15-251: Great Theoretical Ideas In Computer Science

Recitation 11

Subspace madness

Let V be a vector space over some field F. Let U, W be subspaces of V. For the following subsets of V, determine if it is a subspace of V.

(a) $U + W = \{u + w \mid u \in U, w \in W\}$

(b) $a \cdot U = \{a \cdot u \mid u \in U\}$

(c) $U \cup W$

(d) $U \cap W$

(e) $V \setminus U$

Bonus problem : Suppose that $F = \mathbb{R}$ and $V = \mathbb{R}^n$. Recall the dot product operation $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$u \cdot v = \sum_{i=1}^{n} u_i v_i$$

For a subspace U of dimension k, show that the set

$$U^{\perp} = \{v \in V \mid u \cdot v = 0 \text{ for some nonzero } u \in U\}$$

is a subspace and determine its dimension.

More spaces

Show that the following sets are vector spaces over the given field and determine their dimensions.

(a) $M_{m,n}(F)$, the set of $m \times n$ matrices with coefficients from some field F, over F. Here addition is done entrywise and scalar multiplication is applied to all entries.

(b) $F[x]$ over F.

(c) (**Bonus**) $\mathbb{F}_{p^k}$ over $\mathbb{F}_p$, where p is some prime and $k \geq 1$.

MaTricks

Let R be a $m \times n$ matrix, B be a $n \times n$ matrix and A be a $m \times m$ matrix (all with entries over some field F). Show that if $RB = AR$, then $Rf(B) = f(A)R$ for any $f \in F[x]$.

(**Bonus**) Let F be a field and $n \geq 1$. Construct a $n \times n$ matrix A such that no $n \times n$ matrix B satisfies $B^2 = A$.

Rookie Mistake

A rook is placed at the bottom-left corner of an otherwise empty chessboard. For moves $t \geq 1$, the rook makes a move to a destination chosen uniformly at random (from the set of all legal destinations, cannot stay at the current position). What is the expected time to get to the top-right corner?

Random walks

For an undirected graph G, interpret it as a Markov chain where a state transitions to a neighbor chosen uniformly at random. What is the stationary distribution for this Markov chain?