsides as a formal definition:

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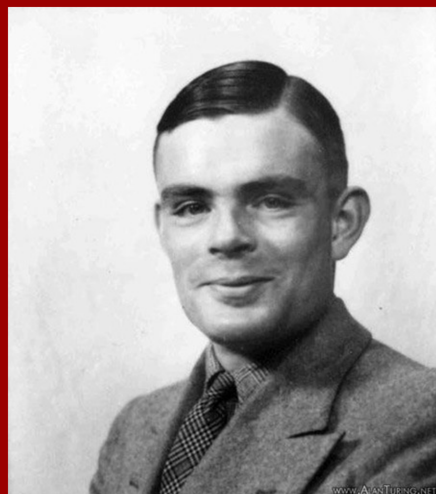
It would be nice to have a **totally minimal** (“**TM**”) programming language such that:

a) can simulate Python, C, Java, SML, etc.;

b) is simple enough to reason about rigorously completely mathematically.

Turing Machine^{TM}



Inspired by

Turing’s mathematical abstraction of a computer

- A (human) computer writes symbols on paper
- WLOG, the paper is a sequence of squares
- No upper bound on the number of squares
- At most finitely many kinds of symbols
- Human observes one square at a time
- Human has only finitely many mental states
- Human can change symbols and change focus to a neighboring square, but only based on its state and the symbol it observes
- Human acts deterministically

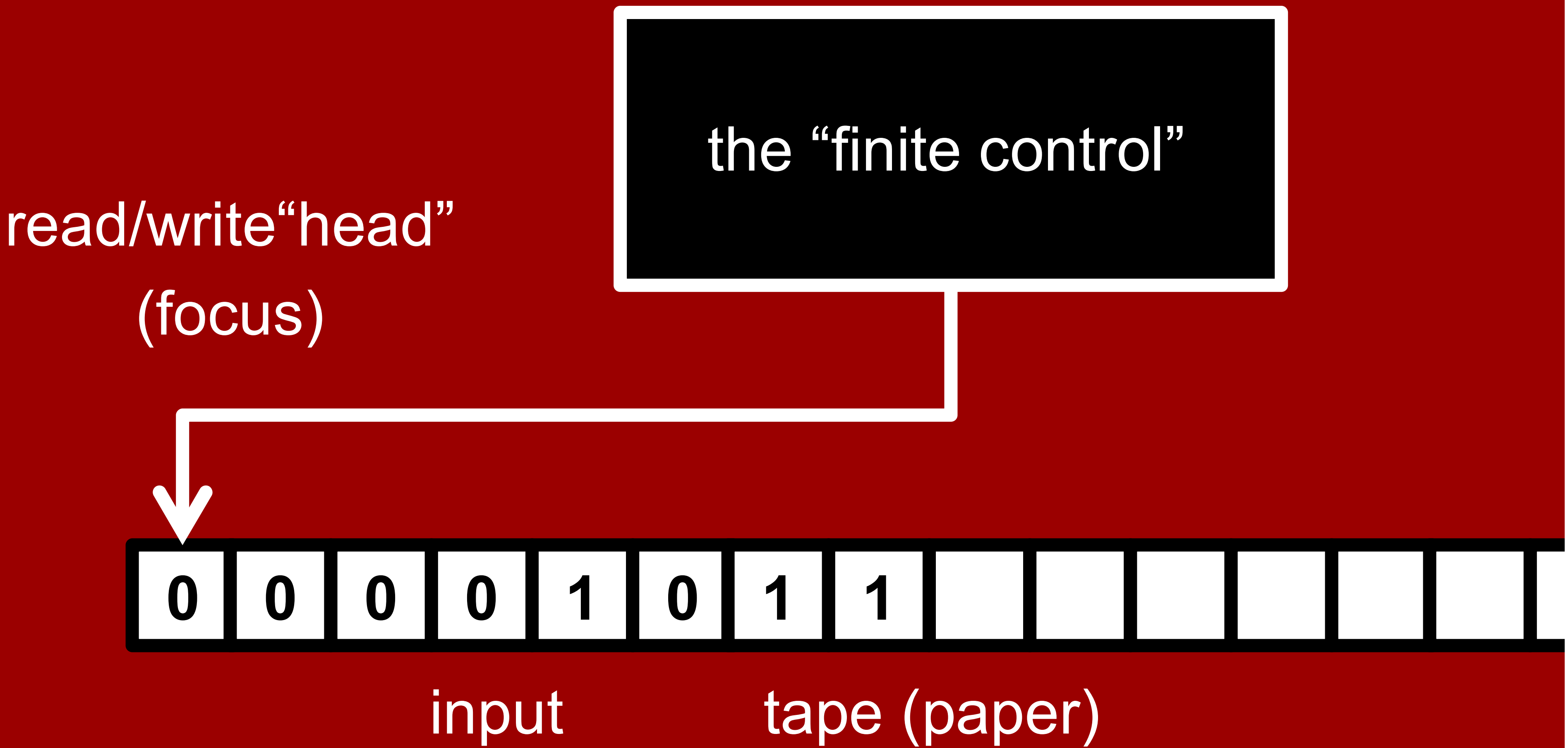
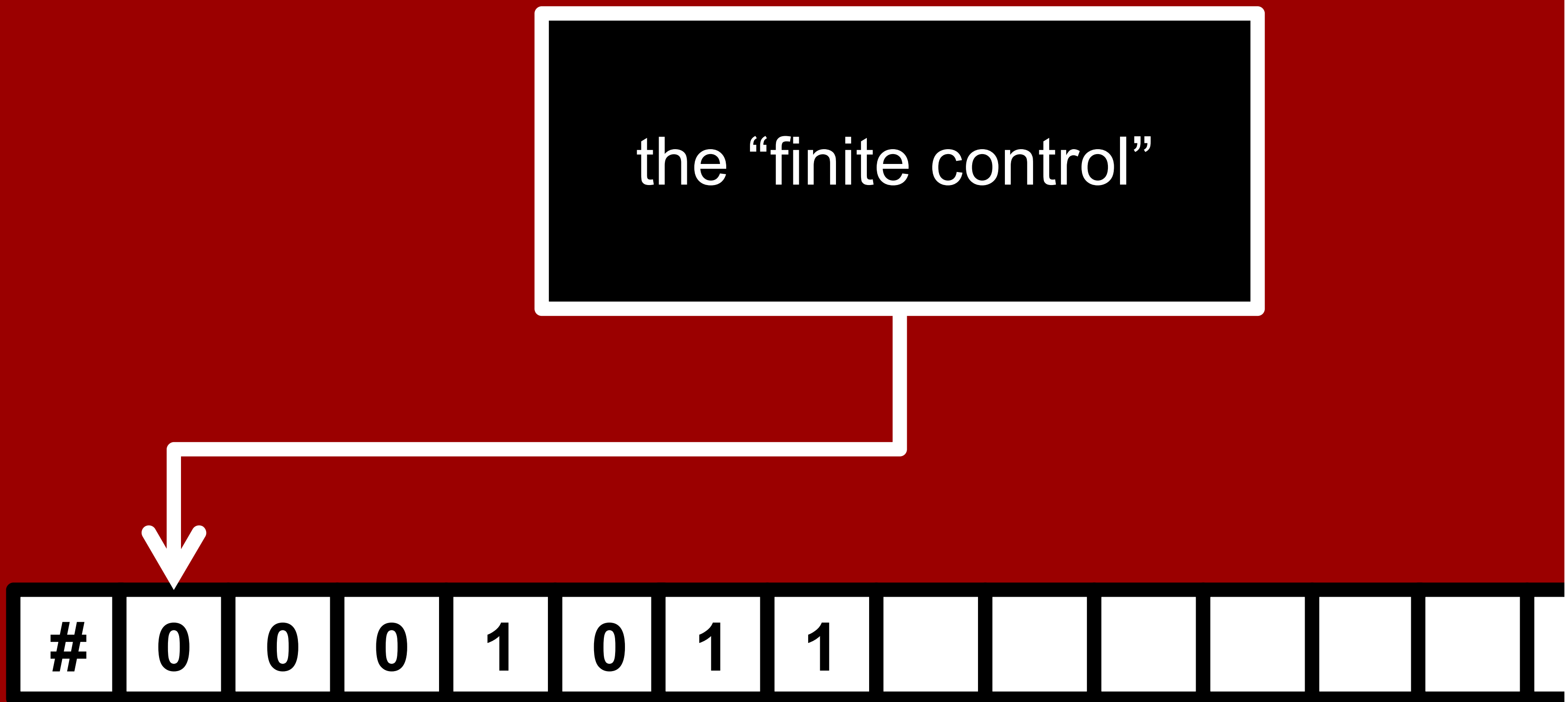
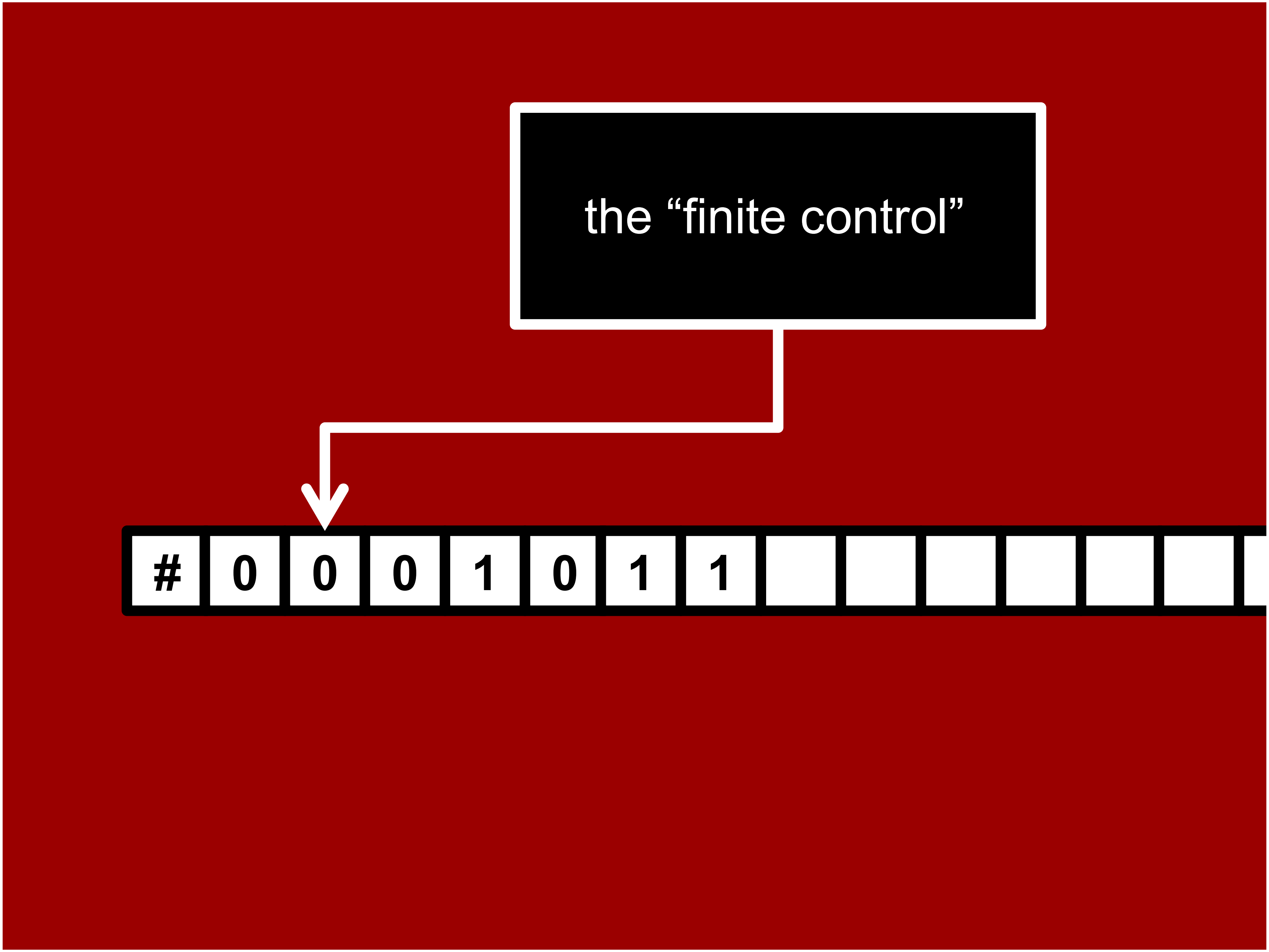


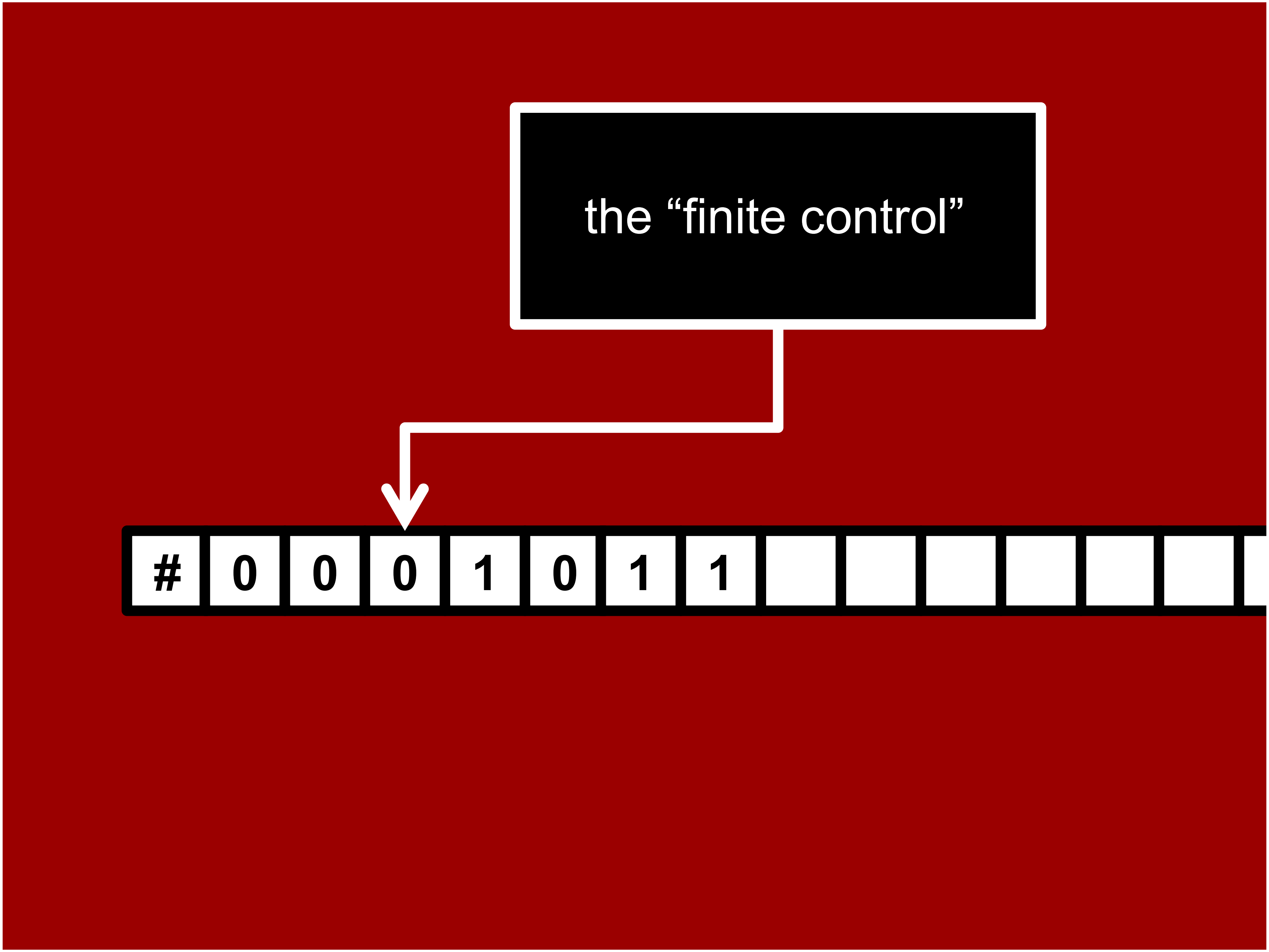
Illustration of a TM trying to decide $\{0^n1^n : n \in \mathbb{N}\}$

We’ll explain the finite control later



the “tape alphabet”
in this example is $\{0,1,\#,\sqcup\}$ (blank)


the "finite control"
0 0 0 1 0 1 1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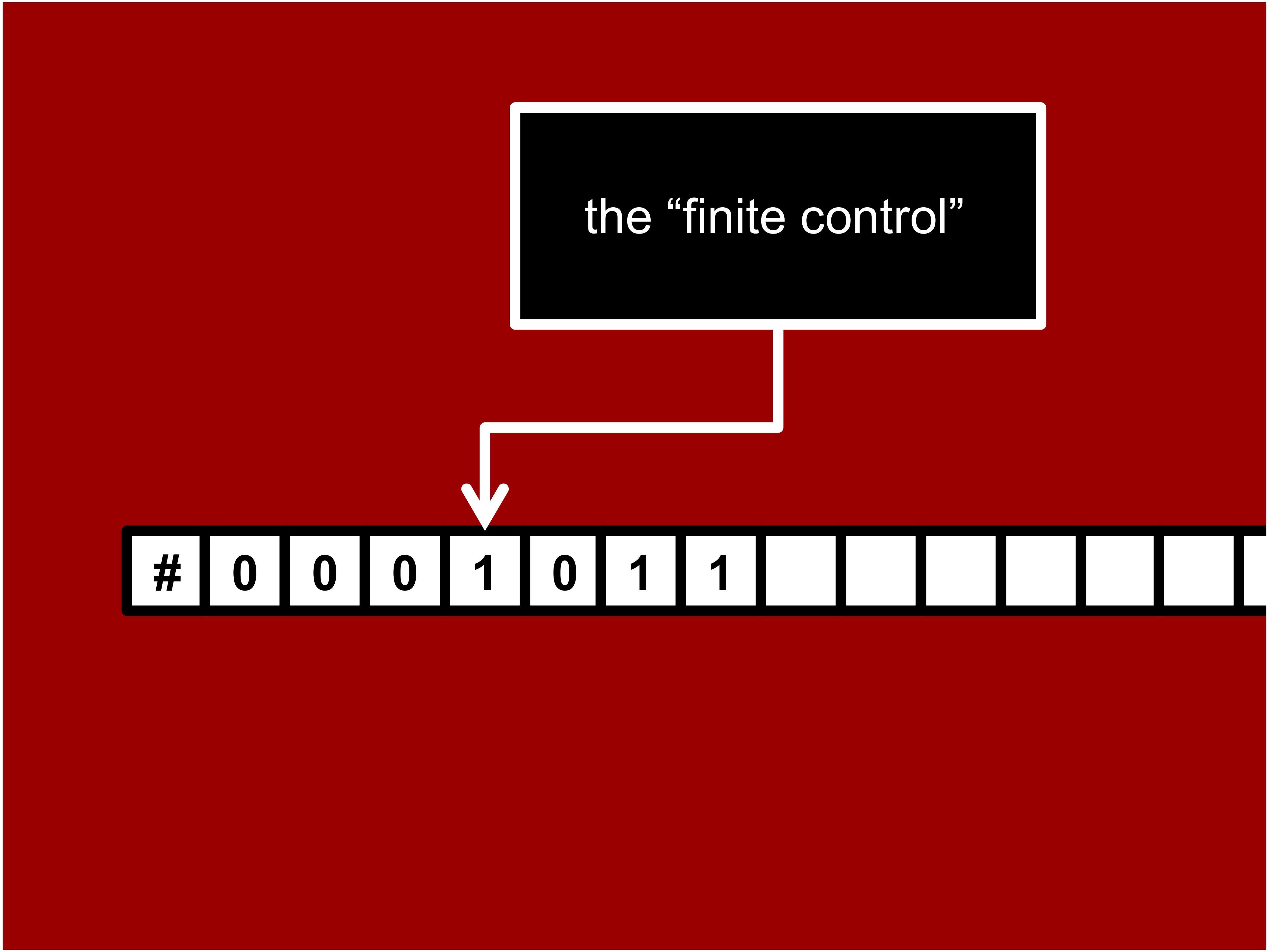

