Subspace madness

Let $V$ be a vector space over some field $F$. Let $U, W$ be subspaces of $V$. For the following subsets of $V$, determine if it is a subspace of $V$.

(a) $U + W = \{ u + w \mid u \in U, w \in W \}$
(b) $a \cdot U = \{ a \cdot u \mid u \in U \}$
(c) $U \cup W$
(d) $U \cap W$
(e) $V \setminus U$

Bonus problem: Suppose that $F = \mathbb{R}$ and $V = \mathbb{R}^n$. Recall the dot product operation $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$u \cdot v = \sum_{i=1}^{n} u_i v_i$$

For a subspace $U$ of dimension $k$, show that the set

$$U^\perp = \{ v \in V \mid u \cdot v = 0 \text{ for some nonzero } u \in U \}$$

is a subspace and determine its dimension.

More spaces

Show that the following sets are vector spaces over the given field and determine their dimensions.

(a) $M_{m,n}(F)$, the set of $m \times n$ matrices with coefficients from some field $F$, over $F$. Here addition is done entrywise and scalar multiplication is applied to all entries.

(b) $F[x]$ over $F$.

(c) (Bonus) $\mathbb{F}_{p^k}$ over $\mathbb{F}_p$, where $p$ is some prime and $k \geq 1$.

MaTricks

Let $R$ be a $m \times n$ matrix, $B$ be a $n \times n$ matrix and $A$ be a $m \times m$ matrix (all with entries over some field $F$). Show that if $RB = AR$, then $Rf(B) = f(A)R$ for any $f \in F[x]$.

(Bonus) Let $F$ be a field and $n \geq 1$. Construct a $n \times n$ matrix $A$ such that no $n \times n$ matrix $B$ satisfies $B^2 = A$. 
**Rookie Mistake**

A rook is placed at the bottom-left corner of an otherwise empty chessboard. For moves $t \geq 1$, the rook makes a move to a destination chosen uniformly at random (from the set of all legal destinations, cannot stay at the current position). What is the expected time to get to the top-right corner?

**Random walks**

For an undirected graph $G$, interpret it as a Markov chain where a state transitions to a neighbor chosen uniformly at random. What is the stationary distribution for this Markov chain?