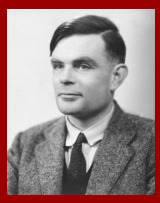
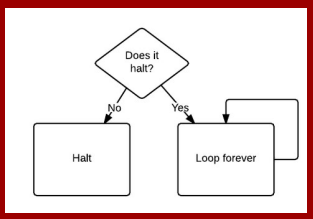
15-251: Great Theoretical Ideas in Computer Science
Lecture 7

Undecidability





Almost all Languages are undecidable

Set of all languages: $|\{S|S \subseteq \{0,1\}^*\}| = |\mathcal{P}(\{0,1\}^*)| = |\mathbb{R}|$

Set of all dec. lang.: $\leq |\{\langle M \rangle \in \{0,1\}^* | M \text{ is a decider TM}\}|$
$= |\{0,1\}^*| = |\mathbb{N}|$

$\Rightarrow$ Most languages do not have a TM deciding them

Question:

Is it just weird languages that no one would care about which are *undecidable*?

Answer (due to Turing, 1936):

Sadly, no.
There are many natural languages one would like to compute but which are undecidable.

Many interesting Languages are undecidable

In particular, any problems related to non-wimpy / Turing equivalent computation are undecidable.

Example: Program Equivalence

Given a program P and a program P' we would like to automatically decide whether both do the same thing.

Formally:

$EQUIV_{TM}$ =
= { $\langle P, P' \rangle$ | P and P' are Python programs and L(P) = L(P')}

Useful for:

- Compiler Optimization
- Matching programs to their specification
- Autograder for 112 or 251 ☺

Decidable Problems

$ACCEPT_{DFA}$ = { $\langle D, x \rangle$ | D is a DFA that accepts x}

$SELF\text{-}ACCEPT_{DFA}$ = {$\langle D \rangle$ | D is a DFA that accepts $\langle D \rangle$ }

$EMPTY_{DFA}$ = { $\langle D \rangle$ | D is a DFA that accepts no x}

$EQUIV_{DFA}$ =
= { $\langle D, D' \rangle$ | D and D' are DFA and L(D) = L(D')}

Theorem:

$ACCEPT_{DFA}$, **$SELF\text{-}ACCEPT_{DFA}$**, **$EMPTY_{DFA}$** and **$EQUIV_{DFA}$** are decideable.

Undecidable Problems

$ACCEPT_{TM}$ = { $\langle M, x \rangle$ | M is a TM that accepts x}

$SELF\text{-}ACCEPT_{TM}$ = { $\langle M \rangle$ | M is a TM that accepts $\langle M \rangle$ }

$EMPTY_{TM}$ = { $\langle M \rangle$ | M is a TM that accepts no x}

$EQUIV_{TM}$ =
= { $\langle M, M' \rangle$ | M and M' are TMs and L(M) = L(M')}

Theorem:

$ACCEPT_{TM}$, **$SELF\text{-}ACCEPT_{TM}$**, **$EMPTY_{TM}$** and **$EQUIV_{TM}$** are undecideable.

A simple undecidable language

Autograder / Hello World problem:

Given a program P, is it terminating and outputting "Hello World"?

HELLO = { $\langle M \rangle$ | M is a TM that outputs "Hello World" when run on the empty intput}

