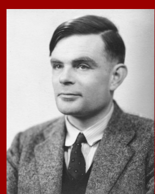
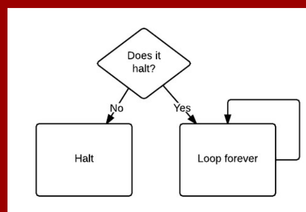
15-251: Great Theoretical Ideas in Computer Science
Lecture 6

Turing's Legacy Continues





Solvable with Python
= Solvable with C
= Solvable with Java
= Solvable with SML

= Decidable Languages
(decidable by Turing Machienes)

Robustness of Decidability

Decidability power is the same for TMs with:

- one-sided or double-sided infinite tape
- ability to stay in addition to going left / right
- even a fixed (oblivious) moving pattern works
- binary or larger finite tape alphabet
- one tape or a finite number of tapes/heads

Decidability power is also the same as:

- Python, C, Java, Assembly (any other language)
- Random Access Machiene + other comp. models
- Lambda-Calculus

Side note: Efficiency

Model details (and encodings) do play a role when it comes to efficiency, e.g., how many computation steps are needed.

Examples:

- a TM with one tape can simulate any multi-tape TM with a quadratic slowdown (sometimes needed)
- Random Access Machines can be simulated by a multi-tape TM with logarithmic slowdown
- Quantum computation can be simulated with exponential slowdown. It is unknown whether a super-polynomial slowdown is needed)

Robustness of Decidability

Most computational models, including those abstracted from any natural phenomenon, tend to be either wimpy or Turing equivalent, i.e., exactly equivalent in computational power to TMs.

No candidates of potentially implementable / natural computational models that are more powerful than a TM have been suggested.

Church–Turing Thesis (1936):
"Any natural / reasonable notion of computation can be simulated by a TM."

Cellular Automata

Most systems / the world can be described as many (tiny) parts interacting with other close-by parts.

Formal computational model:
A Cellural automaton (CA) consists of:

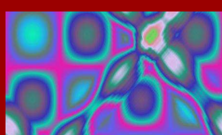
- cells with a finite set of states Q
- a neighborhood relation between cells
- a transition function δ_v: $Q^{deg(v)+1} \rightarrow Q$

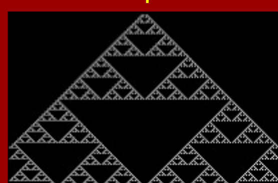
Computation: In every round every cell v (synchronously) transitions its state according to δ_v based on its and its neighbors' state.

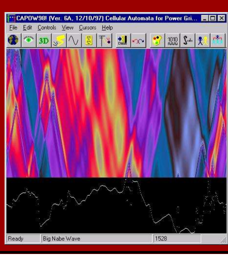
Applications of Cellular Automata

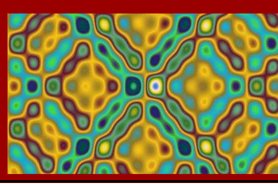
- Simulation of Biological Processes
- Simulation of Cancer cells growth
- Predator – Prey Models
- Art
- Simulation of Forest Fires
- Simulations of Social Movement
- …many more..

Cellular Automata: Examples

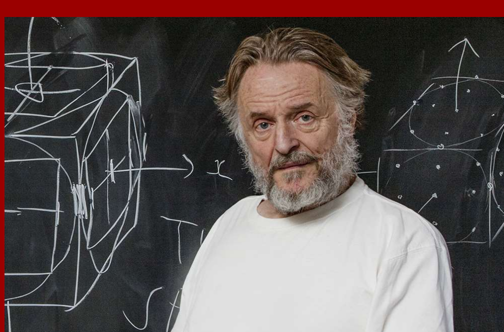






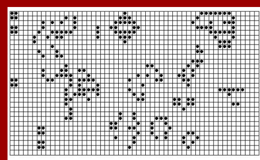


Example CA: Conway's Game of Life



Example CA: Conway's Game of Life

Cells form the infinite 2D-Grid



Q = {alive,dead}

3 transition rules ($\delta: Q^9 \rightarrow Q$):

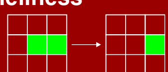
Loneliness: Life cell with fewer than 2 neighbors dies.

Overcrowding: Life cell with at least 4 life neighbors dies.