the “finite control”
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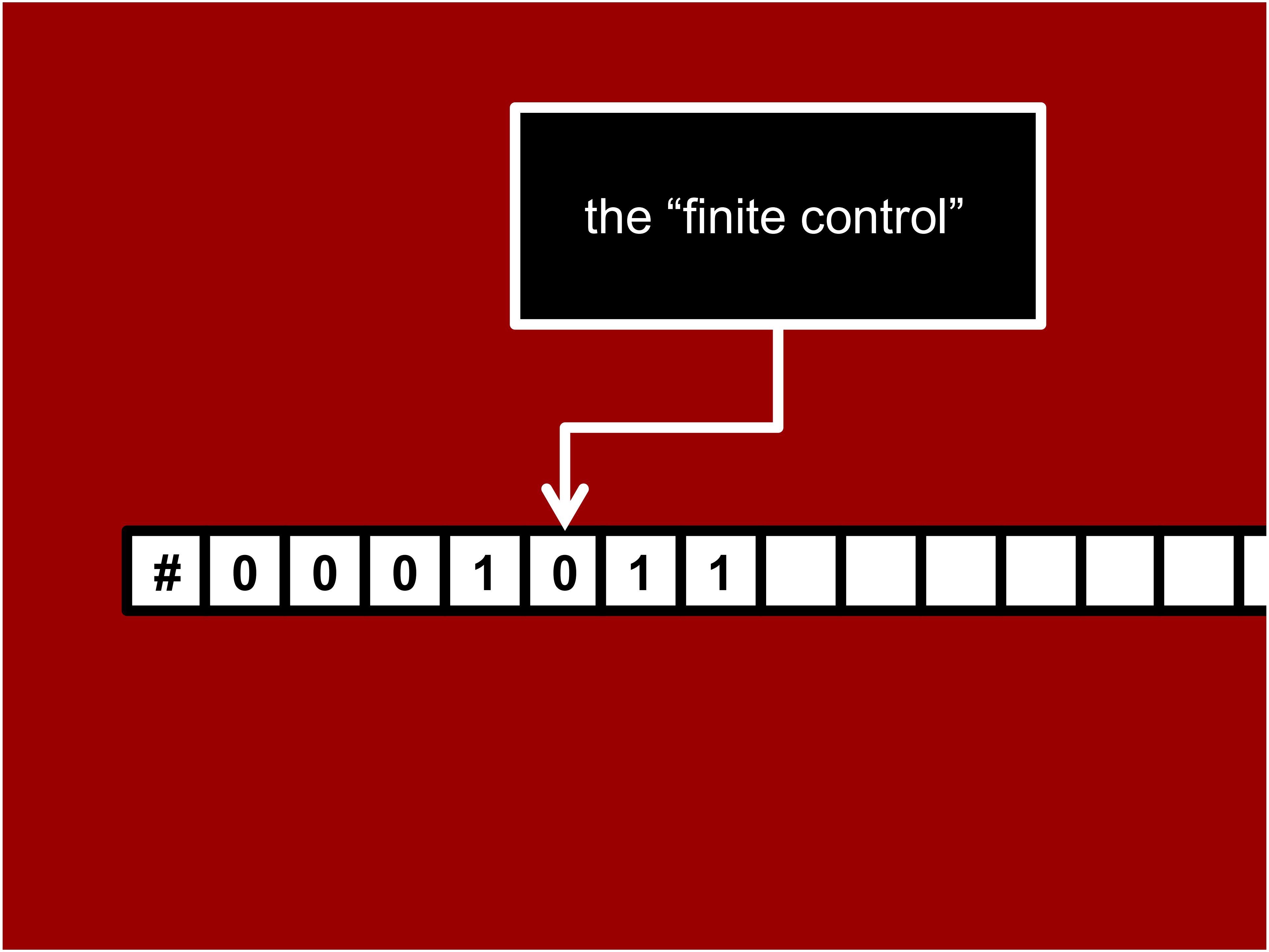
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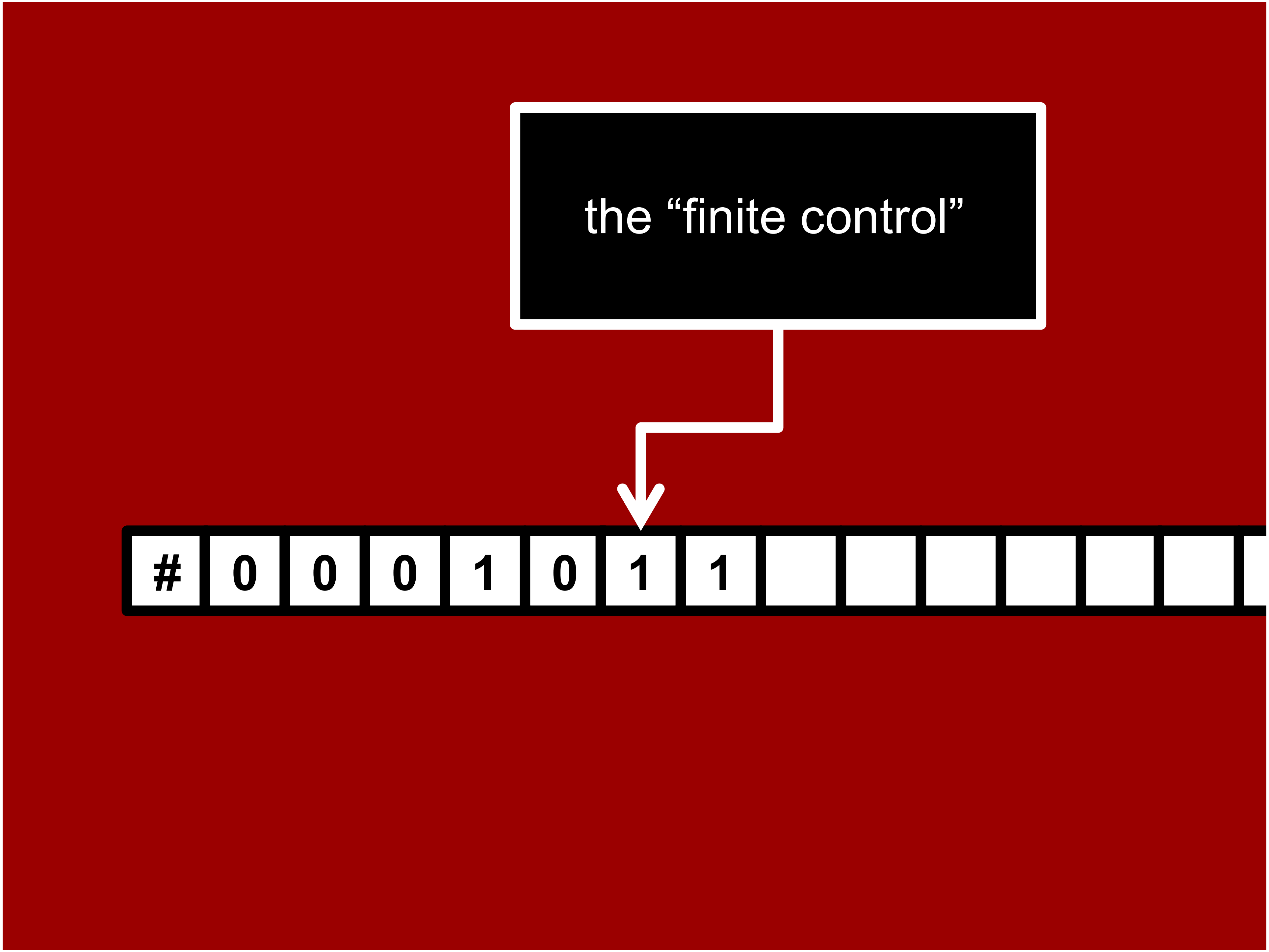
the “finite control”
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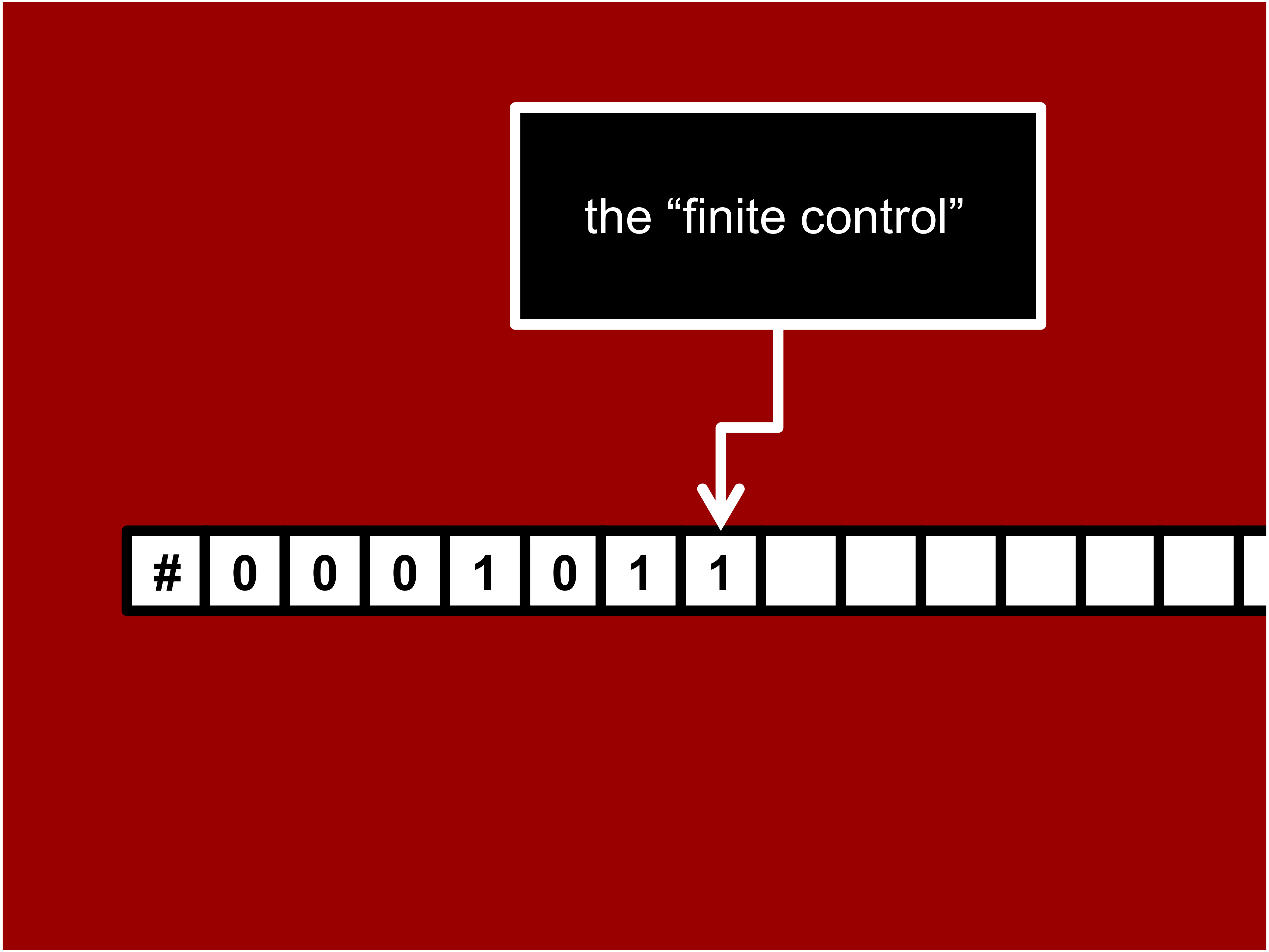
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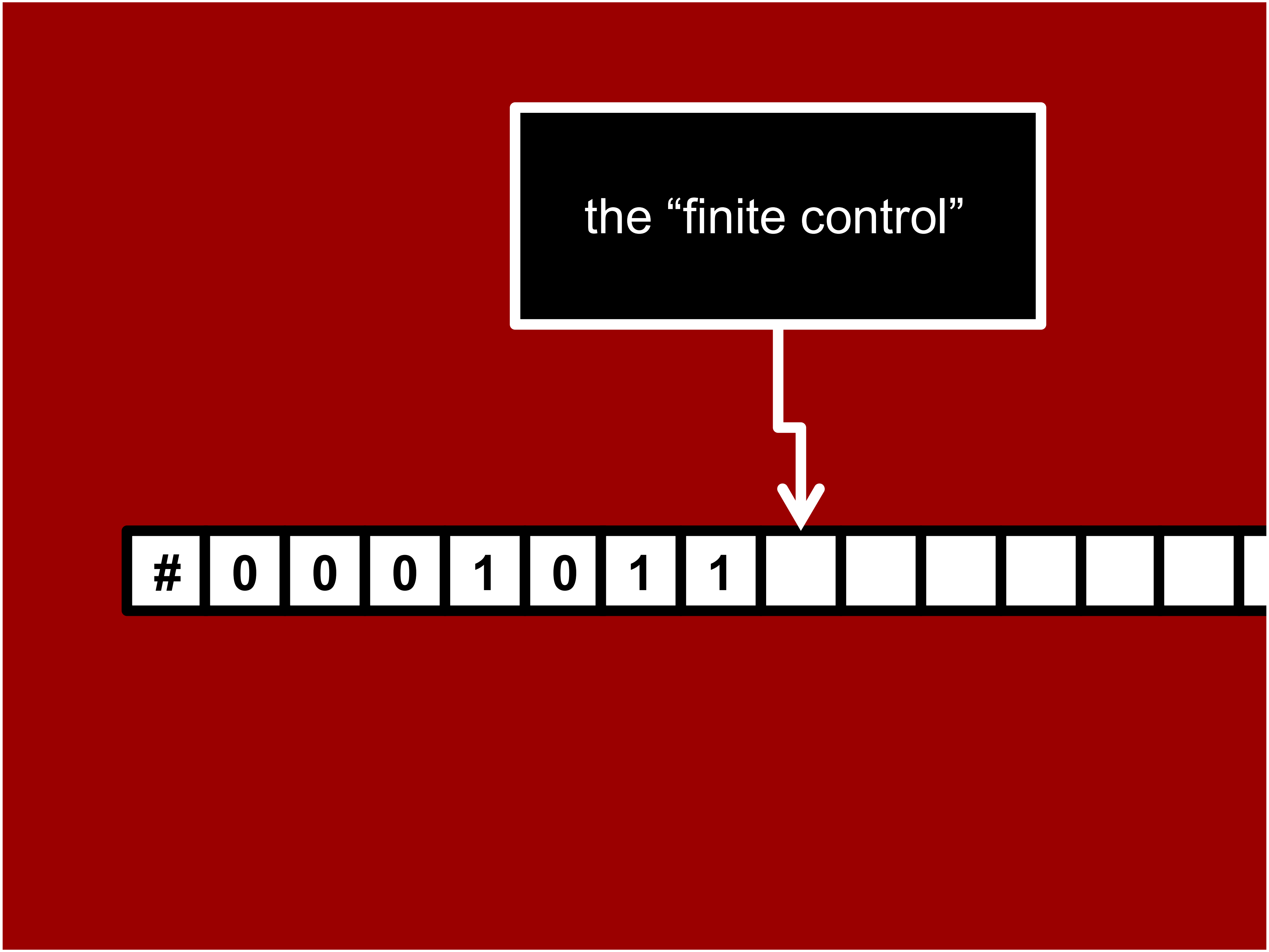
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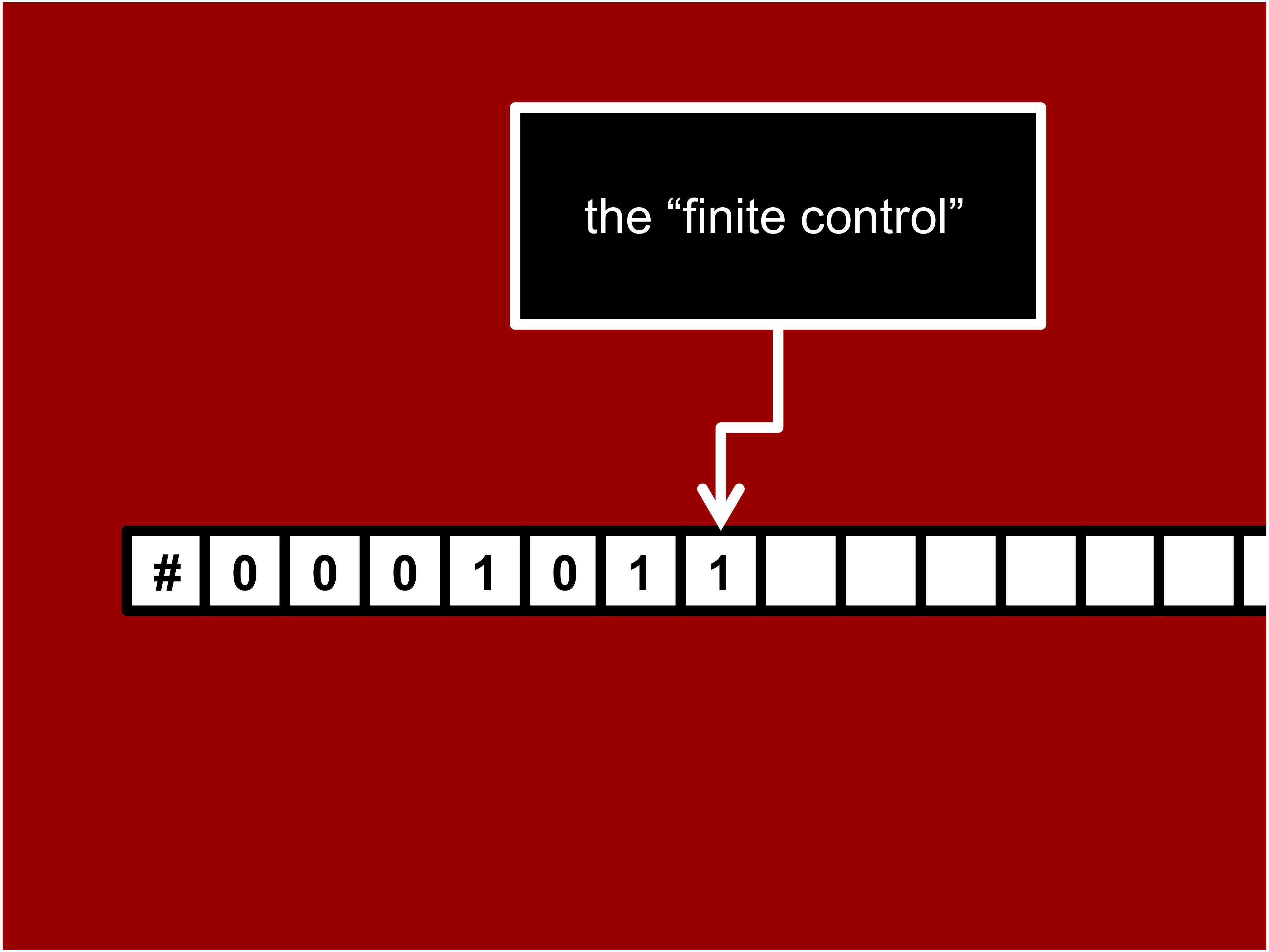
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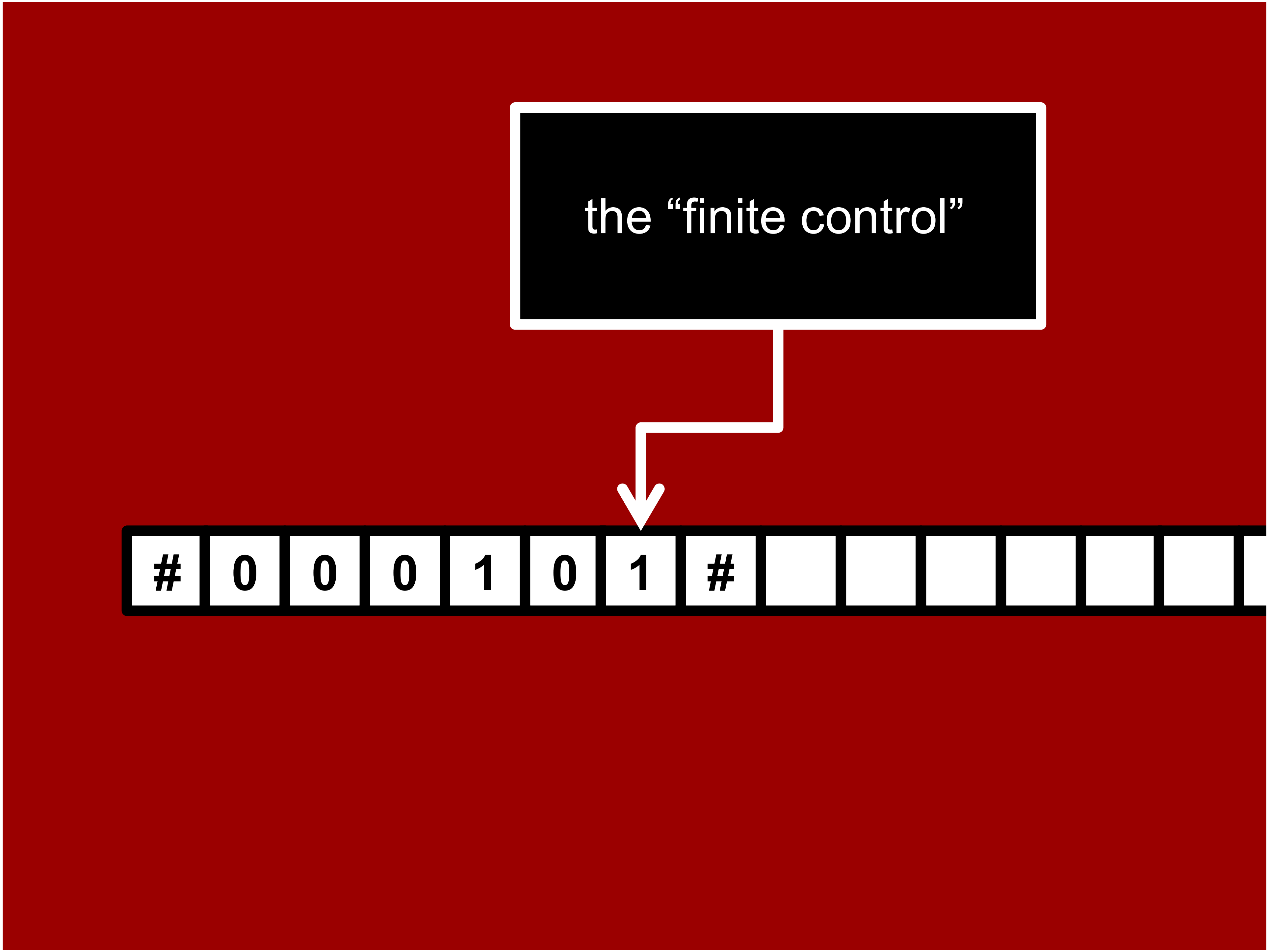
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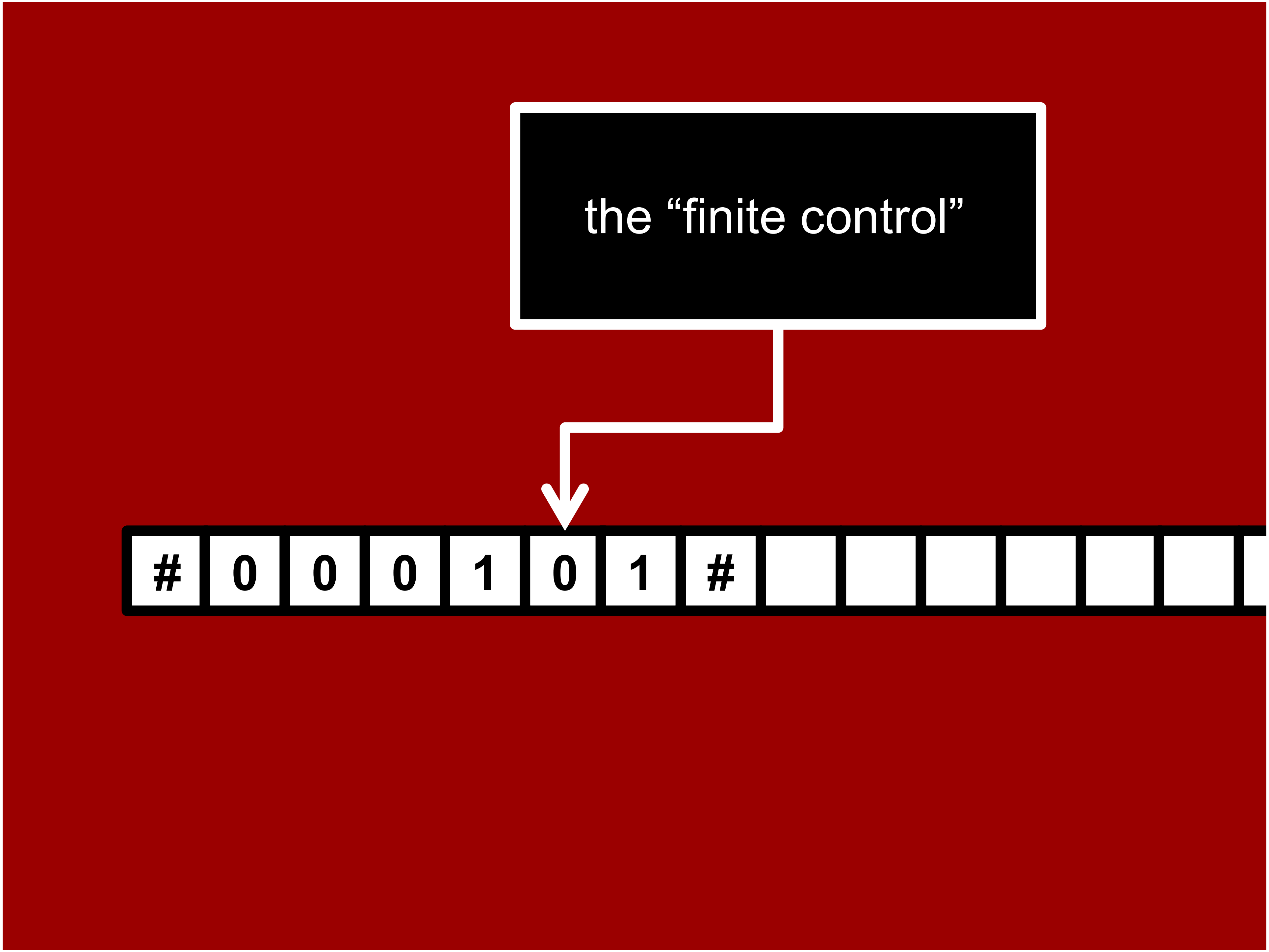
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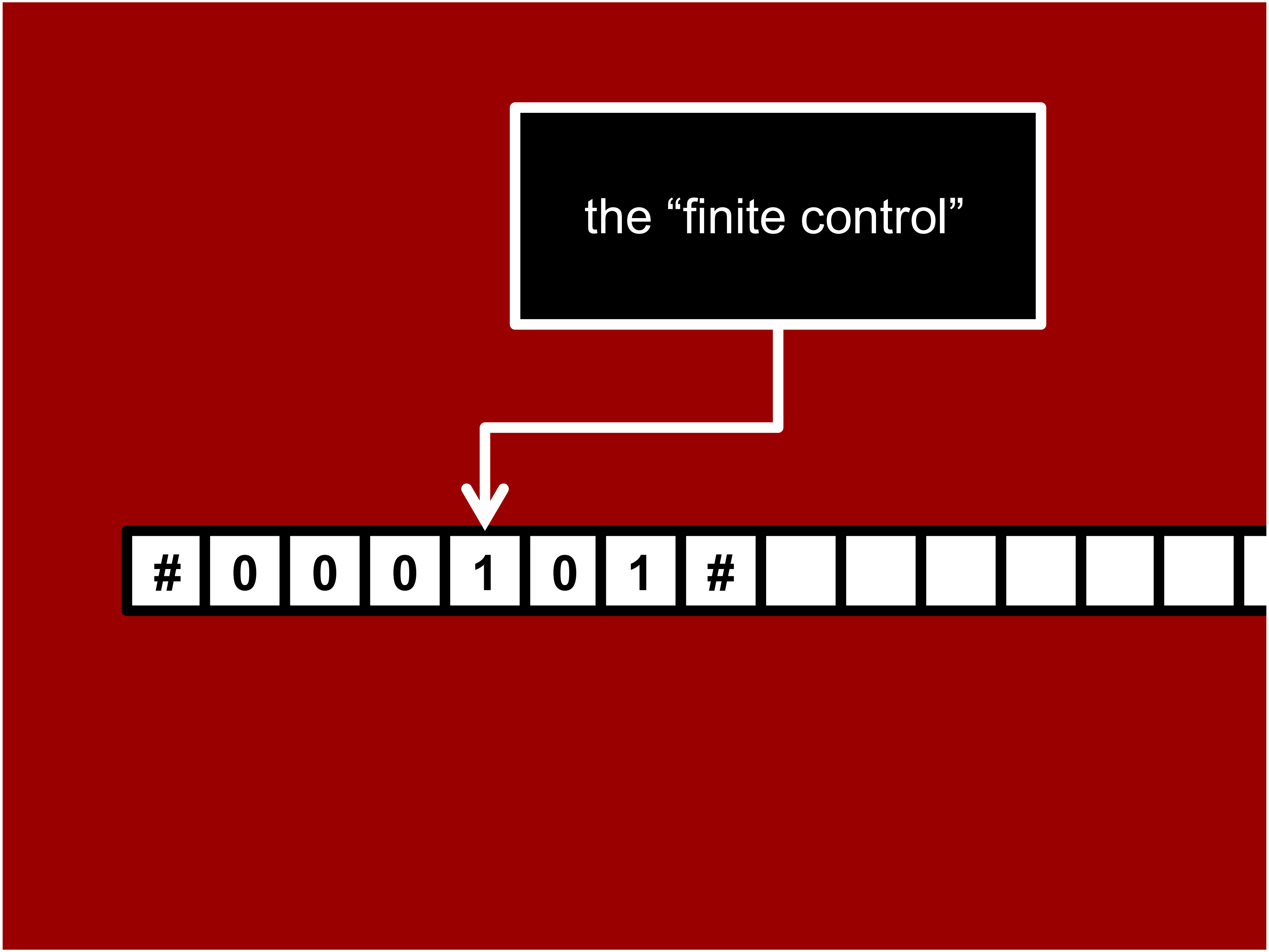
the “finite control”
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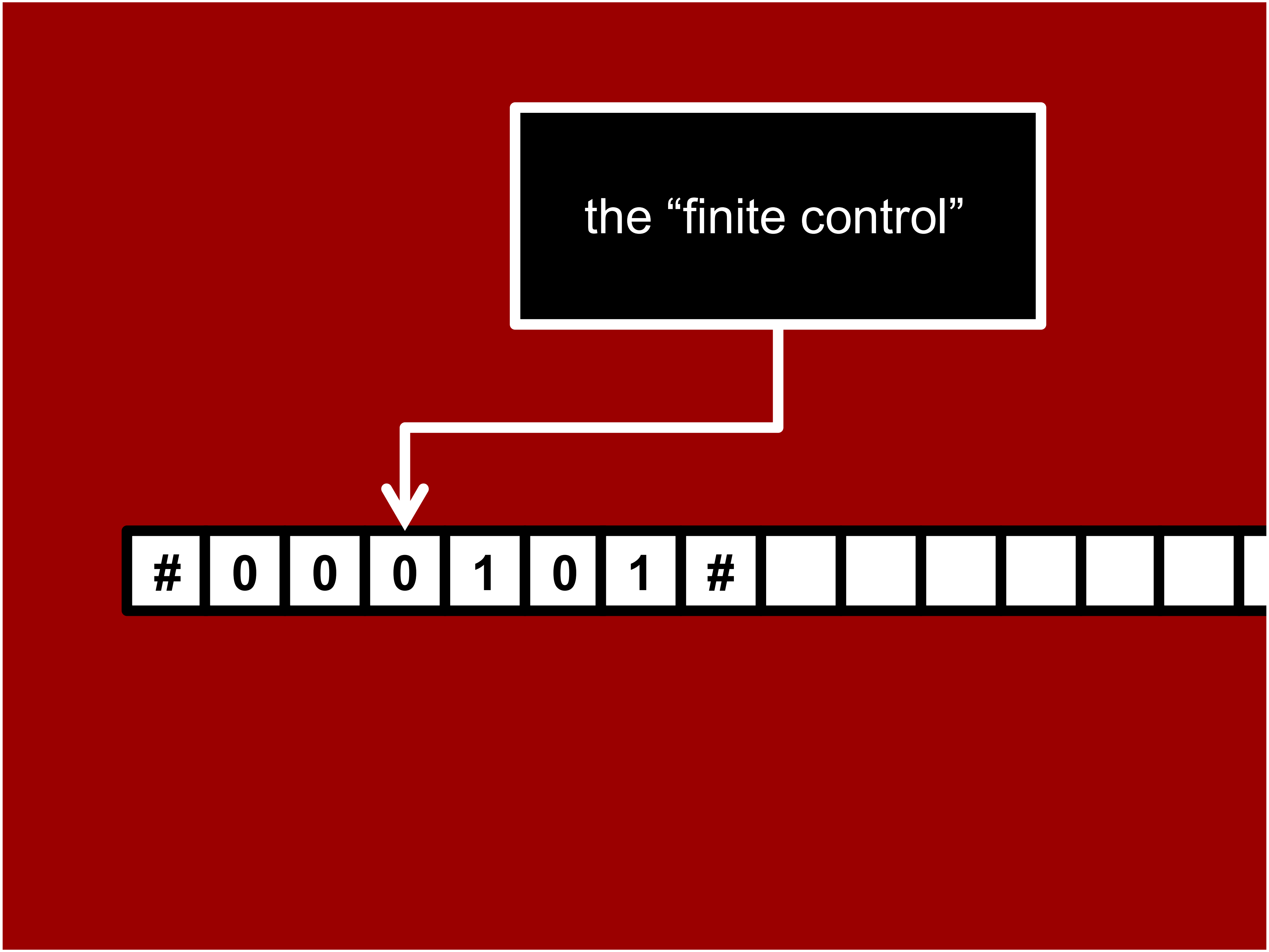
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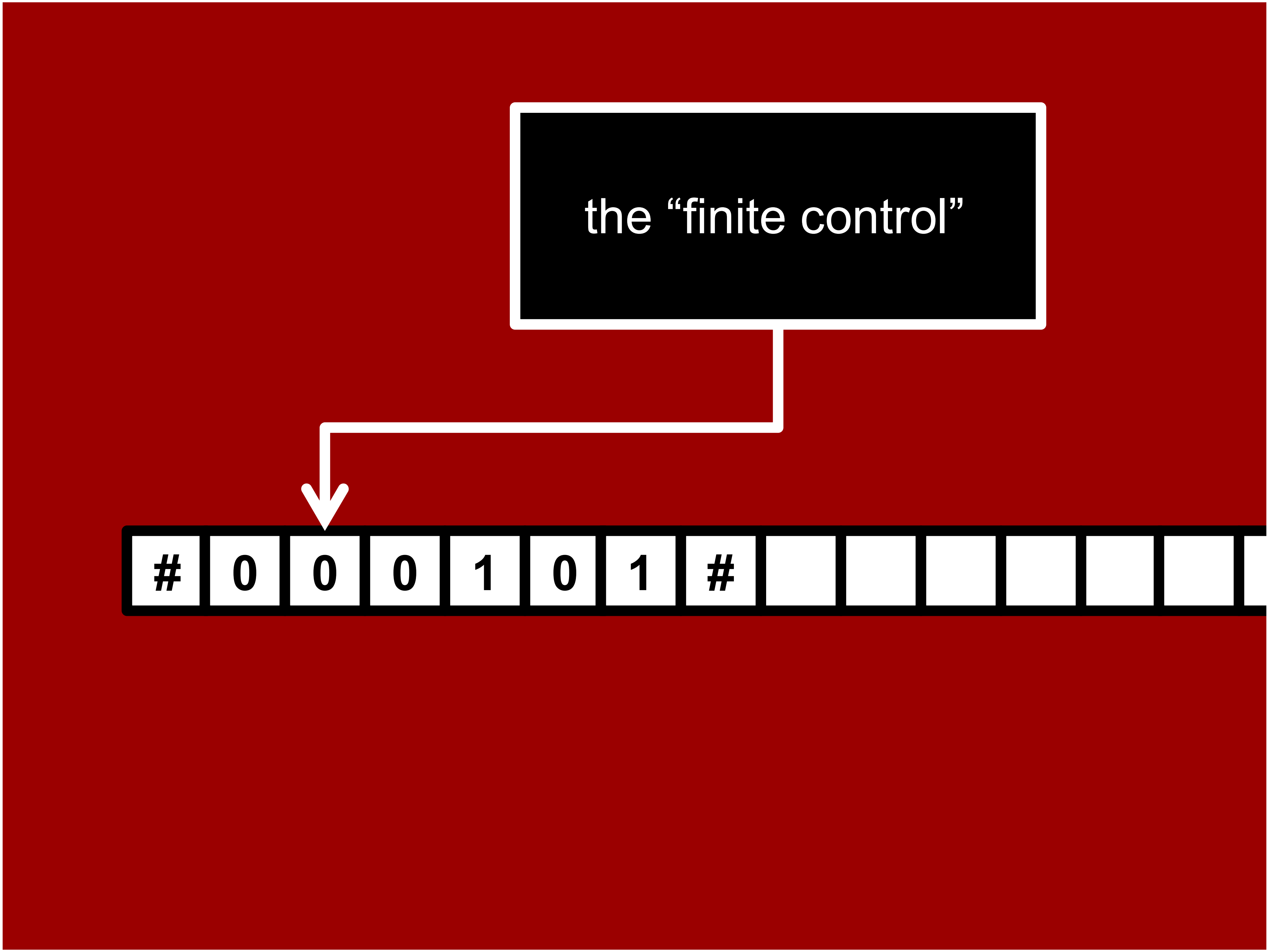
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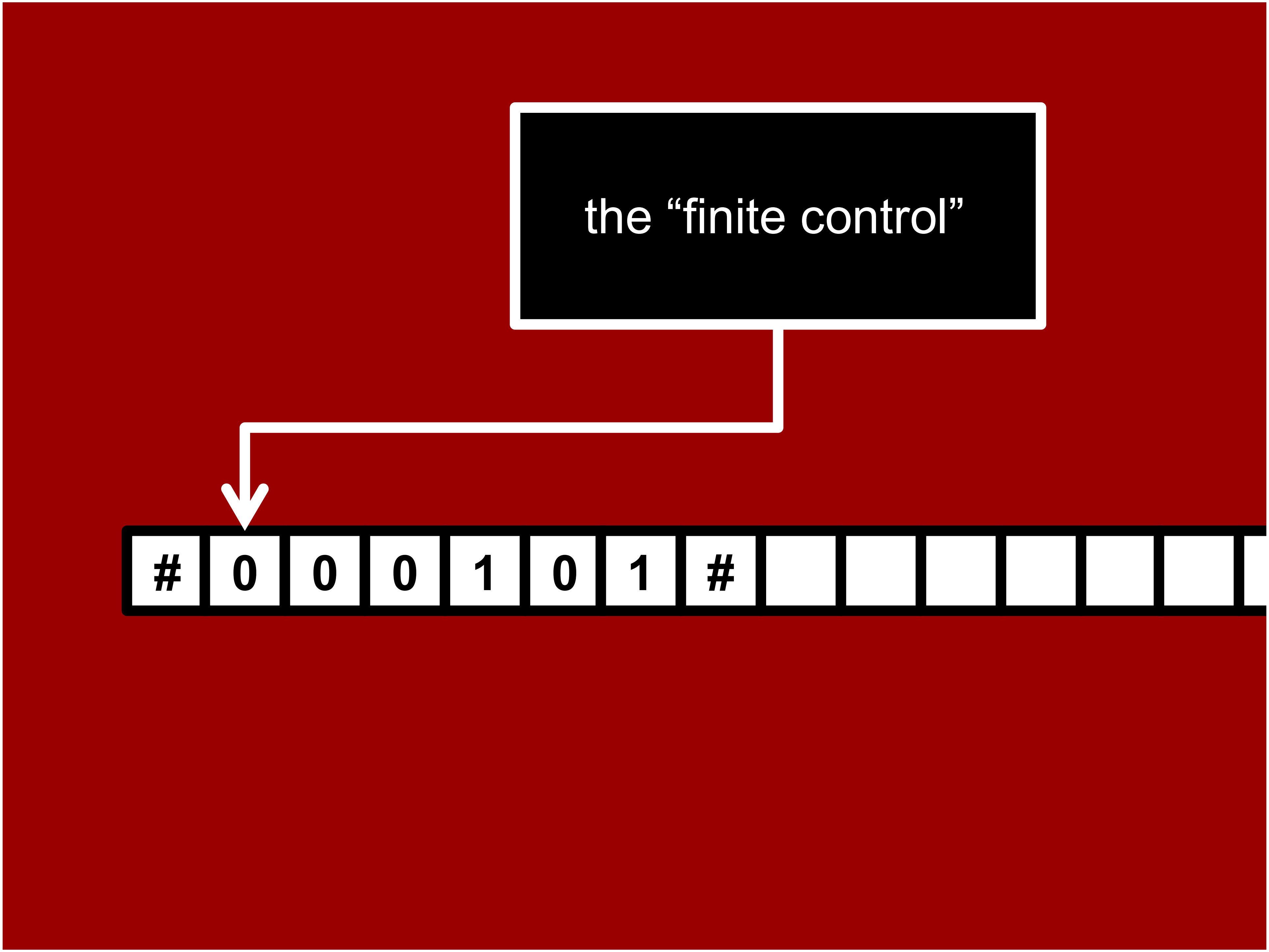
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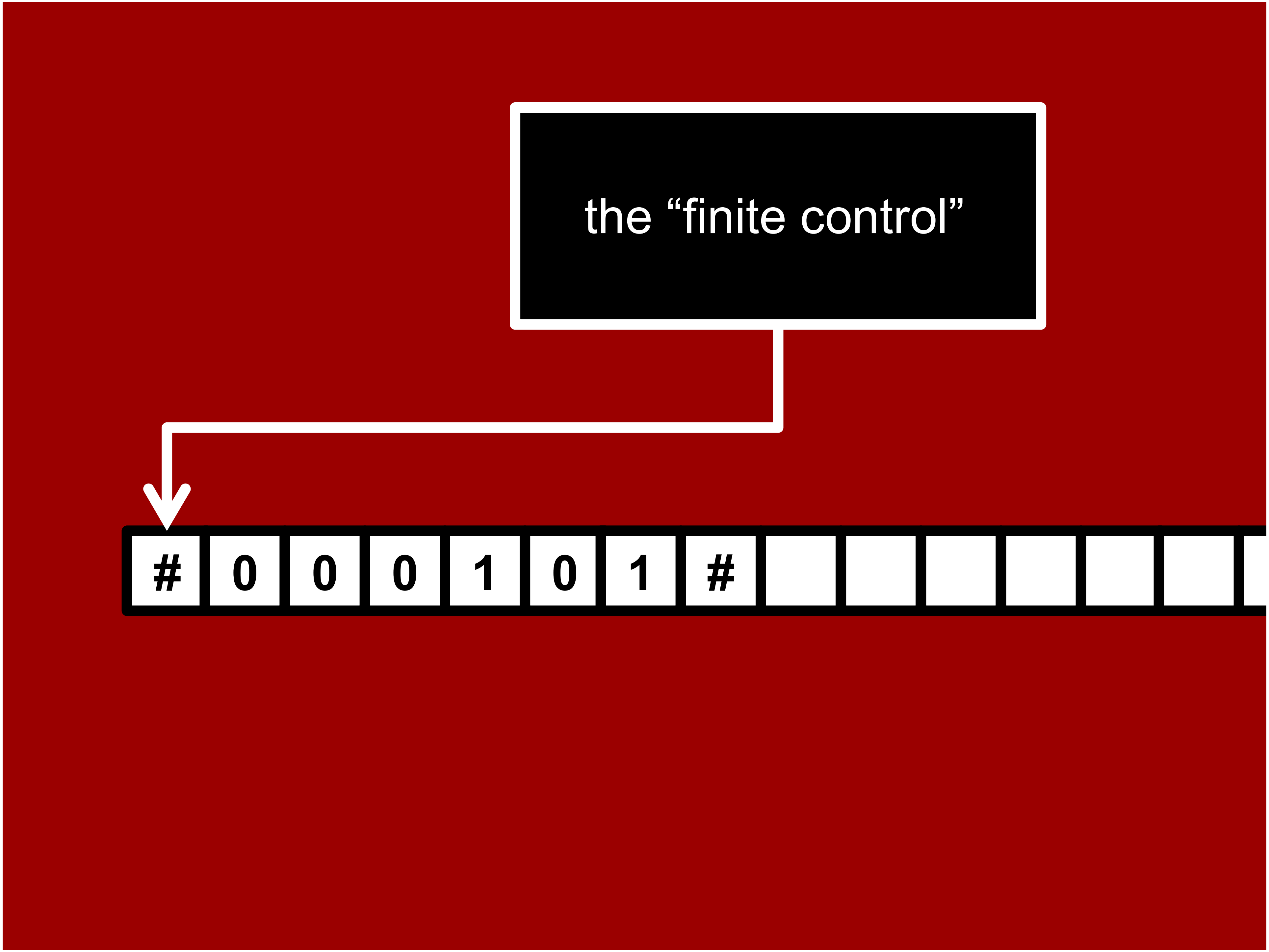
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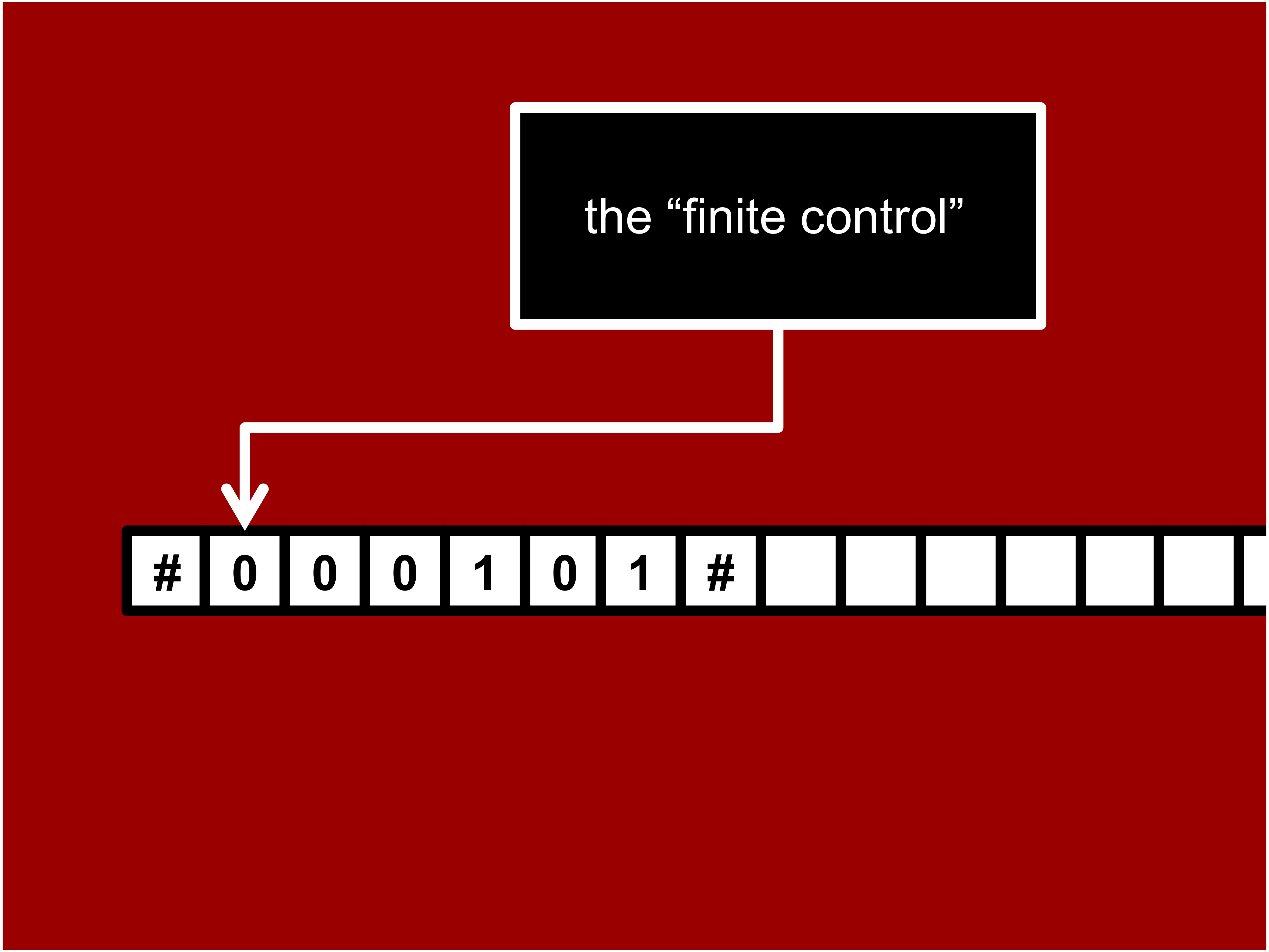
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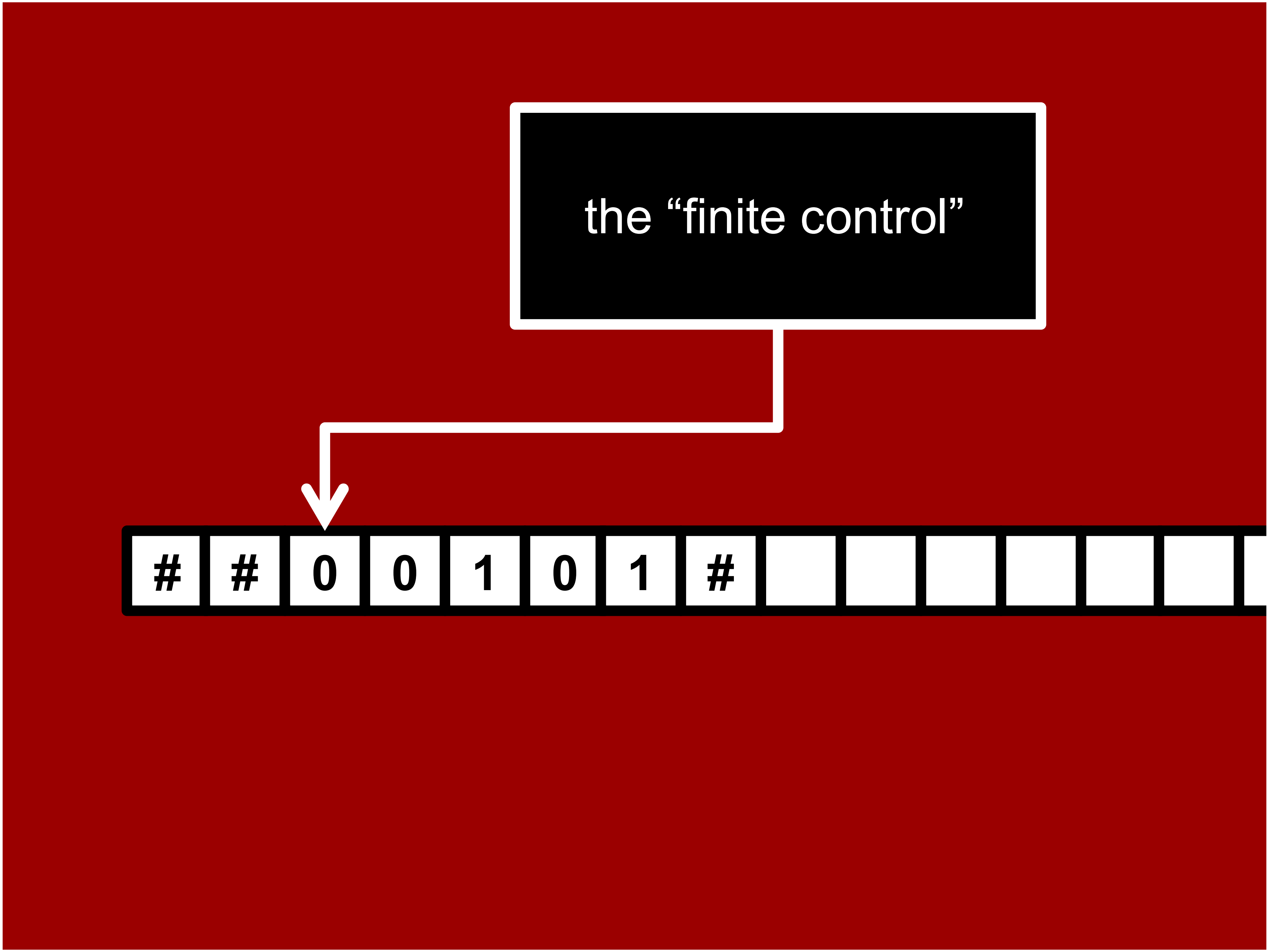