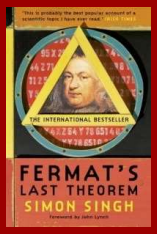
Hello Problem Instance #1

```
main(t,_,a ) char * a; { return!  0<t? t<3?  main(-79,-13,a+ main(-87,1-_, main(-86, 0, a+1 )
+a)):  1, t<_? main( t+1, _, a ) :3,  main ( -94, -27+t, a ) &&t == 2 ?_ <13 ?  main ( 2, _+1, "%s
%d %d\n" )  :9:16: t<0? t<-72? main( _, t,
"@n'+,#'/*{}w+/w#cdnr/+,{}r/*de}+,/*{*+,/w{%+,/w#q#n+,/#{l,+,/n{n+,/+#n+,/#;#q#n+,/+k#;*+,/'r
:'d'*3,}{w+K w'K:'+}e#';dq#'l
q#'+d'K#!/+k#;q#'r}eKK#}w'r}eKK{nl]'/#;#q#n')){}#}w')){}{nl]'/+#n';d}rw' i;# ){nl]!/n{n#'; r{#w'r
nc{nl]'/#{l,+'K {rw' iK{;[{nl]'/w#q#n'wk nw' iwk{KK{nl]!/w{%'l##w#' i;
:{nl]'/*{q#'ld;r'}{nlwb!/*de}'c ;;{nl'-{}rw]'/+,}##'*}#nc,',#nw]'/+kd'+e}+;#'rdq#w! nr'/ ') }+}{rl#'{n'
')# }'+}##(!!/") : t<-50?  _==*a ? putchar(31[a]):  main(-65, _,a+1) : main((*a == '/') + t, _, a + 1 ) :
0<t?  main ( 2, 2 , "%s") :*a=='/'||  main(0,  main(-61,*a, "!ek;dc i@bK'(q)-
[w]*%n+r3#l,{}:\nuwloca-O;m .vpbks,fxntdCeghiry")  ,a+1);}
```

This C program prints out all the lyrics of *The Twelve Days Of Christmas*.

Hello Problem Instance #2

```
def HelloWorld():
    t = 3
    while (True):
        for n in xrange(3, t+1):
            for x in xrange(1, t+1):
                for y in xrange(1, t+1):
                    for z in xrange(1, t+1):
                        if (x**n + y**n == z**n):
                            return "Hello World"
        t += 1
```



Terminates and outputs "Hello World" if and only if Fermat's Last Theorem is false.

Hello Problem Instance #3

```
numberToTest := 2;
flag := 1;
while flag = 1 do
    flag := 0;
    numberToTest := numberToTest + 2;
    for p from 2 to numberToTest do
        if IsPrime(p) and IsPrime(numberToTest−p) then
            flag := 1;
            break;
        end if
    end for
end do
print("HELLO  WORLD")
```

Terminates and outputs "Hello World" if and only if Goldbach's Conjecture is false.

A simple undecidable language

Autograder / Hello World problem:

Given a program P, is it terminating and outputting "Hello World"?

HELLO = {$\langle M \rangle$| M is a TM that outputs "Hello World" on the empty intput ε}

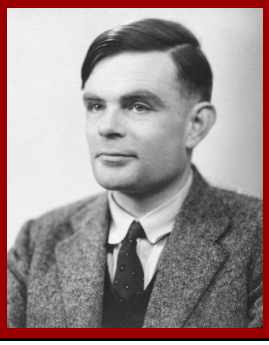
Halting problem:

Given a program P, is it terminating?

$HALT_\varepsilon$ = {$\langle M \rangle$| M is a TM terminating on ε}

HALT = {$\langle M, x \rangle$| M is a TM terminating on x}

The Halting Problem is Undecidable (1936)



The Halting Problem is Undecidable

Theorem:

The language

HALT = { $\langle M, x \rangle$ | M is a TM terminating on x}

is undecidable.

Proof:

Assume for the sake of contradiction that M_{HALT} is a decider TM which decides HALT.

The Halting Problem is Undecidable

Here is the description of another TM called D, which uses M_{HALT} as a subroutine:

D:

Given as input ⟨M⟩, the encoding of a TM M:
D executes M_{HALT}(⟨M, ⟨M⟩⟩).
If this call accepts, D enters an infinite loop.
If this call rejects, D halts (say, it accepts).

In other words… D(⟨M⟩) loops if M(⟨M⟩) halts,
halts if M(⟨M⟩) loops.

The Halting Problem is Undecidable

Assume M_{HALT} is a decider TM which decides HALT.

We can use it to construct a machine D such that

D(⟨M⟩) loops if M(⟨M⟩) halts,
halts if M(⟨M⟩) loops.

Time for the contradiction:

Does D(⟨D⟩) loop or halt?

By definition, if it loops it halts and if it halts it loops.

Contradiction. □

BTW: This is essentially just

Cantor's Diagonal Argument.