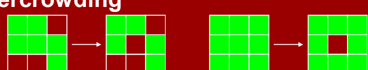
Procreation: Dead cell with exactly 3 neighbors gets born.

Conway's Game of Life: Rule examples

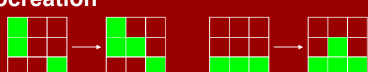
- **loneliness**



- **overcrowding**

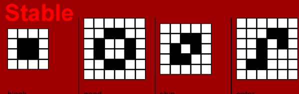


- **procreation**

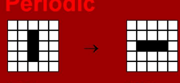


Conway's Game of Life: Patterns

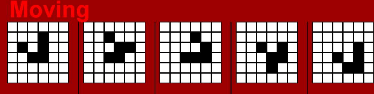
Stable



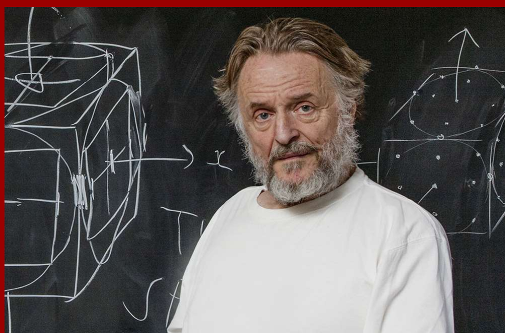
Periodic



Moving



Example CA: Conway's Game of Life



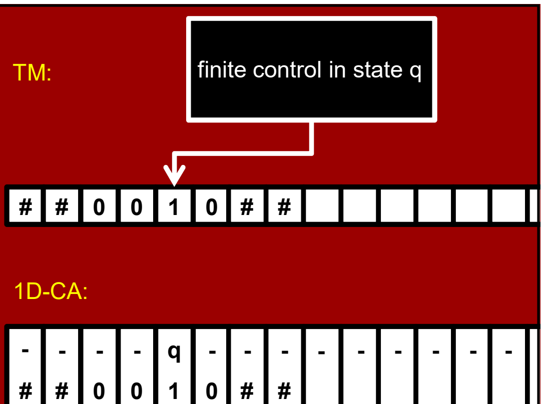
CA Turing Equivalence

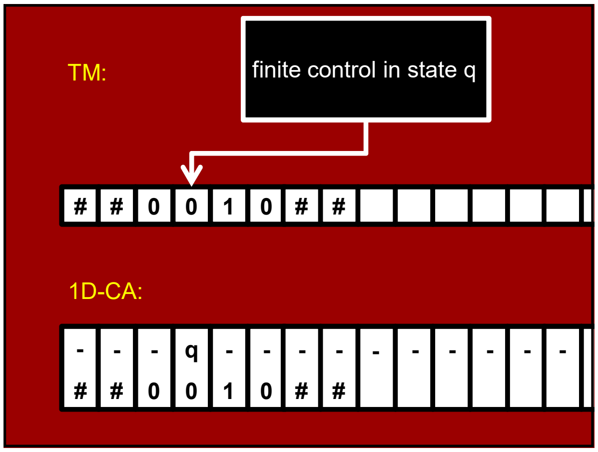
Theorem: Python / a TM can simulate any CA.

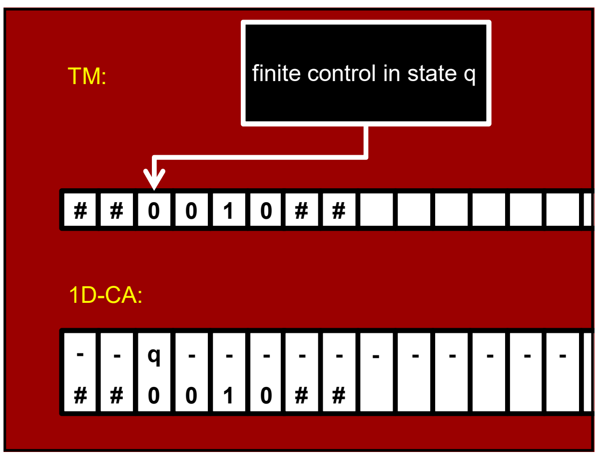
Theorem: For any TM there is a 1D-CA simulating it.