the “finite control”
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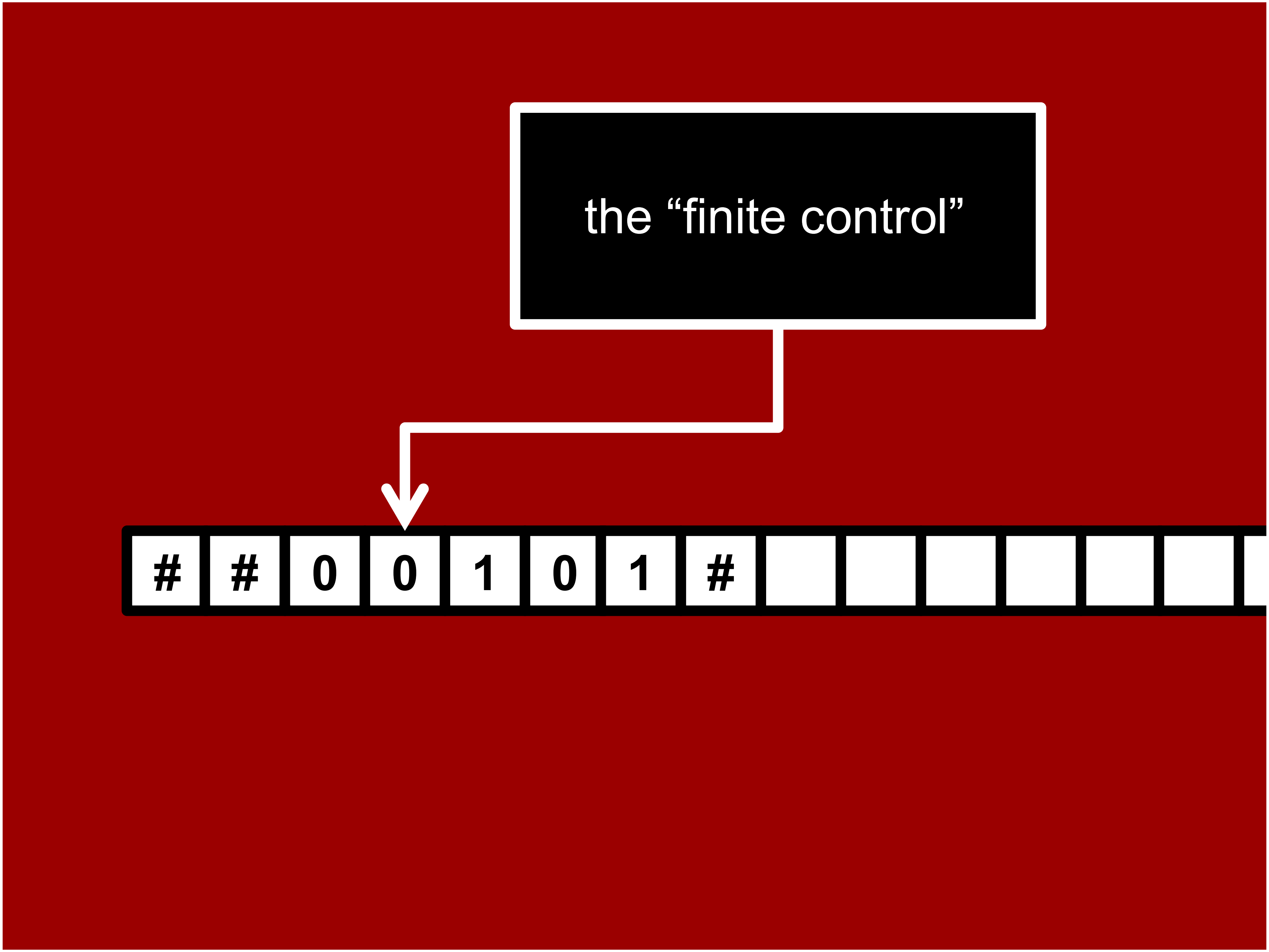
the “finite control”
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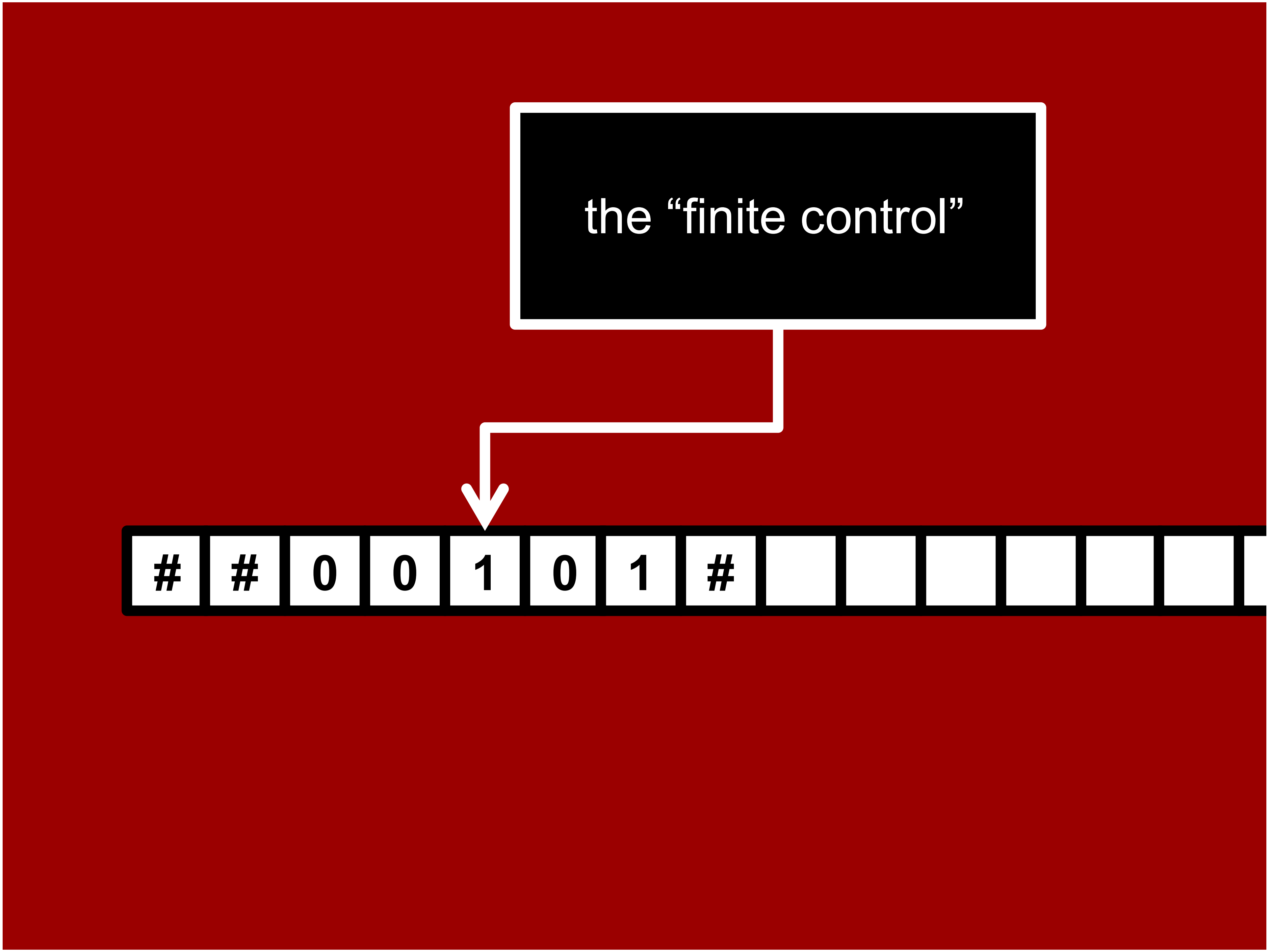
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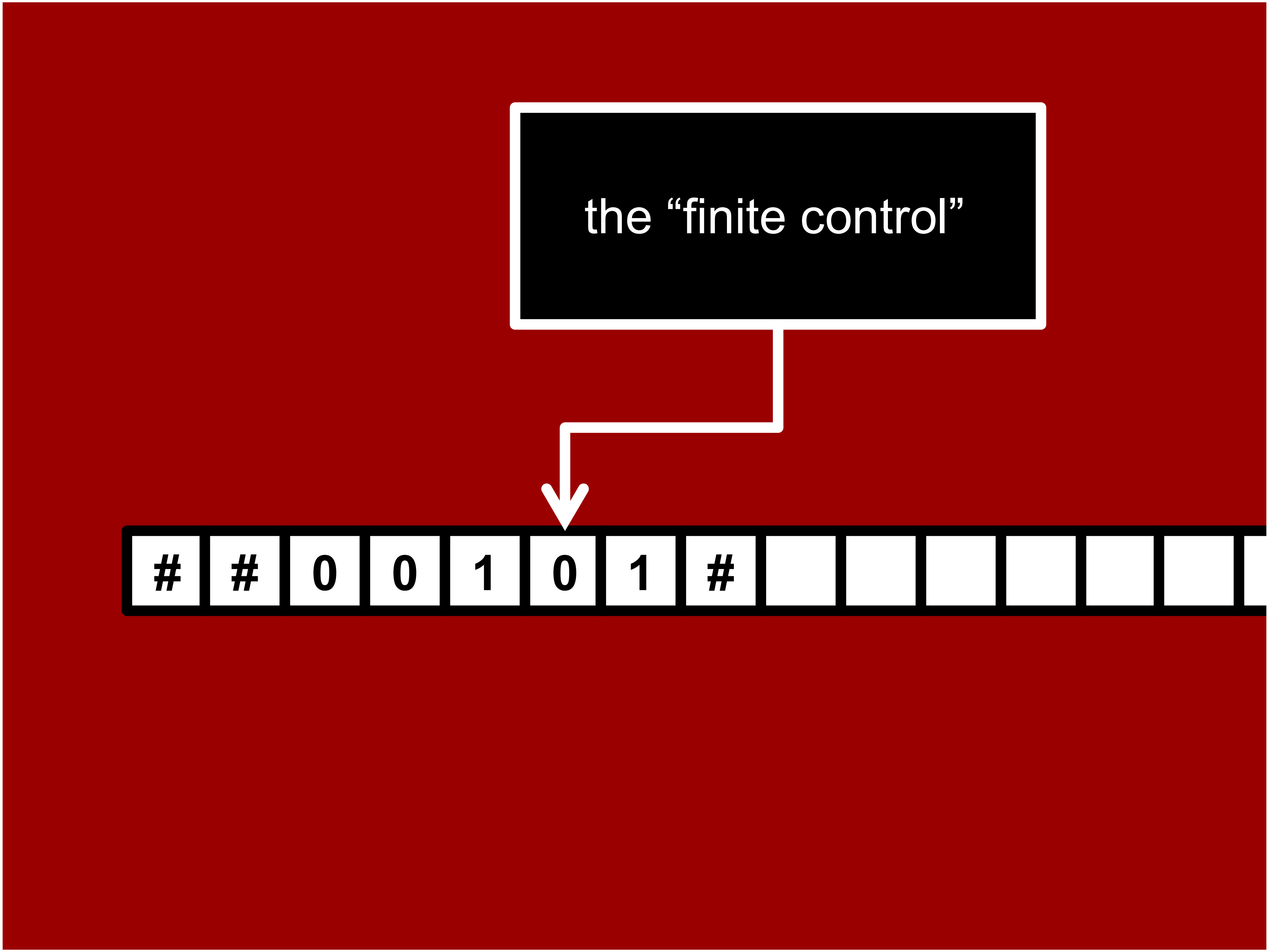
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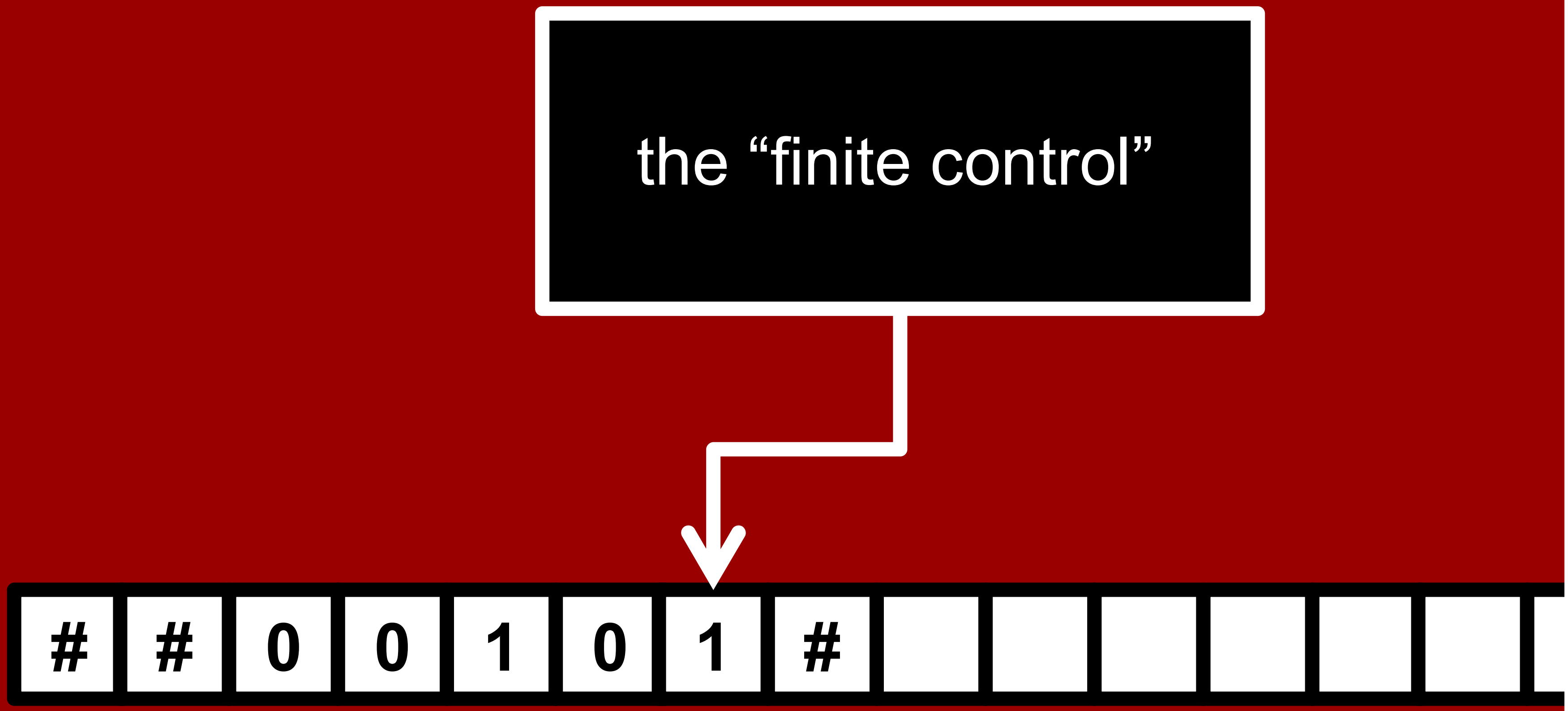
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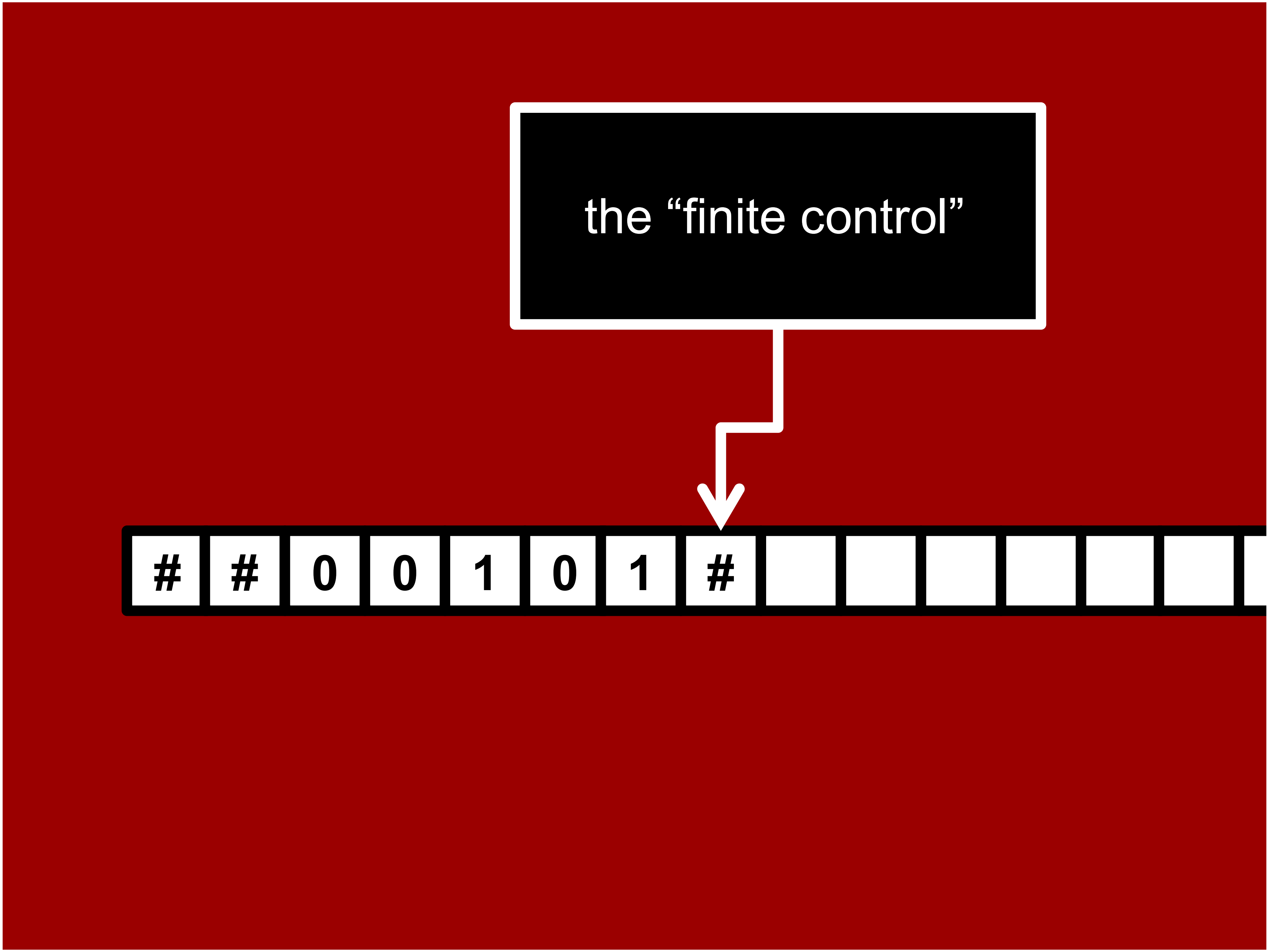