D(⟨M⟩) loops if M(⟨M⟩) halts, halts if M(⟨M⟩) loops

The set of all TM's is countable, so list it:

	⟨M_1⟩	⟨M_2⟩	⟨M_3⟩	⟨M_4⟩	⟨M_5⟩	…
M_1	halts	halts	loops	halts	loops	
M_2	loop	loops	loops	loops	loops	
M_3	halts	loops	halts	halts	halts	
M_4	halts	halts	halts	halts	loops	
M_5	halts	loops	loops	halts	loops	
⋮						

How could D be on this list?
What would the diagonal entry be??

D(⟨M⟩) loops if M(⟨M⟩) halts, halts if M(⟨M⟩) loops

The set of all TM's is countable, so list it:

	⟨M_1⟩	⟨M_2⟩	⟨M_3⟩	⟨M_4⟩	⟨M_5⟩	…
M_1	halts	halts	loops	halts	loops	
M_2	loop	loops	loops	loops	loops	
M_3	halts	loops	halts	halts	halts	
M_4	halts	halts	halts	halts	loops	
M_5	halts	loops	loops	halts	loops	
⋮						

Given some code,
determine if it terminates.

It's not: "we don't know how to solve it efficiently".

It's not: "we don't know if it's a solvable problem".

We know that it is unsolvable by any algorithm.

We know that it is unsolvable by any algorithm, any mechanism, any human being, anything in this world and any (physical) world we can imagine.

ACCEPT is undecidable

Theorem:

ACCEPT = {⟨M, x⟩ | M is a TM which accepts x}
is undecidable.

We could use the same diagonalization proof for ACCEPT.

But maybe there is an easier way …

Particularly, ACCEPT seems clearly harder than HALT.

After all, how can I decide if a program accepts if I don't even know if it halts.

ACCEPT is undecidable

Theorem:

ACCEPT = {⟨M, x⟩ | M is a TM which accepts x}
is undecidable.

New Proof Strategy:

Try to show that:

ACCEPT is at least as hard as HALT

$\Longleftrightarrow$

HALT is at most as hard as ACCEPT

$\Longleftrightarrow$

HALT would be easy if ACCEPT were easy

ACCEPT is undecidable

Theorem:

ACCEPT = {⟨M, x⟩ | M is a TM which accepts x}
is undecidable.

Proof (by contradiction):

Assume ACCEPT is decidable then show that HALT would be also decidable:

Suppose M_{ACCEPT} is a TM deciding ACCEPT.

Here is a description of a TM deciding HALT:

"Given ⟨M, x⟩, run $M_{ACCEPTS}$(⟨M, x⟩). If it accepts, then accept.

Reverse the accept & reject states in ⟨M⟩, forming ⟨M′⟩.

Run $M_{ACCEPTS}$(⟨M′, x⟩). If it accepts (i.e., M rejects x), then accept.

Else reject."

New Proof Strategy summarized:

Want to show:

Problem L is undecidable

New Proof Strategy:

Deciding L is at least as hard as deciding HALT

$\Longleftrightarrow$

HALT would be easy if L were easy

$\Longleftrightarrow$

HALT reduces to L

$\Longleftrightarrow$

HALT $\leq_T$ L

Reductions

Definition:

Language A *reduces to* language B means:
"It is possible to decide A using an algorithm for deciding B as a subroutine."

Notation: $A \leq_T B$ (T stands for Turing).

Think, "A is no harder than B".

Reductions

Fact:

Suppose $A \leq_T B$; i.e., A reduces to B.
If B is decidable, then so is A.

We actually used the contrapositive:

Fact:

Suppose $A \leq_T B$; i.e., A reduces to B.
If A is undecidable, then so is B.

Note that "$A \leq_T B$" is a stronger statement than proving that A is decidable under the assumption that B is decidable.

Reductions

Reductions are the main technique for showing undecidability.

Interesting:
We use a positive statement, i.e., the existence of a reduction algorithm, in order to prove a negative (impossibility) result.

Reductions (HALT $\leq_T$ ACCEPT)

Theorem:

HALT $\leq_T$ ACCEPT.