Construction Sketch:

- For TM with state set Q and tape alphabet Γ create 1D-CA with state space Γ x $(Q \cup \{-\})$.
- Cells simulate the tape and exactly one cell indicates the position of the a head and the TM state.
- Cells only transition if a neighboring cell contains the head.
- Transitions are based on the TM transition function.




TM:
finite control in state q
0 0 1 0 #
1D-CA:
- - - q - - - - - - - - - -
0 0 1 0 #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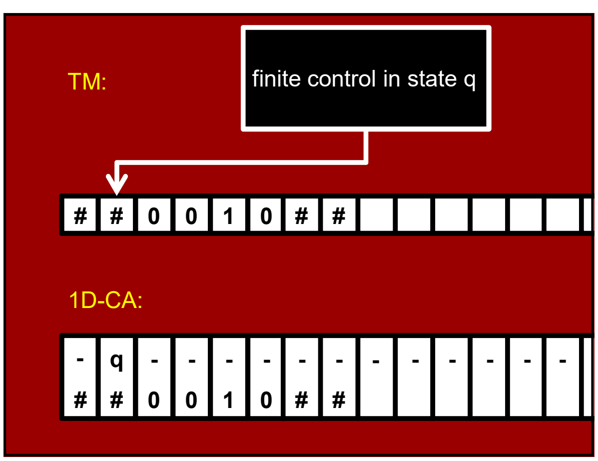
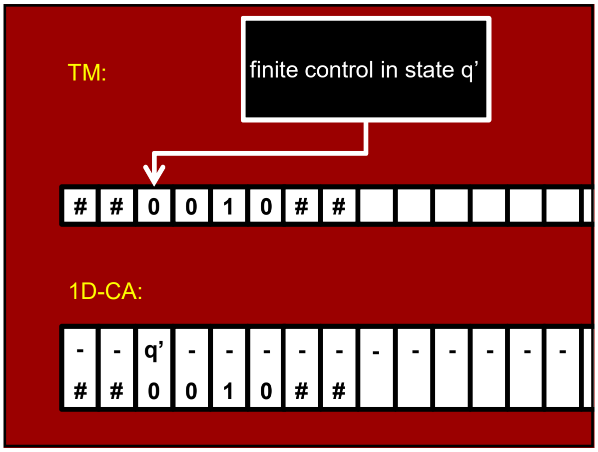
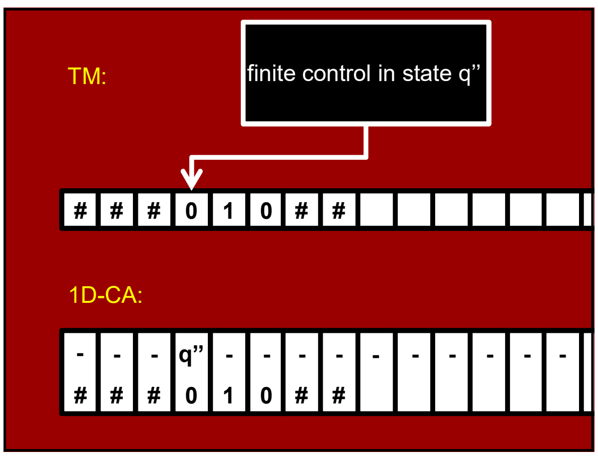
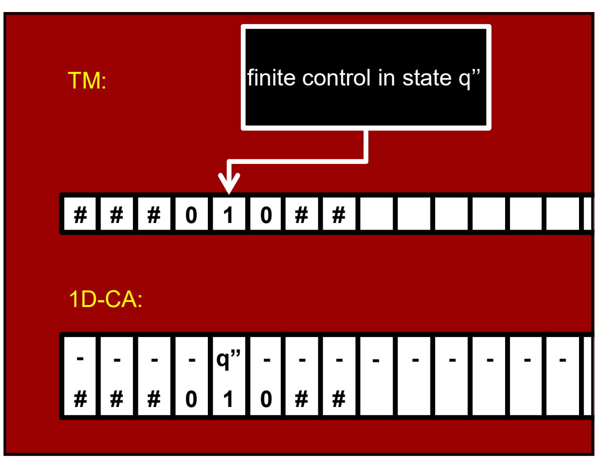

TM:
finite control in state q
0 0 1 0 #
1D-CA:
- q - - - - - - - - - - - -
0 0 1 0 #