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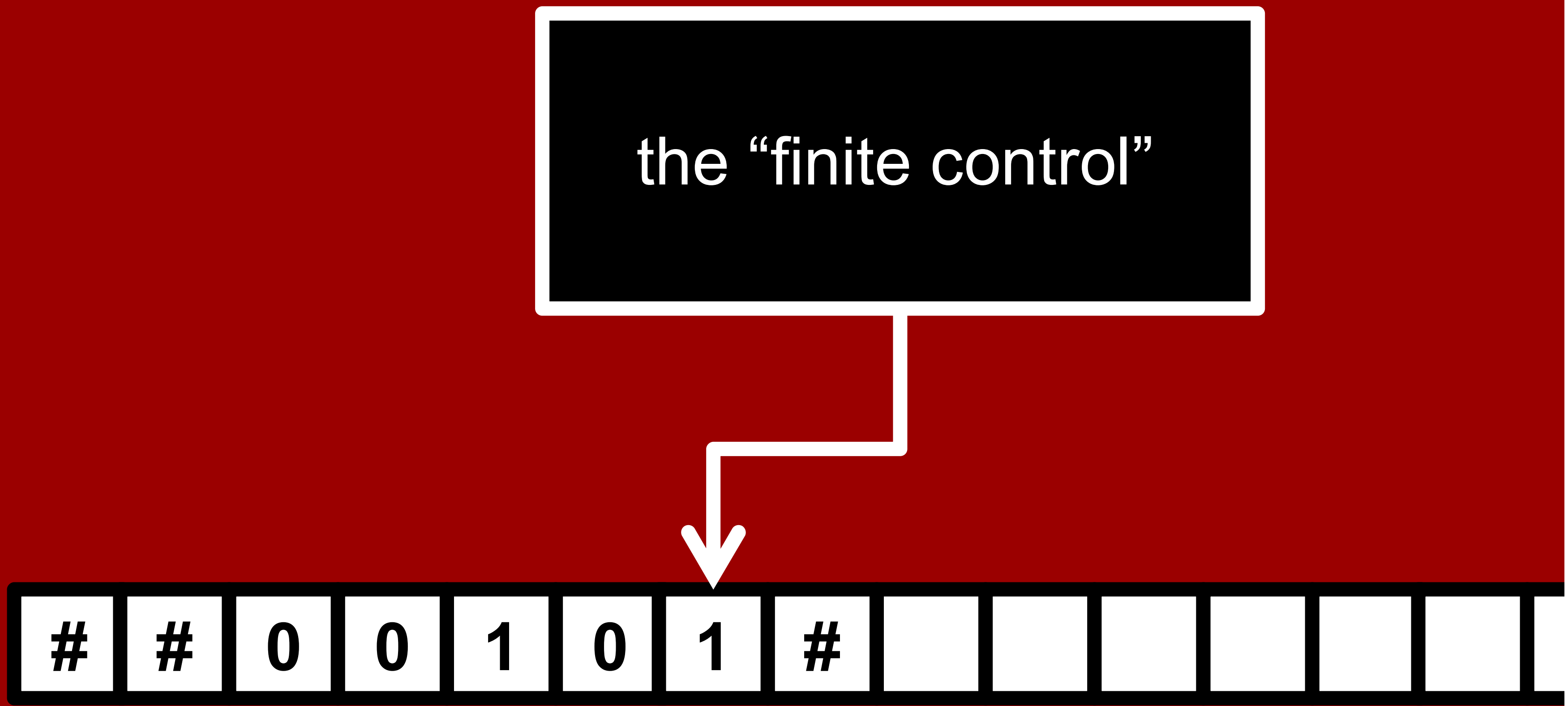
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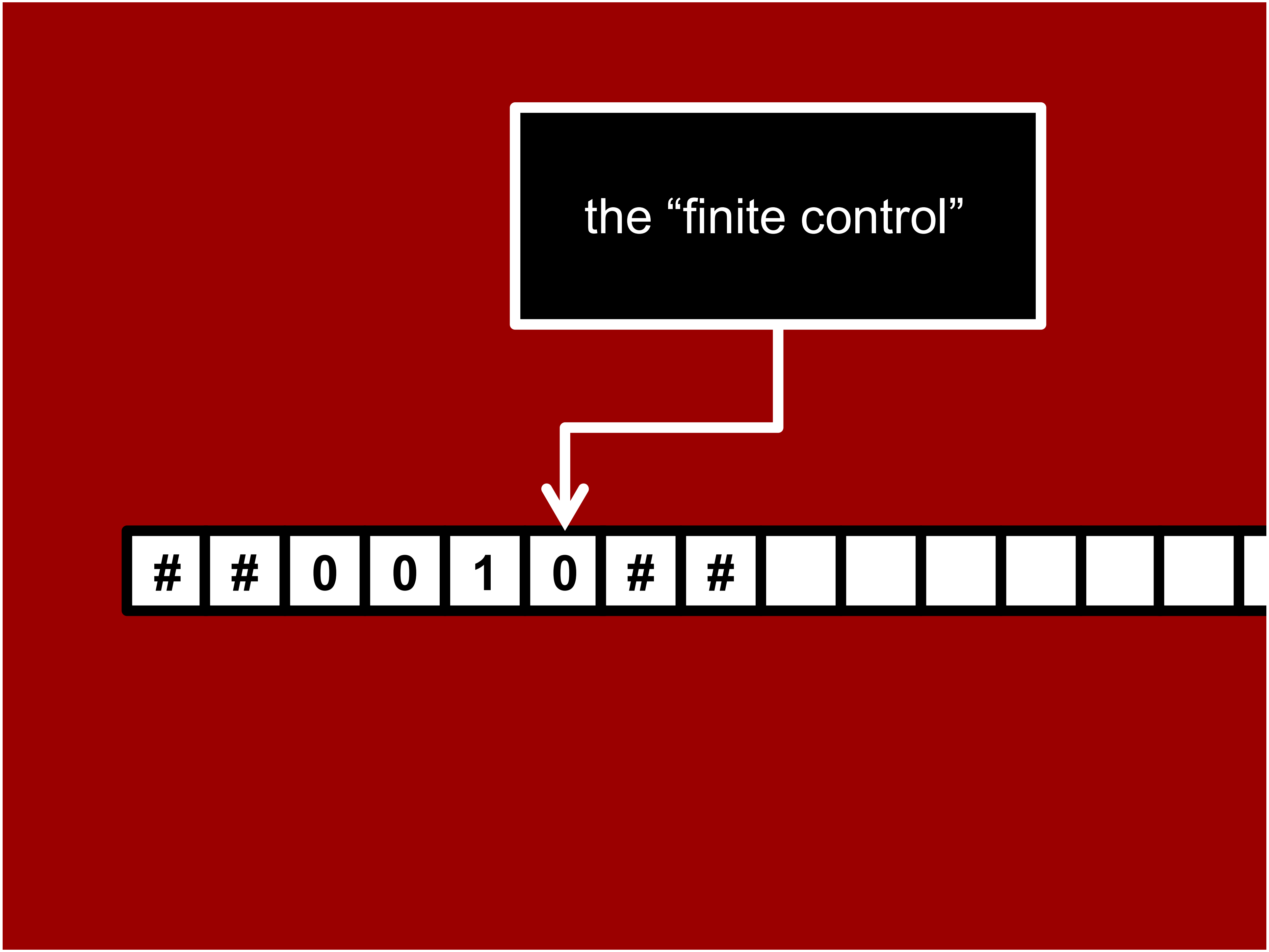
the "finite control"
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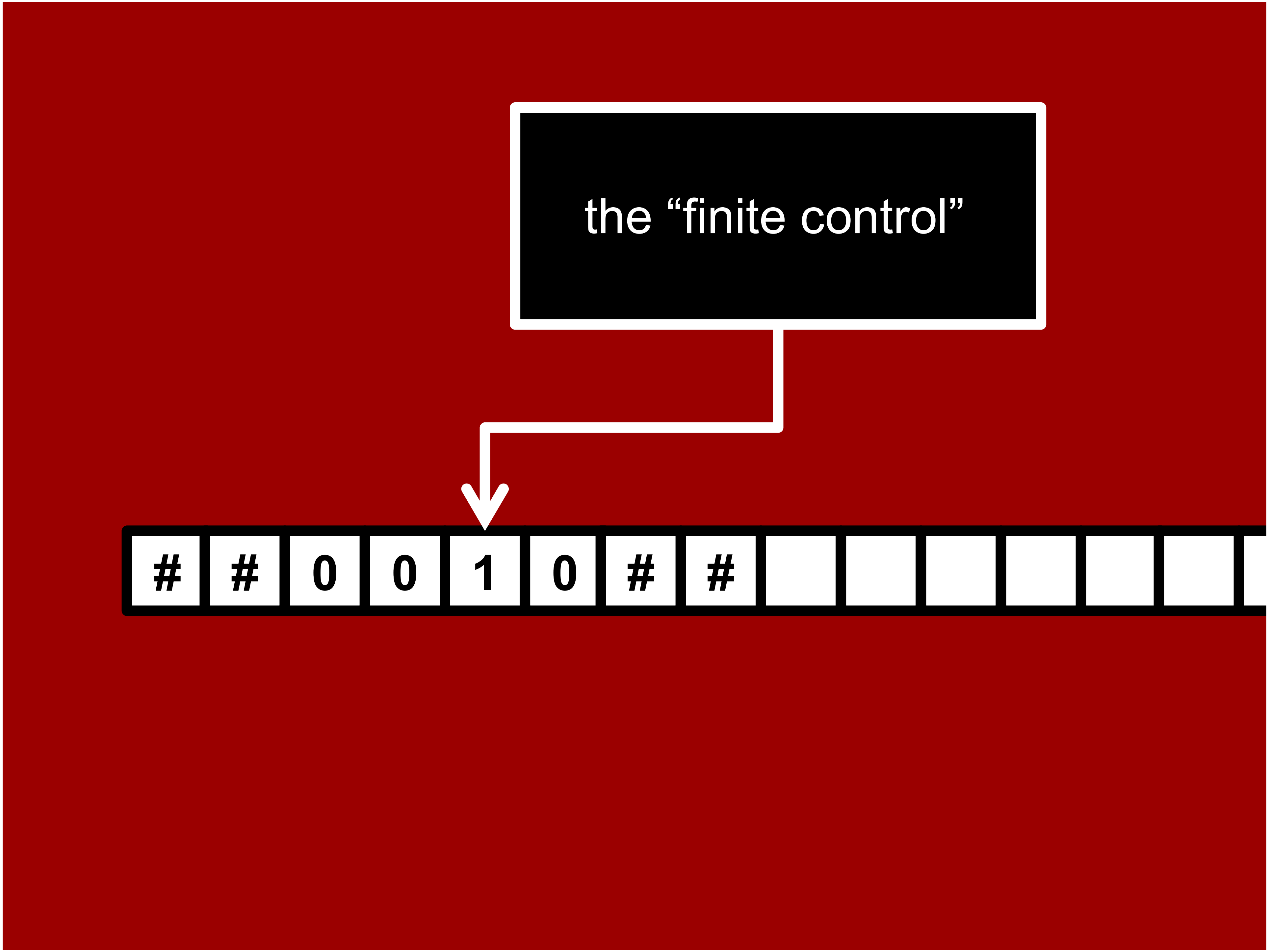
the “finite control”
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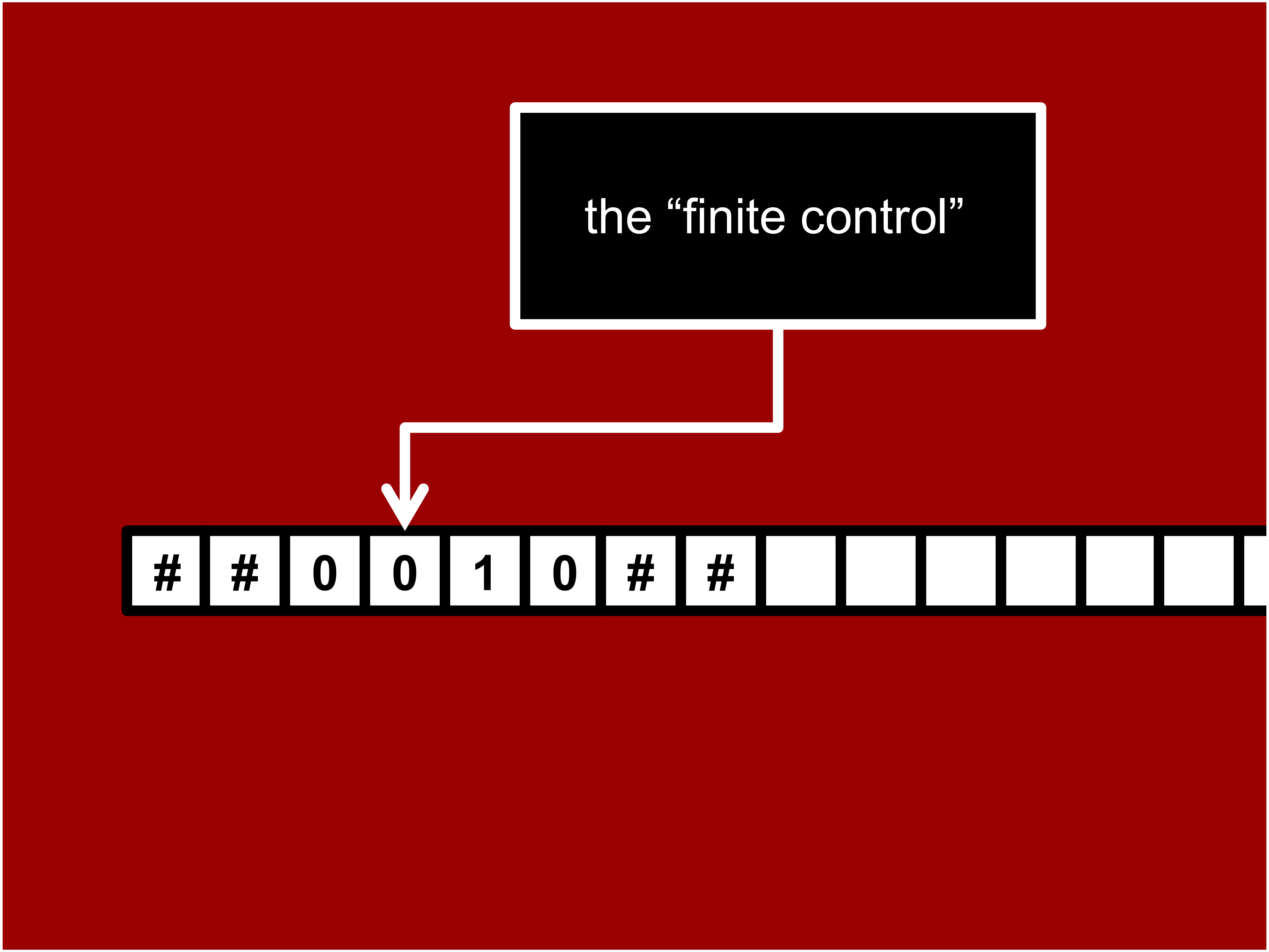