Proof:

Suppose M_{ACCEPT} is a subroutine deciding ACCEPT.
Here is a description of a TM deciding HALT:
"Given $\langle M, x \rangle$, run $M_{ACCEPTS}(\langle M, x \rangle)$. If it accepts, then accept.
Reverse the accept & reject states in $\langle M \rangle$, forming $\langle M' \rangle$.
Run $M_{ACCEPT}(\langle M', x \rangle)$. If it accepts (i.e., M rejects x), then accept.
Else reject."

More Reductions (ACCEPT $\leq_T$ ALL)

Theorem:

ALL = {⟨M⟩ | M accepts all strings} is undecidable.

Proof: (ACCEPT $\leq_T$ ALL)

Suppose M_{ALL} is a subroutine deciding ALL.

Here is a description of a TM deciding ACCEPT:

"Given ⟨M, x⟩, write down the description ⟨M_x⟩ of a TM M_x which does this:

"Overwrite the input with x and then run M."

Call subroutine M_{ALL} on input ⟨M_x⟩. Accept if it accepts, reject otherwise"

(Note that M_x behaves the same on all inputs and in particular we have that M_x accepts all strings if and only if M accepts x.)

More Reductions (ACCEPT $\leq_T$ EMPTY)

Theorem:

We also have ACCEPT $\leq_T$ EMPTY.

Proof: (ACCEPT $\leq_T$ EMPTY)

Suppose M_{EMPTY} is a subroutine deciding EMPTY.

Here is a description of a TM deciding ACCEPT:

"Given ⟨M, x⟩, write down the description ⟨M_x⟩ of a TM M_x which does this:

"Overwrite the input with x and then run M."

Call subroutine M_{EMPTY} on input ⟨M_x⟩. Reject if it accepts else reject."

More Reductions (ALL,EMPTY $\leq_T$ EQUIV)

Theorem:

EQUIV = {⟨M,M'⟩ | L(M) = L(M')} is undecidable.

Proof: (ALL $\leq_T$ EQUIV and EMPTY $\leq_T$ EQUIV)

Suppose M_{EQUIV} is a subroutine deciding EQUIV.

Here is a description of a TM deciding ALL:

"Given ⟨M⟩ write down the description ⟨M'⟩ of a TM M' which always accepts / rejects.

Then call subroutine M_{EQUIV} on input ⟨M,M'⟩."

Poll – Test your Intuition

We just showed:

$HALT \leq_T ACCEPT \leq_T EMPTY \leq_T EQUIV$

and

$ACCEPT \leq_T ALL \leq_T EQUIV$

Which of the following, do you believe also hold?

$HALT \leq EMPTY$
$HALT \leq EQUIV$
$EMPTY \leq ACCEPT$
$EQUIV \leq EMPTY$
$EQUIV \leq HALT$

More Reductions ($EMPTY \leq_T HALT$)

Theorem:

HALT, ACCEPT, EMPTY are all equally hard.

Proof: ($EMPTY \leq_T HALT$)

Suppose M_{HALT} is a subroutine deciding HALT.

Here is a description of a TM deciding EMPTY:

"Given ⟨M⟩, write down the description ⟨M'⟩ of a TM M' which does this:

"For t=1 to ∞

run M on each string of length at most t for t steps

If any execution terminates and accepts then terminate (+ accept)"

Then call subroutine M_{HALT} on input ⟨M', ε⟩ but reverse the accept/reject."

More Undecidability

Theorem:

HALT, ACCEPT, EMPTY are all equally hard.

What about EQUIV and ALL?

Fun Fact #1:

EQUIV and ALL are harder than HALT and so are

TOTAL = {⟨M⟩ | M halts on all inputs x}

FINITE = {⟨M⟩ | L(M) is finite}

and in fact all these problems are equally hard.

Fun Fact #2:

There is an infinite hierarchy of harder and harder undecidable languages.

More Undecidability

Fun Fact #2:

There is an infinite hierarchy of harder and harder undecidable languages. (which however still only covers countably many languages)

How does one define / construct this hierarchy?