TM:
finite control in state q'
0 0 1 0 #
1D-CA:
- - q' - - - - -
0 0 1 0 #


TM:
finite control in state q"
0 1 0 #
1D-CA:
- - - q" - - - -
0 1 0 #


TM:
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- - - - q" - - -
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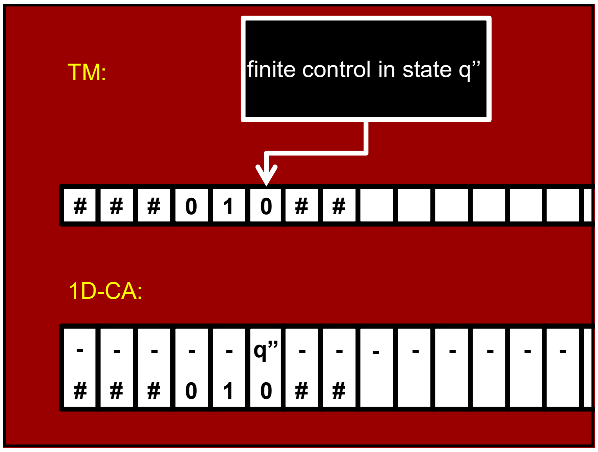

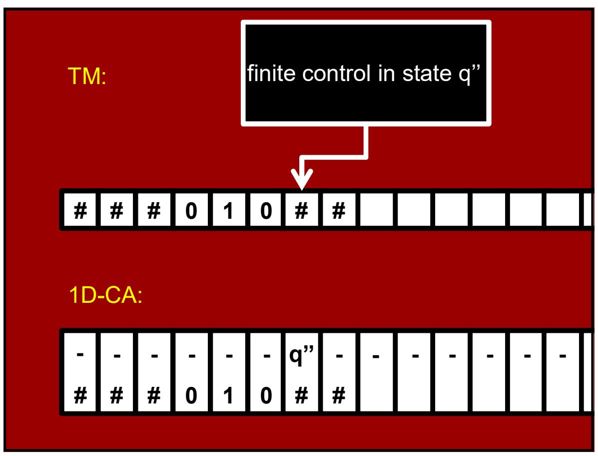

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Construction Sketch:

For TM with state set Q and tape alphabet Γ create 1D-CA with state space Γ x $(Q \cup \{-\})$..

Cells simulate the tape and exactly one cell indicates the position of the a head and the TM state. Cells only transition if a neighboring cell contains the head. Transitions are based on the TM transition function.

Theorem: Game of Life can simulate a universal TM.

Church–Turing Thesis:

"Any natural / reasonable notion of computation can be simulated by a TM."

Decidability

Decidable languages

Definition:

A language $L \subseteq \Sigma^*$ is **decidable** if there is a Turing Machine M which:

1. **Halts on every input** $x \in \Sigma^*$.
2. Accepts inputs $x \in L$ and rejects inputs $x \notin L$.

Such a Turing Machine is called a **decider**. It 'decides' the language L.

We like deciders. We don't like TM's that sometimes loop.

Encoding different objects with strings

Fix some alphabet Σ .
We use the $\langle \cdot \rangle$ notation to denote the encoding of an object as a string in Σ^* .

Examples:

$\langle M \rangle \in \Sigma^*$ is the encoding a TM M

$\langle D \rangle \in \Sigma^*$ is the encoding a DFA D

$\langle M_1, M_2 \rangle \in \Sigma^*$ is the encoding of a pair of TMs

$\langle M, x \rangle \in \Sigma^*$ is the encoding a pair (M, x) , where M is a TM, and $x \in \Sigma^*$.

Decidability: Poll

$\text{ACCEPT}_{\text{DFA}}$ = { $\langle D, x \rangle$ | D is a DFA that accepts x}

$\text{SELF-ACCEPT}_{\text{DFA}}$ = { $\langle D \rangle$ | D is a DFA that accepts $\langle D \rangle$ }

$\text{EMPTY}_{\text{DFA}}$ = { $\langle D \rangle$ | D is a DFA that accepts no x}

$\text{EQUIV}_{\text{DFA}}$ =
= { $\langle D, D' \rangle$ | D and D' are DFA and L(D) = L(D')}

Decidability: Examples

$\text{ACCEPT}_{\text{DFA}}$ = { $\langle D, x \rangle$ | D is a DFA that accepts x}

$\text{SELF-ACCEPT}_{\text{DFA}}$ = { $\langle D \rangle$ | D is a DFA that accepts $\langle D \rangle$ }

Theorem:

$\text{ACCEPT}_{\text{DFA}}$ is decideable.
$\text{SELF-ACCEPT}_{\text{DFA}}$ is decideable.