the “finite control”
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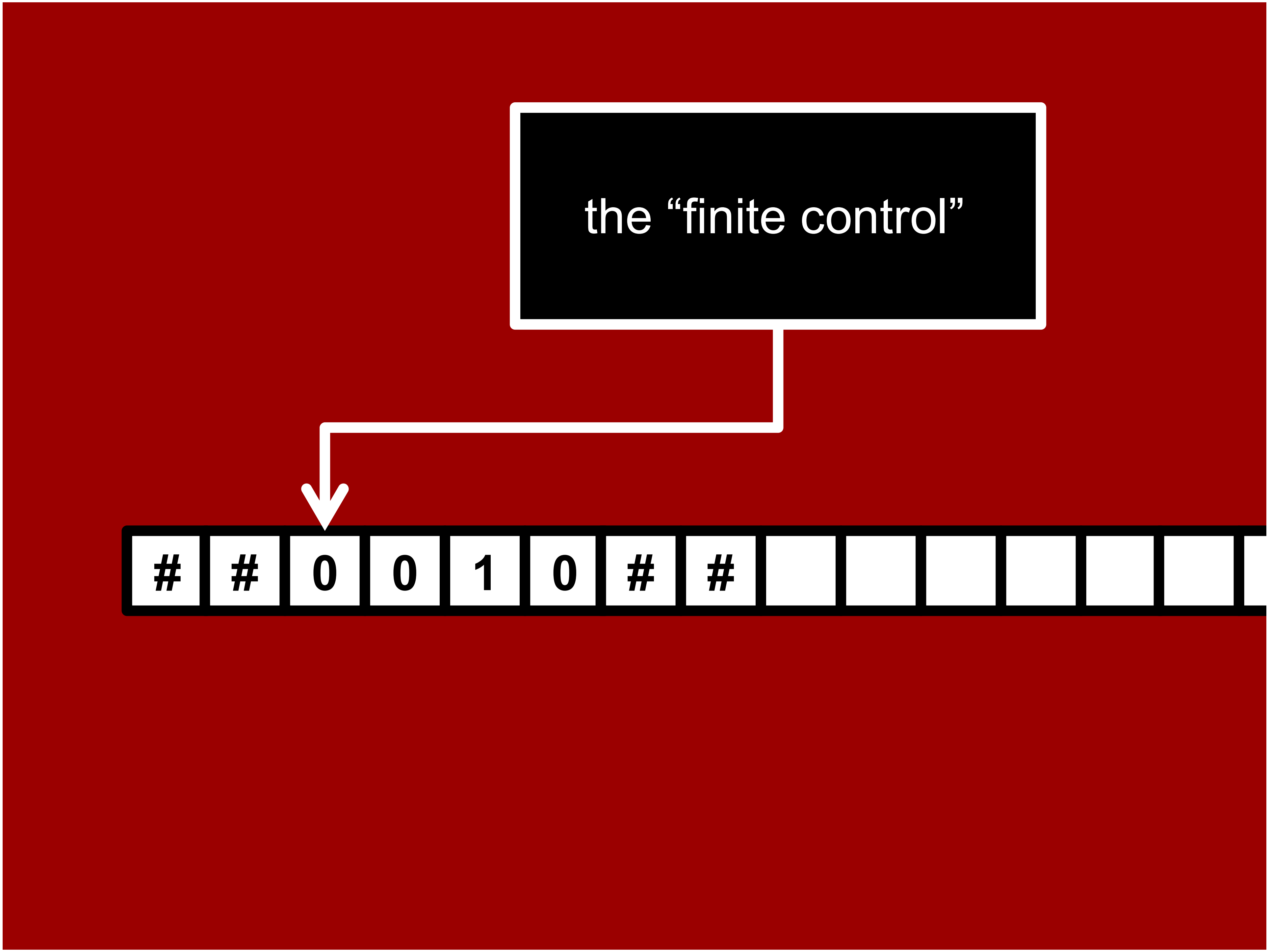
the "finite control"
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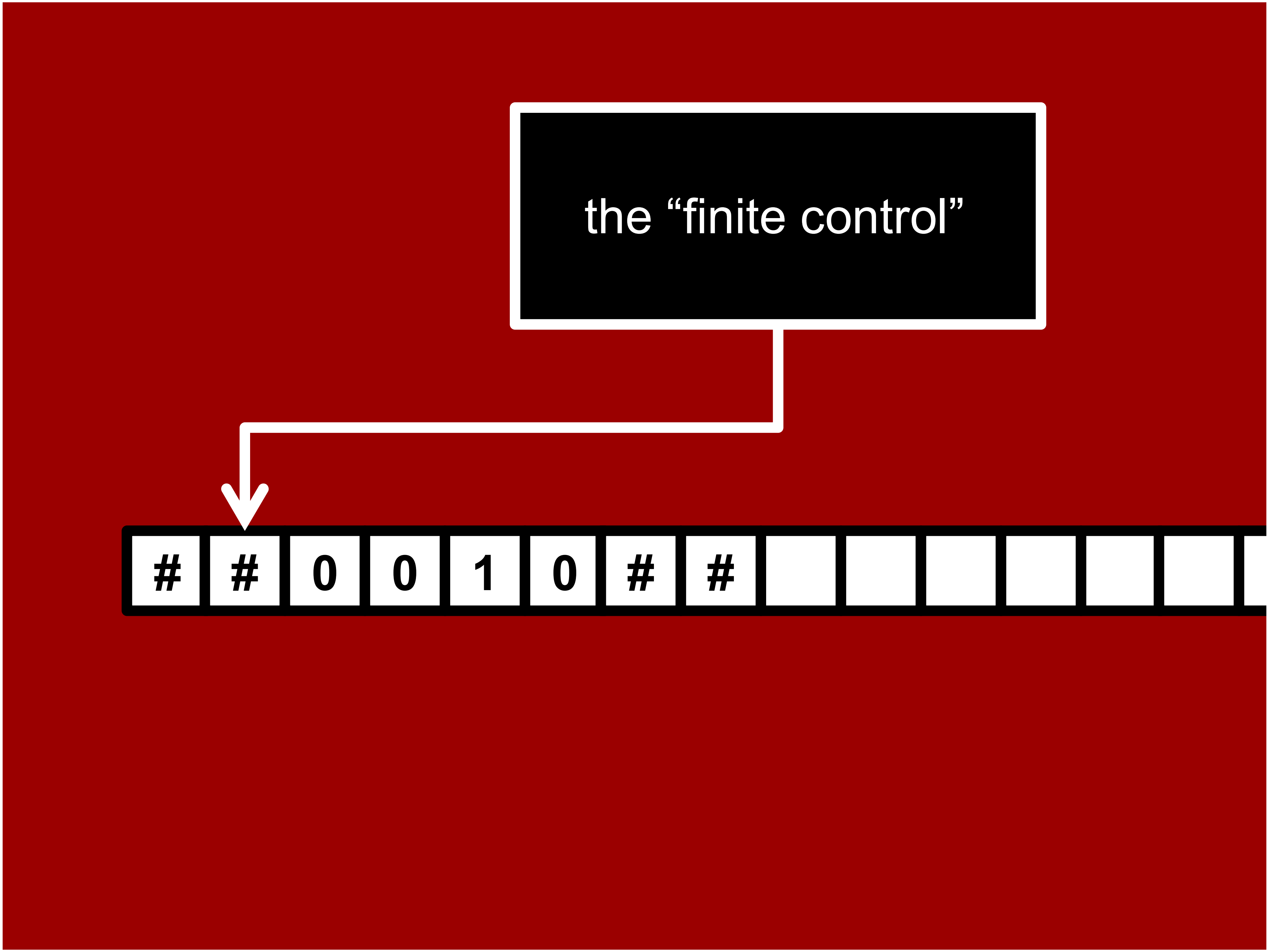
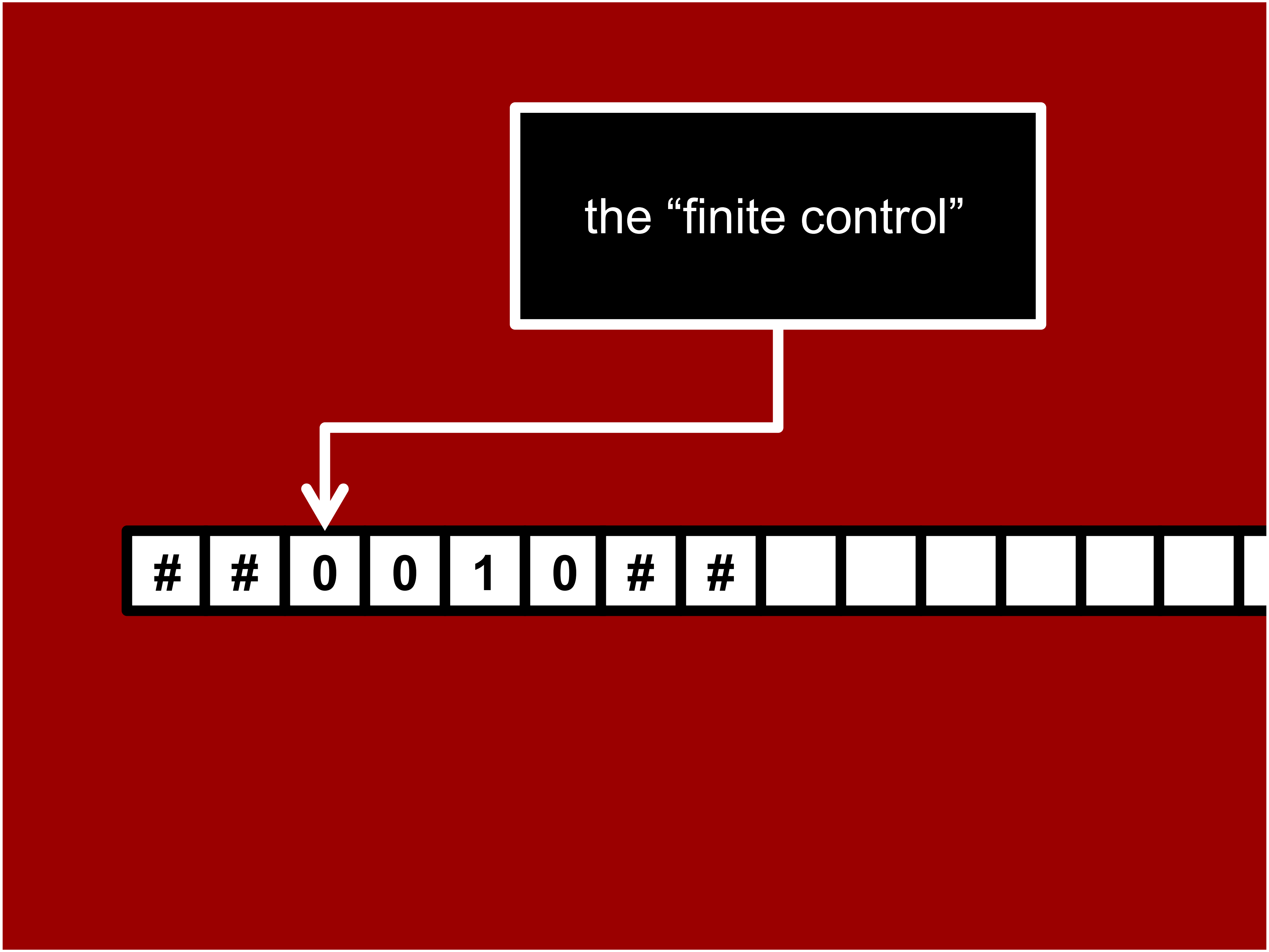