Look at TMs which have a subroutine/oracle that solves HALT. These oracle TMs can solve ACCEPT and other equivalent problems easily BUT they cannot decide if an oracle TM given to them halts. This makes the HALTing problem for oracle TMs even harder. …

Question:

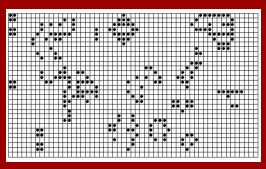
Do all undecidable problems involve TM's?

Answer:

No! Some very different problems are undecidable!

Cellular Automata

Input: A CA with its initial configuration.
E.g. a game of life pattern



Theorem: Deciding whether the input CA loops is an undecidable problem.

Post's Correspondence Problem

Input: A finite collection of "dominoes", having strings written on each half.

E.g.:

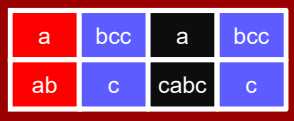
Definition: A match is a sequence of dominoes, repetitions allowed, such that top string = bottom string.

Post's Correspondence Problem

Input: A finite collection of "dominoes", having strings written on each half.

E.g.:

Match:



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Post's Correspondence Problem

Input: A finite collection of "dominoes", having strings written on each half.

Task: Output YES if and only if there is a match.

Theorem (Post, 1946): **Undecidable**. There is no algorithm solving this problem.

(More formally, PCP = {⟨Domino Set⟩ : there's a match} is an undecidable language.)

Post's Correspondence Problem

Input: A finite collection of "dominoes", having strings written on each half.

Task: Output YES if and only if there is a match.

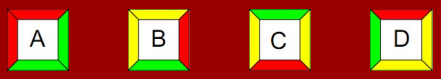
Theorem (Post, 1946): **Undecidable**.

Two-second proof sketch:

Given a TM M, you can make a domino set such that the only matches are execution traces of M which end in the accepting state. Hence ACCEPTS $\leq_T$ PCP.

Wang Tiles

Input: Finite collection of "Wang Tiles" (squares) with colors on the edges. E.g.,



Task: Output YES if and only if it's possible to make an infinite grid from copies of them, where touching sides must color-match.

Theorem (Berger, 1966): **Undecidable.**

Modular Systems

Input: Finite set of rules of the form "from ax+b, can derive cx+d", where a,b,c,d∈ℤ. Also given is a starting integer u and a target v.

Task: Decide if v can be derived starting from u.

E.g.: "from 2x derive x", "from 2x+1 derive 6x+4", target v = 1. Starting from u, this is equivalent to asking if the "3n+1 problem" halts on u.

Theorem (Börger, 1989): **Undecidable.**

Richardson's Problem

Input: A set S of rational numbers.

What you can do: Make an expression E using the numbers in S, the numbers π and ln(2), the variable x, and operations +, −, ·, sin, exp, abs.

Question: Can you make an E such that E ≡ 0?

Theorem (Richardson, 1968): **Undecidable.**

Mortal Matrices

Input: Two 21×21 matrices of integers, A & B.

Question: Is it possible to multiply A and B together (multiple times in any order) to get the 0 matrix?

Theorem (Halava, Harju, Hirvensalo, 2007): **Undecidable.**

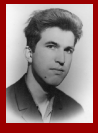
Hilbert's 10th problem

Input: Multivariate polynomial w/ integer coeffs.

Question: Does it have an integer root?

Theorem (1970): **Undecidable.**

Matiyasevich Robinson Davis Putnam




Hilbert's 10th problem

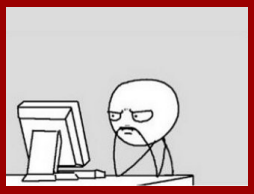
Input: Multivariate polynomial w/ integer coeffs.

Question: Does it have an integer root?
Undecidable.

Question: Does it have a real root?
Decidable.

Tarski, 1951.

Question: Does it have a rational root?
Not known if it's decidable or not.



Study Guide

Definitions:
- Halting and other Problems

Theorems/proofs:
- Undecidability of HALT
- many reduction proofs

Practice:
- Diagonalization
- Reductions
- Programming with TM's