Proof: Simulate DFA step by step.

Decidability: Examples

$EMPTY_{DFA}$ = { $\langle D \rangle$ | D is a DFA that accepts no x}

Theorem:

$EMPTY_{DFA}$ is decidable.

Proof:

A DFA D accepts the empty language iff
no accepting state is reachable from the start state
via a simple sequence of states.
Try all |Q|! possible such sequences.

Decidability: Examples

$EQUIV_{DFA}$ =
= { $\langle D, D' \rangle$ | D and D' are DFA and L(D) = L(D')}

Theorem:

$EQUIV_{DFA}$ is decidable.

Proof:

Create a DFA D'' for the symmetric difference
$L(D'') = (L(D) \cap \overline{L(D')}) \cup (\overline{L(D)} \cap L(D'))$
using the Union and Intersection theorem for DFA.
Run the decider TM for $EMPTY_{DFA}$ on $\langle D'' \rangle$.

Reductions

*Using one problem as a **subroutine** to solve another is a powerful algorithmic technique.*

Definition:

Language A *reduces to* language B means:
"It is possible to decide A using an algorithm for deciding B as a subroutine."

Notation: $A \leq_T B$ (T stands for Turing).

Think, "A is no harder than B".

Reductions

Fact:

Suppose $A \leq_T B$; i.e., A reduces to B.
If B is decidable, then A is also decidable.

Here:
$EQUIV_{DFA} \leq_T EMPTY_{DFA}$ and $EMPTY_{DFA}$ is decidable.
This makes $EQUIV_{DFA}$ decidable.
Indeed, $EQUIV_{DFA}$ is at most as hard as $EMPTY_{DFA}$ because solving $EQUIV_{DFA}$ is easy given a solution to $EMPTY_{DFA}$.

Undecidability

Undecidability

Definition:

A language $L \subseteq \Sigma^*$ is **undecidable** if there is no Turing Machine M which:

1. **Halts on every input** $x \in \Sigma^*$.
2. Accepts inputs $x \in L$ and rejects inputs $x \notin L$.

Poll

Let A be the set of all languages over $\Sigma = \{0,1\}$.
Select all correct ones:

- A is finite
- A is infinite
- A is countable
- A is uncountable

Poll

Let A be the set of all languages over $\Sigma = \{0,1\}$.
Select all correct ones:

- A is finite
- ✓ A is infinite
- A is countable
- ✓ A is uncountable $|A| = |\mathcal{P}(\Sigma^*)| = |\mathcal{P}(\mathbb{N})|$

Question:

Is every language in $\{0,1\}^*$ decidable?
⇔ Is every function $f : \{0,1\}^* \rightarrow \{0,1\}$ computable?

Answer: No!

Every TM is encodable by a finite string.
Therefore the set of all TM's is countable.
So the subset of all *decider* TM's is countable.
Thus the set of all decidable languages is countable.

But the set of **all** languages is the power set of $\{0,1\}^*$
which is uncountable.

Question:

Is every language in $\{0,1\}^*$ decidable?

⇔ Is every function f : $\{0,1\}^* \rightarrow \{0,1\}$ computable?

Answer:

Essentially all (decision) functions are uncomputable!

Question:

Is it just weird languages that no one would care about which are *undecidable*?

Answer (due to Turing, 1936):

Sadly, no.
There are many natural languages one would like to compute but which are undecidable.

Example: Program Equivalence

Given a program P and a program P' we would like to automatically decide whether both do the same thing.

Formally:

$EQUIV_{TM}$ =
= { $\langle P, P' \rangle$ | P and P' are Python programs and L(D) = L(D')}

Useful for:

- Compiler Optimization
- Matching programs to their specification
- Autograder for 112 or 251 ☺

Example: 112 Autograder

First 112 assignment: Write a “Hello World” program.

Given a program P submitted by a student we want to automatically decide whether P does the right thing.