the “finite control”
0 0 1 0 #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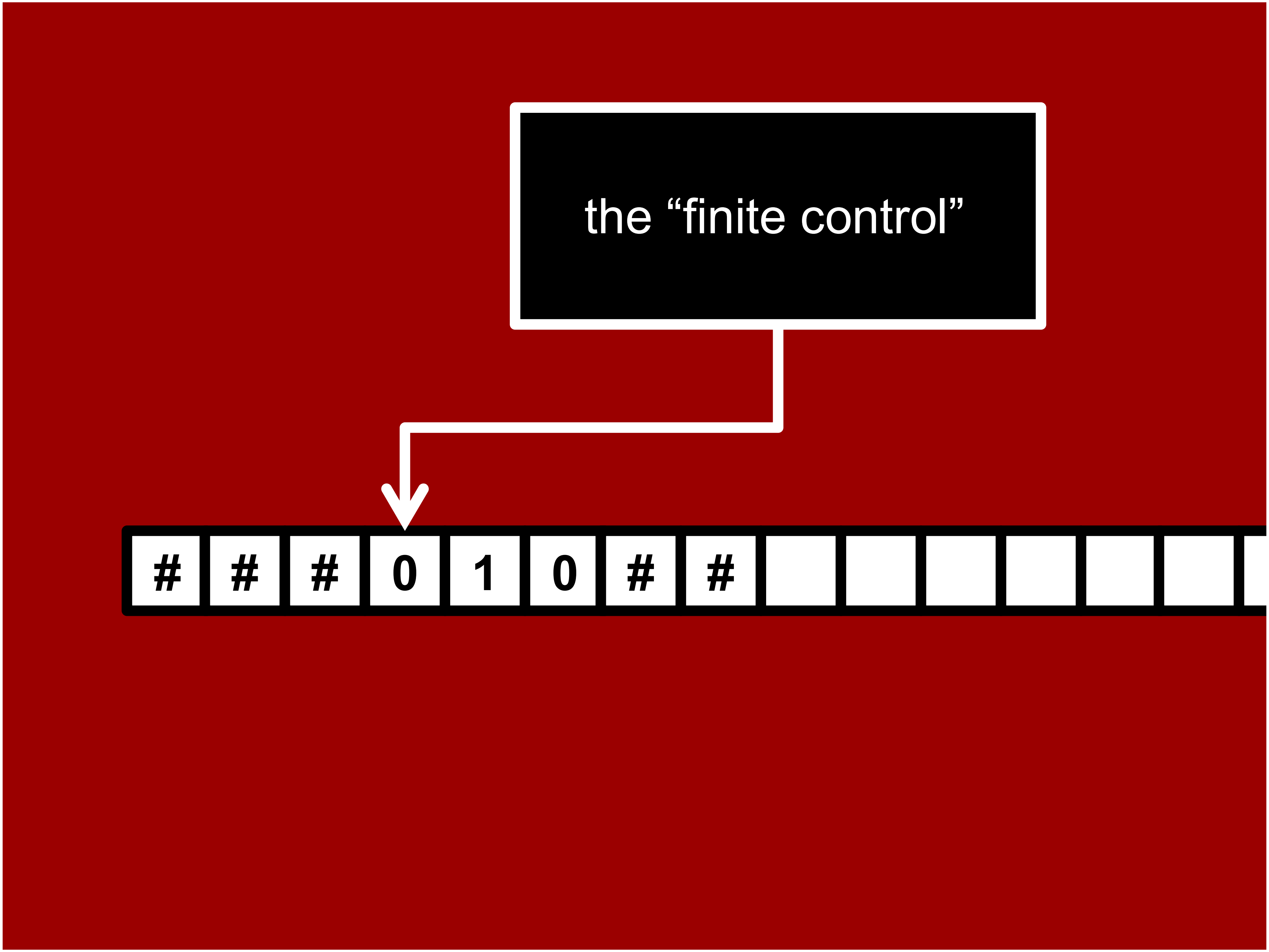
the “finite control”
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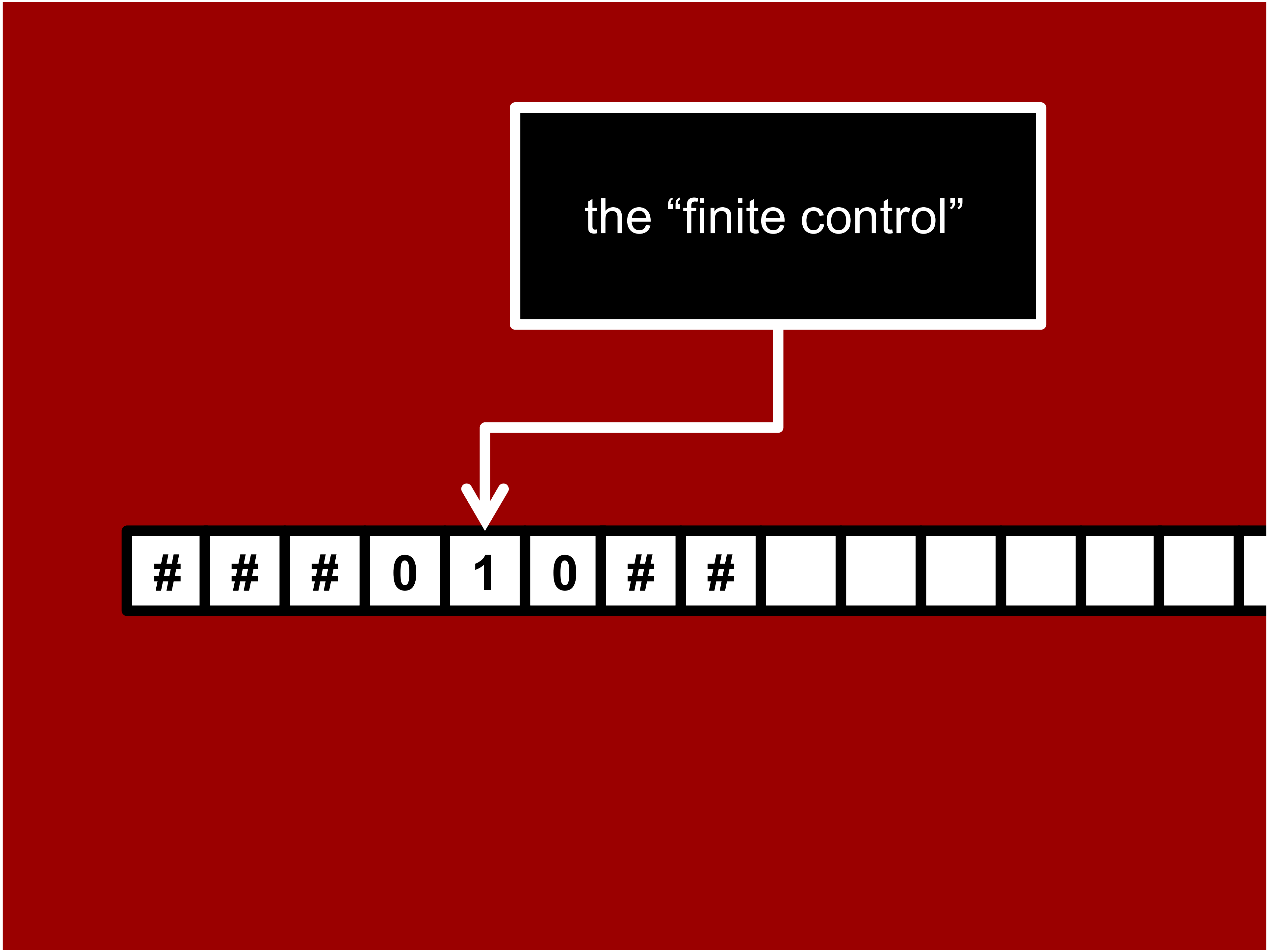
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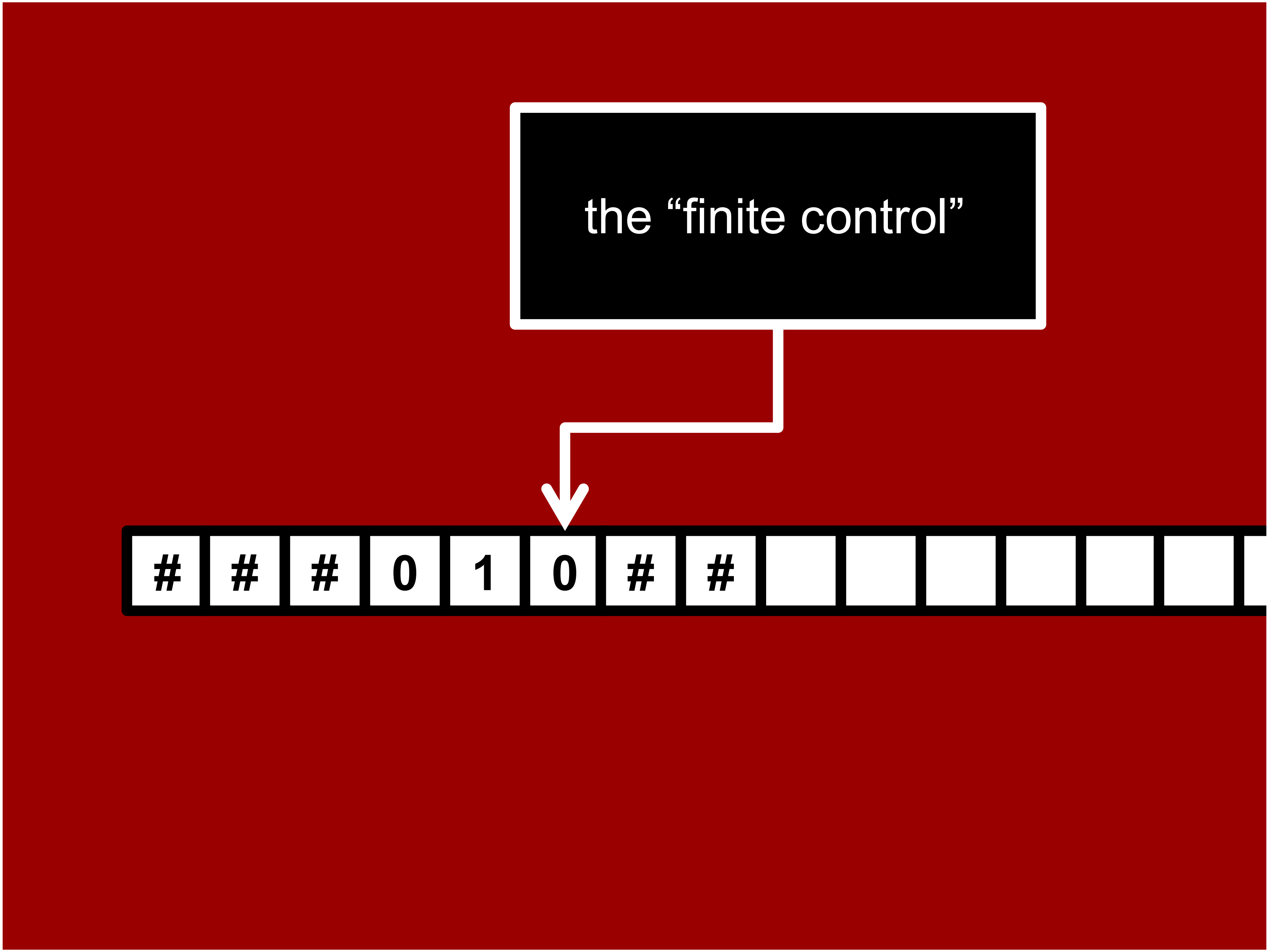
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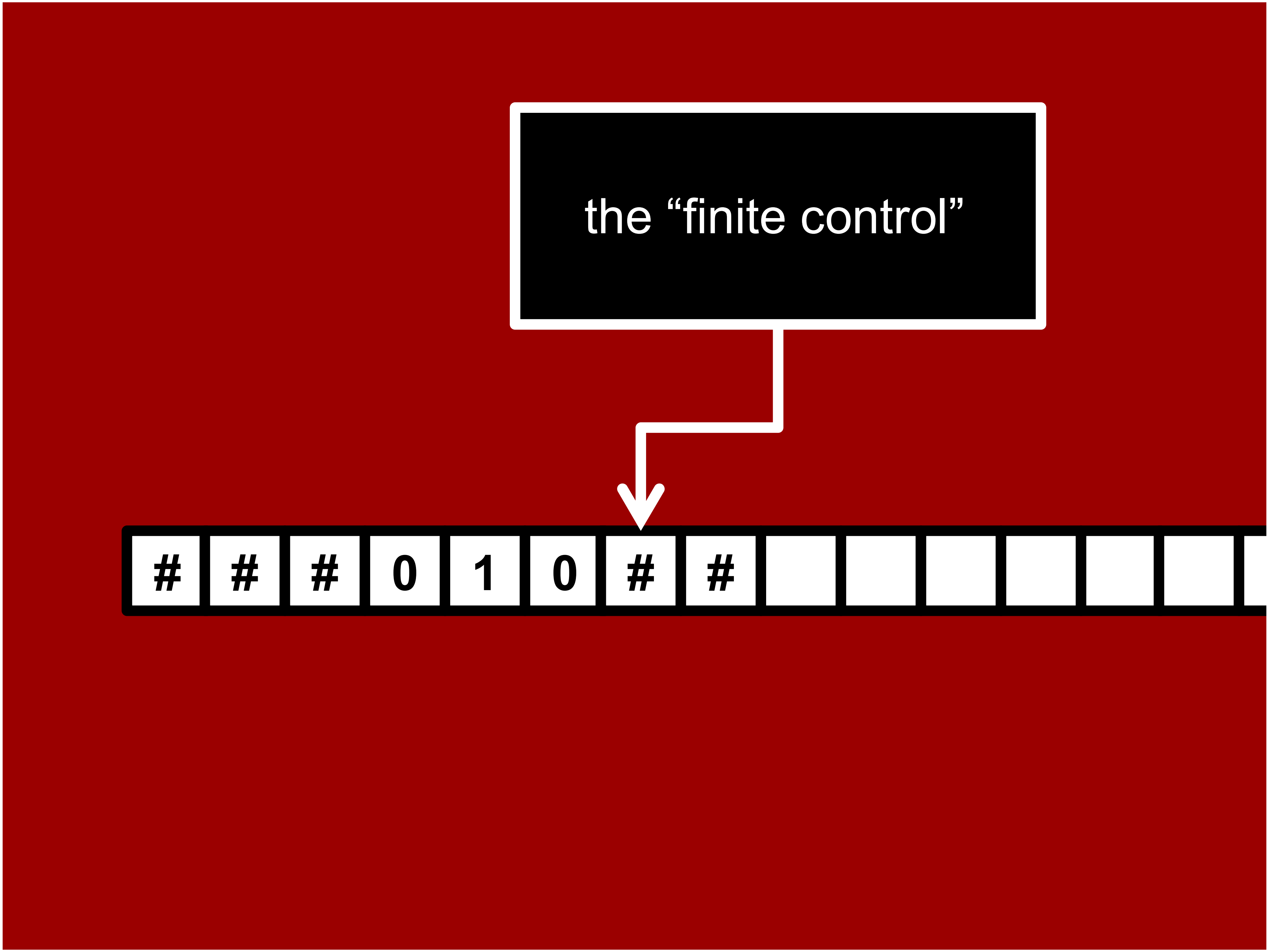

