15-251: Great Theoretical Ideas In Computer Science

Recitation 8

Definitions

- The sample space Ω is the set of all outcomes, each of which has some nonnegative probability, and the sum of these probabilities is equal to 1.
- An event is a subset of outcomes.
- A conditional probability $Pr(A \mid B) = \dfrac{Pr(A \cap B)}{Pr(B)}$
- The Law of Total Probability states that given an event A and a partition of the sample space $B_1, \cdots B_k$, $Pr(A) = \sum_{i=1}^{k} Pr(A \mid B_i) Pr(B_i)$
- Two events are independent if $Pr(A \cap B) = Pr(A)Pr(B)$, or if $Pr(A \mid B) = Pr(A)$, or if $Pr(B \mid A) = Pr(B)$. The latter two definitions require nonzero probability of what you condition on.
- A random variable X is a function from $\Omega \to \mathbb{R}$.
- Random variables X, Y are independent if for all $x, y \in \mathbb{R}$, events $X = x$ and $Y = y$ are independent.
- An indicator random variable for an event A is 1 when A happens and 0 otherwise.
- The expected value of a random variable X is $\sum_{l \in \Omega} Pr(l) X(l)$
- If $X = \sum_{i=1}^{k} X_i$ for random variables X_i, linearity of expectation states that $\mathbb{E}[X] = \sum_{i=1}^{k} \mathbb{E}[X_i]$

Total Expectation

In lecture, you learned about the Law of Total Probability. Here, we will prove a similar theorem, the Law of Total Expectation. Given a random variable X and an event A, define the conditional expectation $\mathbb{E}[X \mid A] = \sum_{l \in \Omega} Pr(l \mid A) X(l)$. The Law of Total Expectation states that:

$$\mathbb{E}[X] = \mathbb{E}[X \mid A] Pr(A) + \mathbb{E}[X \mid A^C] Pr(A^C)$$

Prove this.

NOTE: The Law of Total Expectation generalizes in the same way the Law of Total Probability does to a partition of the sample space $A_1, \cdots, A_k$. You may use this on the rest of the recitation and homework without proof.

Geometric Expectation

Let $X \sim Geometric(p)$. Compute $\mathbb{E}[X]$.

Caution with Independence

Suppose we have some die (not necessarily fair) and roll it three times independently, with results A, B, C. Prove or disprove that the events $A \neq B$ and $B = C$ are independent.

Your Intuition is Probably Wrong

(a) Suppose we flip a fair coin until we get two heads in a row, or a heads then a tails. What is the probability we terminate due to a double heads?

(b) Suppose we flip a fair coin until we get two heads in a row. What is the expected number of coin flips?

(c) Suppose we flip a fair coin until we get a heads then a tails. What is the expected number of coin flips?

(d) Prove or disprove that given random variables A and B such that $\mathbb{E}[A] < \mathbb{E}[B]$, $Pr(A < B) > \frac{1}{2}$.

Expected Cost

Suppose the numbers from 1 to n are given to you in some order. You need to keep track of the minimum of the numbers you've seen so far. If the minimum changes, it costs $1.

(a) What is the best possible cost? Worst?

(b) If the permutation of 1 to n is chosen uniformly at random, what is the expected cost of keeping track of the minimum.

Partner Troubles

(a) The 251 class has learned that you are very very good at probability. So good, in fact, that a large number of people want to partner up with you for the next homework assignment. You want to be fair, so you want to pick your partner uniformly, so each person is your partner with probability $\frac{1}{n}$, where n is the number of people who want to partner up with you. Unfortunately, you don't know n in advance! Instead people will come up to you one at a time, and you have to choose whether to partner up with them now (and forget about your old partner) or send them on their way. How can you solve this problem fairly?

(b) As you figure out the solution to the problem, you remember that in 251, we let you have groups of 3! Now you want to choose a pair of people to group up with, again with uniform probability, so everyone has the same probability of being in your group. Solve this new, related problem!