We want an algorithm A such that:

$$A(\langle P \rangle) = \begin{cases} \text{pass} & \text{iff P outputs “Hello World” and} \\ \text{fail} & \text{otherwise} \end{cases}$$

Example: 112 Autograder

Given a program P and a program P’ we would like to automatically decide whether both do the same thing.

Formally:

$EQUIV_{TM}$ =
= { $\langle P, P' \rangle$ | P and P’ are Python programs and L(D) = L(D’)}

Useful for:

- Compiler Optimization
- Matching programs to their specification
- Autograder for 112 or 251 ☺

112 Autograder Submission 1

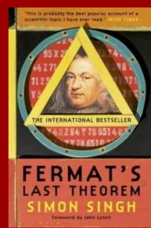
```
main(t,_,a ) char * a; { return!  0<t? t<3?  main(-79,-13,a+ main(-87,1-_, main(-86, 0, a+1 )
+a)):  1, t<_? main( t+1, _, a ) :3,  main ( -94, -27+t, a ) &&t == 2 ?_ <13 ?  main ( 2, _+1, "%s
%d %d\n" )  :9:16: t<0? t<-72? main( _, t,
"@n'+,#'/*{}w+/w#cdnr/+,{}r/*de}+,/*{*+,/w{%+,/w#q#n+,/#{l,+,/n{n+,/+#n+,/#;#q#n+,/+k#;*+,/'r
:'d*'3,}{w+K w'K:'+}e#';dq#'l
q#'+d'K#!/+k#;q#'r}eKK#}w'r}eKK{nl]'/#;#q#n'){)#}w'){){nl]'/+#n';d}rw' i;# ){nl]!/n{n#'; r{#w'r
nc{nl]'/#{l,+'K {rw' iK{;[{nl]'/w#q#n'wk nw' iwk{KK{nl]!/w{%'l##w#' i;
:{nl]'/*{q#'ld;r'}{nlwb!/*de}'c ;;{nl'-{}rw]'/+,}##'*}#nc,',#nw]'/+kd'+e}+;#'rdq#w! nr'/ ') }+}{rl#'{n'
')# }'+}##(!!/") : t<-50? _==*a ? putchar(31[a]):  main(-65,_,a+1) : main((*a == '/') + t, _, a + 1 ) :
0<t?  main ( 2, 2 , "%s") :*a=='/'||  main(0,  main(-61,*a, "!ek;dc i@bK'(q)-
[w]*%n+r3#l,{}:\nuwloca-O;m .vpbks,fxntdCeghiry")  ,a+1);}
```

This C program prints out all the lyrics of *The Twelve Days Of Christmas*.

Ok, so let just run the program P and check the output.

112 Autograder Submission 2

```
def HelloWorld():
  t = 3
  while (True):
    for n in xrange(3, t+1):
      for x in xrange(1, t+1):
        for y in xrange(1, t+1):
          for z in xrange(1, t+1):
            if (x**n + y**n == z**n):
              return "Hello World"
    t += 1
```



This program terminates and outputs "Hello World" if and only if Fermat's Last Theorem is false.

112 Autograder Submission 3

```
numberToTest := 2;
flag := 1;
while flag = 1 do
  flag := 0;
  numberToTest := numberToTest + 2;
  for p from 2 to numberToTest do
    if IsPrime(p) and IsPrime(numberToTest−p) then
      flag := 1;
      break;
    end if
  end for
end do
print("HELLO  WORLD")
```

Terminates with "Hello World" output if and only if Goldbach's Conjecture is false.

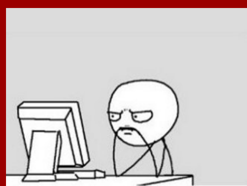
Some uncomputable functions

Given two TM descriptions, ⟨M_1⟩ and ⟨M_2⟩, do they act the same (accept/reject/loop) on all inputs?

Given the description of an algorithm, ⟨M⟩, does it print out "HELLO WORLD"?

Given a TM description ⟨M⟩ and an input x, does M halt on input x?

Given a TM description ⟨M⟩, does M halt when the input is a blank tape?



Study Guide

Definitions:

- Cellular Automata (CA)
- Reductions
- Undecidability

Theorems/proofs:

- Turing equivalency of CA
- Decidability of several languages
- Existence of undecidable problems

Practice:

- Decidability Proofs (via Reductions)