the “finite control”
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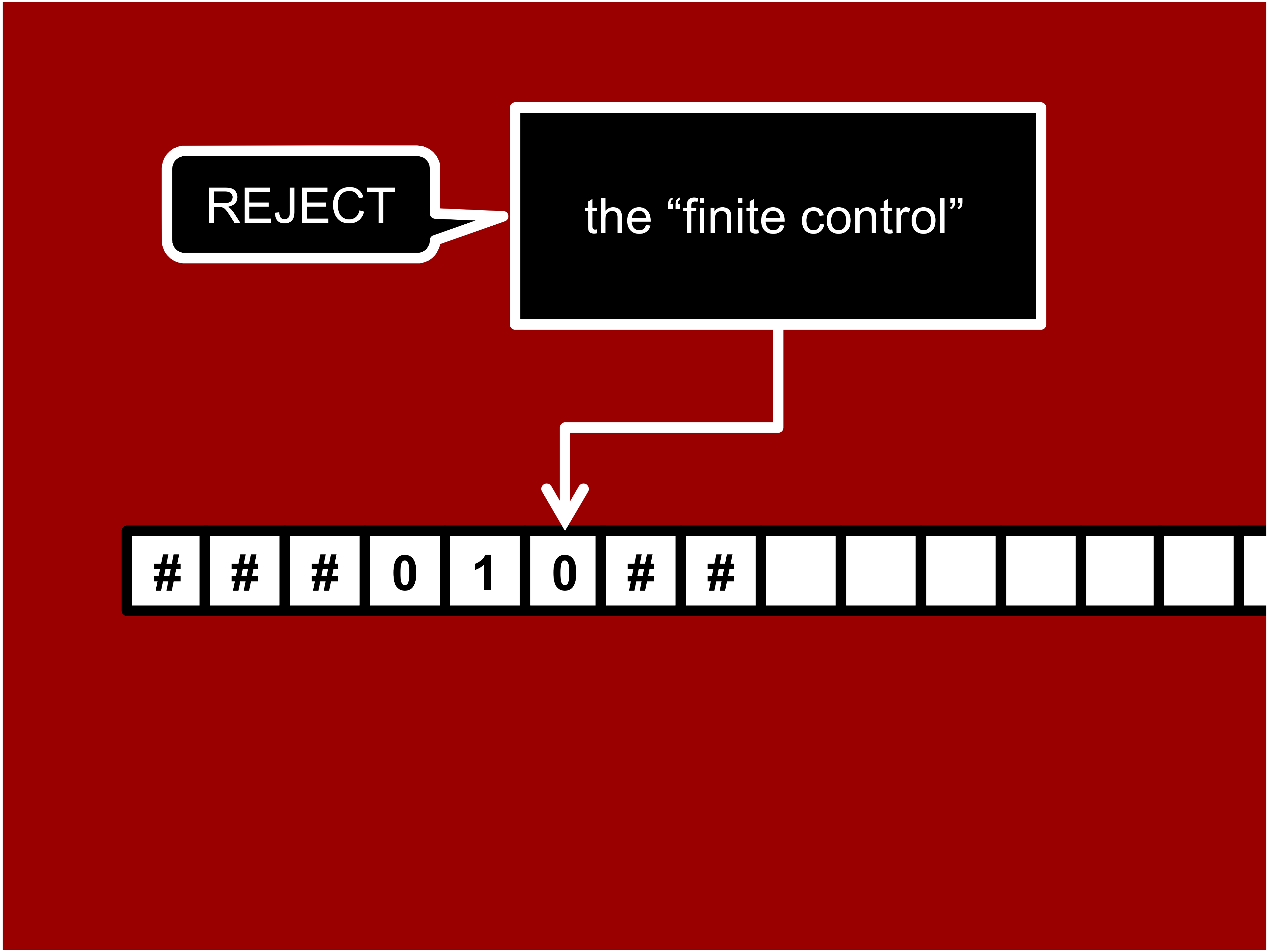
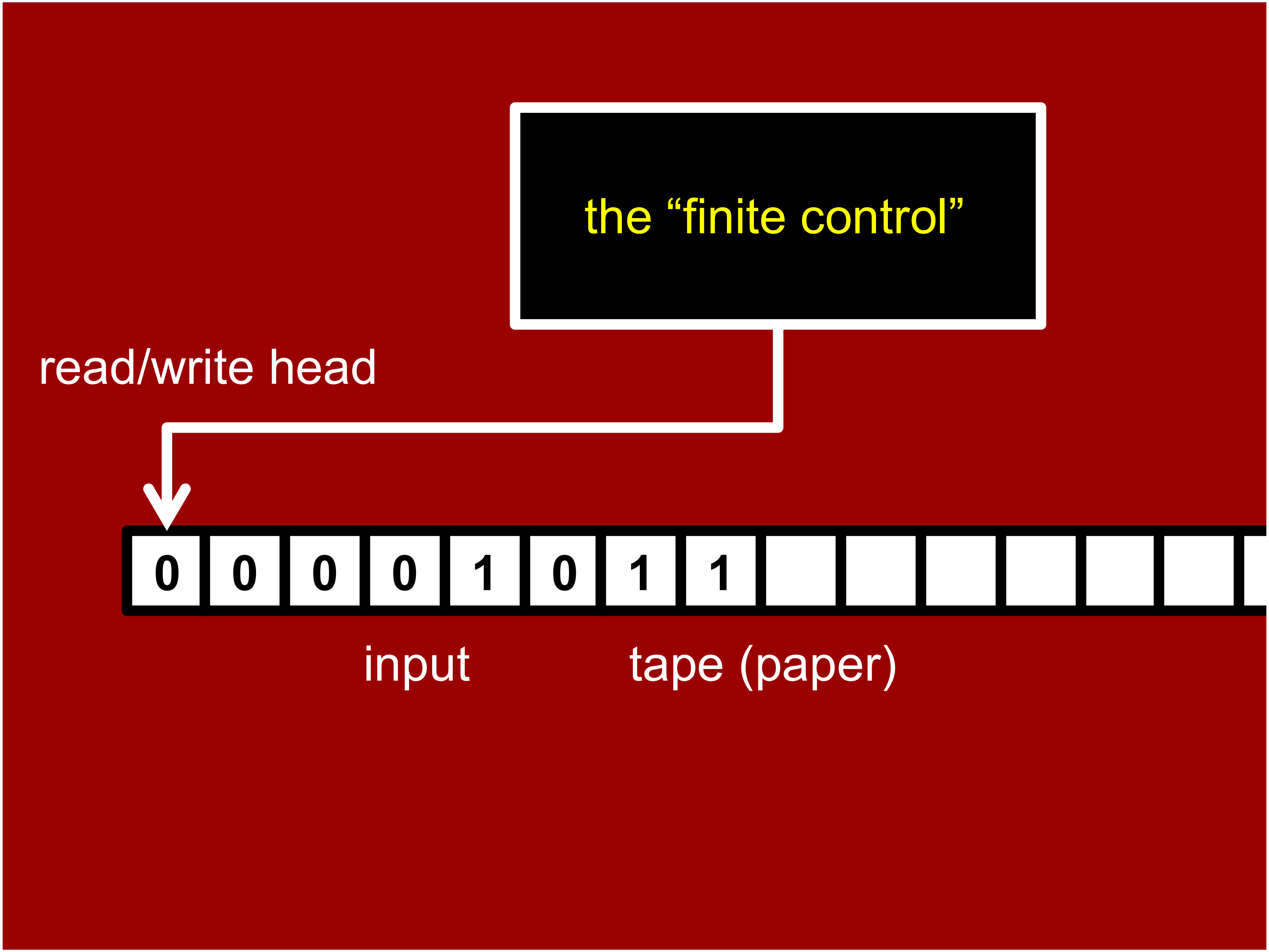


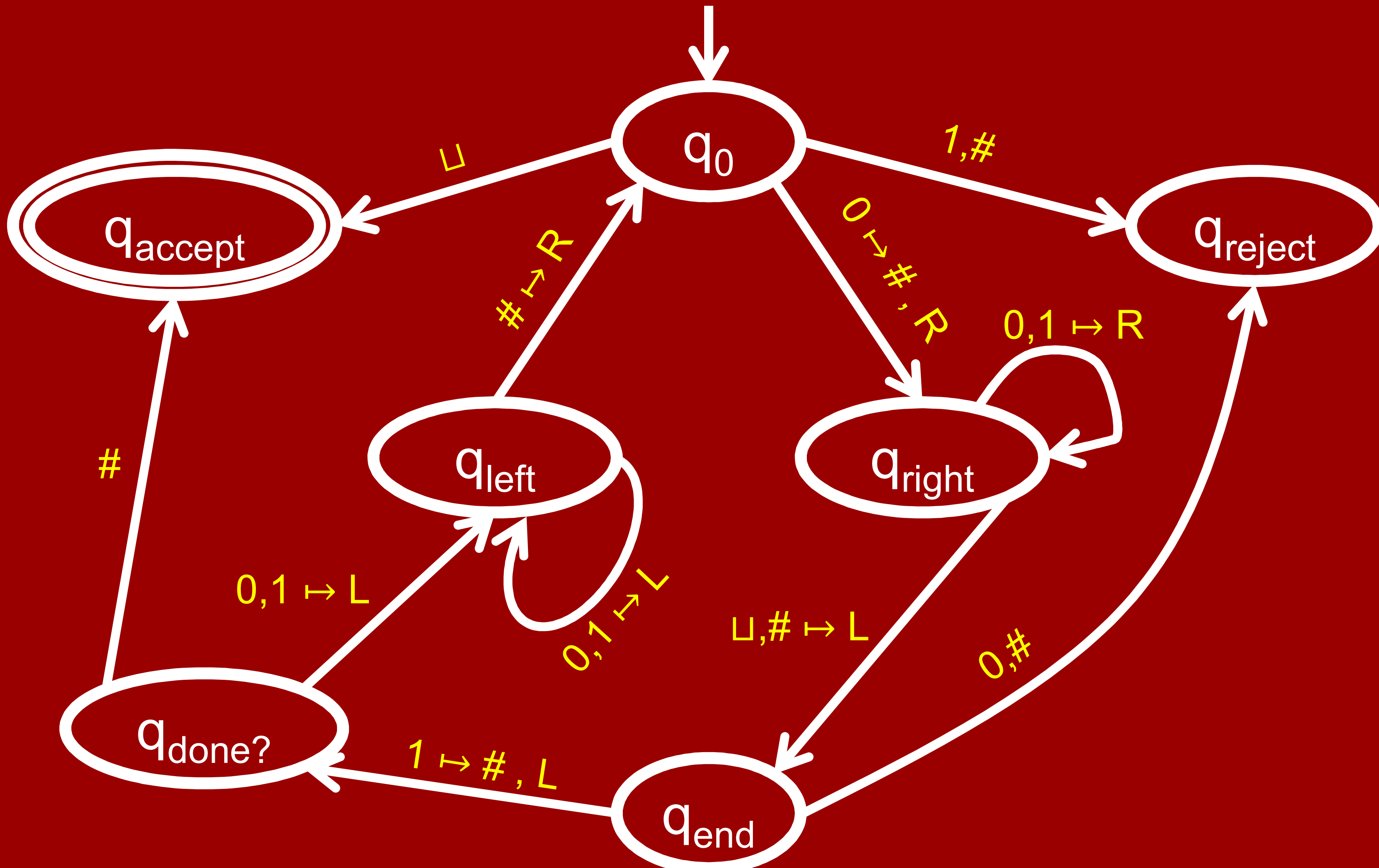
REJECT
the “finite control”
0 1 0 #


the "finite control"
read/write head
0 0 0 0 1 0 1 1
input
tape (paper)

Turing's mathematical abstraction of a computer

- A (human) computer writes symbols on paper
- WLOG, the paper is a sequence of squares
- No upper bound on the number of squares
- At most finitely many kinds of symbols
- Human observes one square at a time
- Human has only finitely many mental states
- Human can change its state, change symbols, and change focus to a neighboring square, but only based on its state and the symbol it observes
- Human acts deterministically

The **finite control** (aka transition rules)



Formal definition of Turing Machines

A Turing Machine is a 7-tuple

$M = (Q, q_0, q_{accept}, q_{reject}, \Sigma, \Gamma, \delta)$:

Q is a finite set of **states**,

$q_0 \in Q$ is the **start state**,

$q_{accept} \in Q$ is the **accept state**,

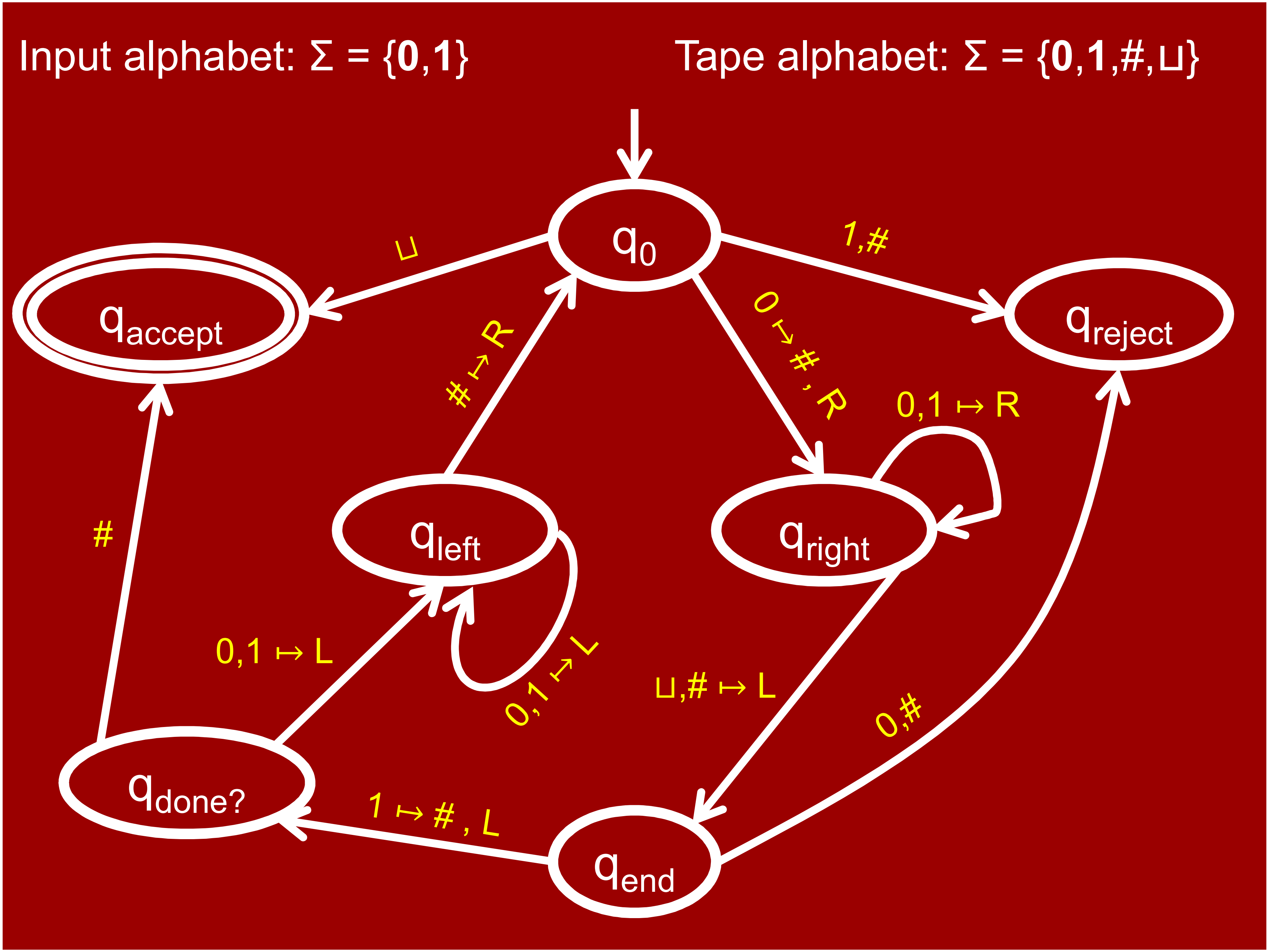
$q_{reject} \in Q$ is the **reject state**, $q_{reject} \neq q_{accept}$.

Σ is a finite **input alphabet** (with $\sqcup \notin \Sigma$),

Γ is a finite **tape alphabet** (with $\sqcup \in \Gamma$, $\Sigma \subseteq \Gamma$),

$\delta : Q' \times \Gamma \to Q \times \Gamma \times \{L,R\}$ is the **transition function**

(here $Q' = Q \setminus \{q_{accept}, q_{reject}\}$)


Input alphabet: Σ = {0,1}
Tape alphabet: Σ = {0,1,#,⊔}
q0
⊔
1,#
qaccept
qreject
0 ↦ #, R
↦ R
0,1 ↦ R
qleft
qright
#
0,1 ↦ L
0,1 ↦ L
⊔,# ↦ L
0,#
qdone?
1 ↦ #, L
qend

Formal definition of Turing Machines

Rules of computation:

Tape starts with input $x \in \Sigma^*$, followed by infinite ⊔'s.

Control starts in state q_0, head starts in leftmost square.

If the current state is q and head is reading symbol $s \in \Gamma$,
 the machine transitions according to $\delta(q,s)$, which gives:
  the next state,
  what tape symbol to overwrite the current square with,
  and whether the head moves Left or Right.

Technicality: moving left from the leftmost square ≡ staying put.

Continues until either the accept state or reject state reached.

When accept/reject state is reached, M **halts**.

M might also never halt, in which case we say it **loops**.

Decidable languages

Definition:

A language $L \subseteq \Sigma^*$ is **decidable** if there is a Turing Machine M which:

1. **Halts on every input** $x \in \Sigma^*$.
2. Accepts inputs $x \in L$ and rejects inputs $x \notin L$.

Such a Turing Machine is called a **decider**.
It 'decides' the language L.

We like deciders. We don't like TM's that sometimes loop.

Computable functions

Definition:

A function $f : \Sigma^* \to \{0,1\}$ is computable
if $L = \{x \in \Sigma^* : f(x) = 1\}$ is decidable

A function $f: \Sigma^* \to (\Gamma \setminus \{\sqcup\})^*$ is **computable** if there is
a Turing Machine M which:

Halts on every input $x \in \Sigma^*$ with
the tape containing $f(x)$ followed by $\sqcup$'s.

Decidable languages

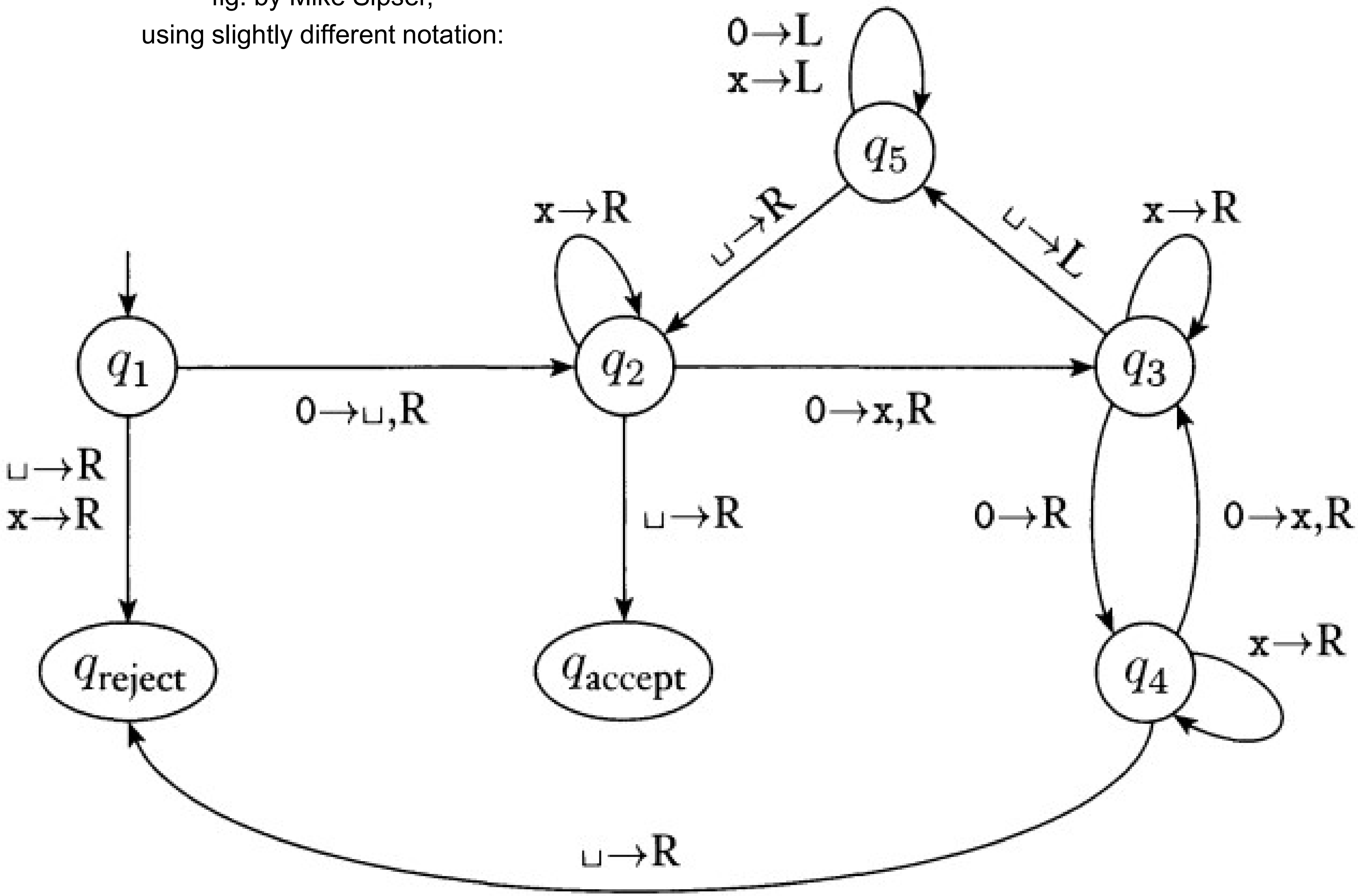
Examples:

Hopefully you're convinced that $\{0^n 1^n : n \in \mathbb{N}\}$
is decidable. (Recall it's not "regular".)

The language $\{0^{2^n} : n \in \mathbb{N}\} \subseteq \{0\}^*$,
i.e. {0, 00, 0000, 00000000, …},
is decidable.

Proof: I'll show you a decider TM for it…

fig. by Mike Sipser,
using slightly different notation:



Describing Turing Machines

Low Level:

Explicitly describing all states and transitions.

Medium Level:

Carefully describing in English how the TM operates. Should be 'obvious' how to translate into a Low Level description.

High Level:

Skips 'standard' details, just highlights 'tricky' details. **For experts only!**

$\{0^{2^n} : n \in \mathbb{N}\}$ is decidable

Medium Level description:

1. Sweep from left to right across the tape, overwriting a # over top of every *other* 0.
2. If you saw one 0 on the sweep, **accept**.
3. If you saw an odd number of 0's, **reject**.
4. Move back to the leftmost square.
   (Say you write ⊔ on the leftmost square at the very beginning so that you can recognize it later.)
5. Go back to step 1.

TM programming exercises & tricks

1. Move right (or left) until first ⊔ encountered.
2. Shift entire input string one cell to the right
3. Convert input $x_1x_2x_3\cdots x_n$ to $x_1\sqcup x_2\sqcup x_3\sqcup\cdots\sqcup x_n$
4. Simulate large tape alphabet Γ with just {0,1,⊔}
5. Ability to "mark" cells (e.g., replace symbol a by å)
6. Copy a stretch of tape between two marked cells into another marked section
7. Increment or Decrement an input in binary.
8. Implement basic string and arithmetic operations

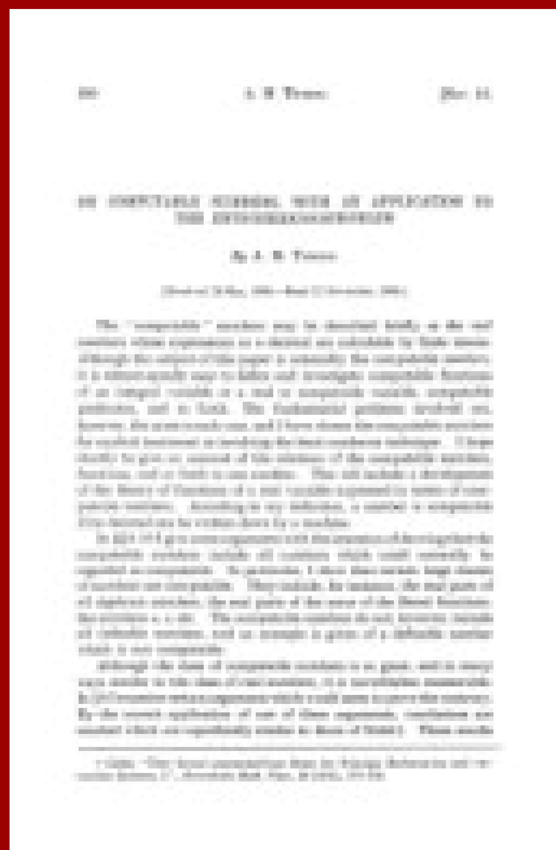
TM programming exercises & tricks

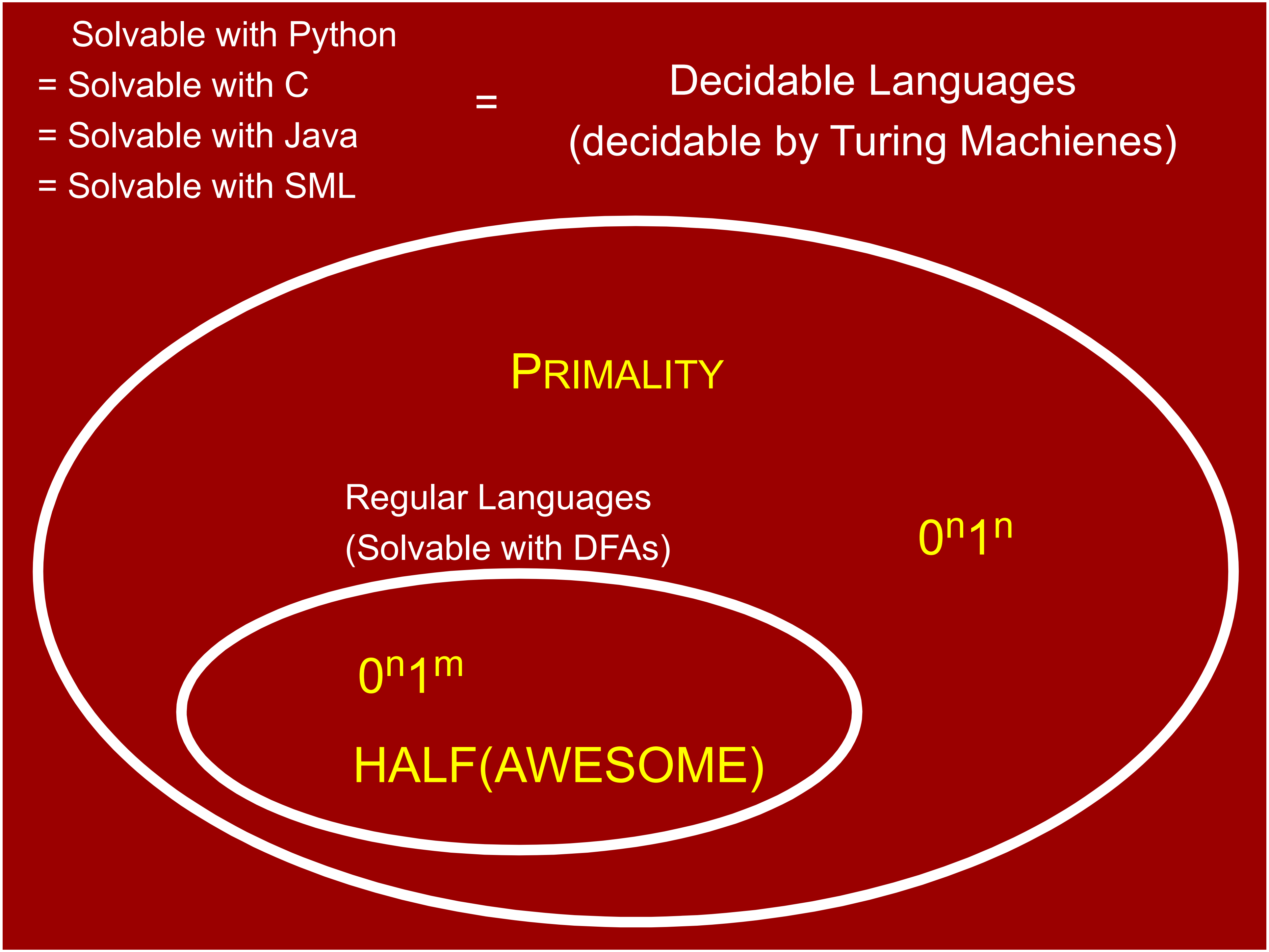
9. Simulate a TM with 2 tapes and read/write heads
10. Implement a dictionary data structure
11. Simulate "random access memory"
12. ….
13. Simulate an assembly language interpreter
14. Simulate a C interpreter
15. Create a Turing Machine interpreter or Universal TM, i.e., a Turing Machine U whose input is
    - ⟨M⟩, the **encoding** of a TM M,
    - x, a string

   and which **simulates** the execution of M on x.

Universal Turing Machine

If you get stuck on the last exercise, you can look up the answer in Turing's 1936 paper!




Solvable with Python
= Solvable with C
= Solvable with Java
= Solvable with SML
=
Decidable Languages
(decidable by Turing Machienes)
PRIMALITY
Regular Languages
(Solvable with DFAs)
0^n1^n
0^n1^m
HALF(AWESOME)

Church–Turing Thesis:

"Any natural / reasonable notion of computation can be simulated by a TM."

Describing Turing Machines

Low Level:

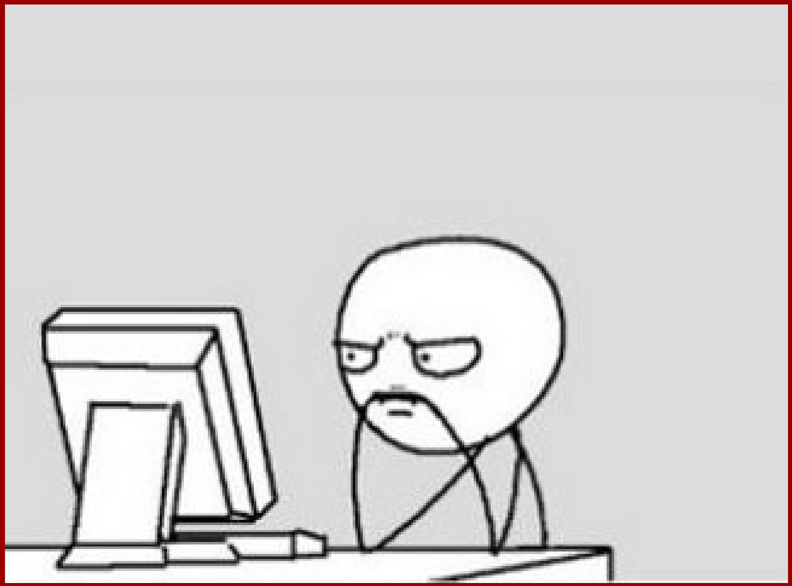
Medium Level:

High Level:

Super-high Level:

Just describe an algorithm / pseudocode.

Assuming the Church–Turing Thesis
there exists a TM which executes that algorithm.



Study Guide

Definitions:

- Turing Machines
- Decidable languages/
  computable functions
- Universal TM
- Church–Turing Thesis

Theorems/proofs:

- $\{0^{2^n} : n \in \mathbb{N}\}$ is decidable
- $\{0^n 1^n : n \in \mathbb{N}\}$ is decidable
- Equivalence of Solvability
  (between Python, C, TM)

Practice:

- Programming with TM's