15-251

Great Theoretical Ideas in Computer Science

Introduction to Computational Complexity I

February 3rd, 2015

Reminder

Midterm 1

February 11th, 18:30 - 21:30

Covers Lectures 1-6 (first 3 homeworks)

What have we done so far?

> Introduction to the course

Computer science is no more about computers than astronomy is about telescopes.

> Logic

Foundation of mathematics

> Formalization of computation/algorithm

Deterministic Finite Automata

Turing Machines

> The study of computation

Computability

What have we done so far?

> The study of computation

Computability

- Most problems are undecidable.
- Some very interesting problems undecidable.

But most interesting problems are decidable!

What is next?

> The study of computation

Computability

Computational Complexity (Practical Computability)

- How do we define complexity?
- What is the right level of abstraction to use?
- How do we analyze complexity?
- What are some interesting problems to study?
- What can we do to better understand the complexity of problems?

⋮

What is next?

CMI

ABOUT PROGRAMS MILLENNIUM PROBLEMS PEOPLE PUBLICATIONS EUCLID EVENTS

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

1 million dollar question

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

(or maybe 6 million dollar question)

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

P = NP ???

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

Introduction to Computational Complexity

Computational complexity of an algorithm.

Computational complexity of a problem.

- complexity of the best algorithm computing the problem.

Complexity with respect to what?

- time (number of steps)
- space (memory)
- randomness
- quantum resources

Our focus for now

Introduction to Computational Complexity

Church-Turing Thesis:

With respect to computability
the particular computational model doesn't matter.

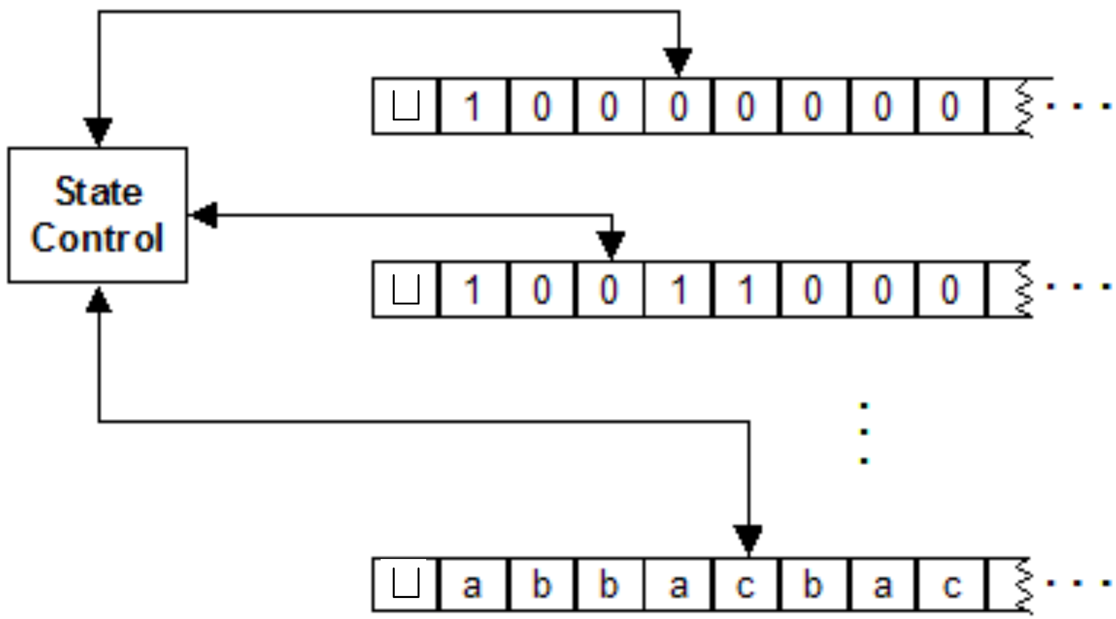
Unfortunately, this is **not** true with respect to computational complexity.

The model makes a difference.

How the model can affect running time

A multitape Turing machine

Ordinary TM with multiple tapes, each with its own head.



Multiple Tape/Head Turing Machines

Number of tapes is fixed (cannot grow with input size).

$$\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$$

e.g. $(q_i, a_1, \ldots, a_k) \mapsto (q_j, b_1, \ldots, b_k, R, L, \cdots, L)$

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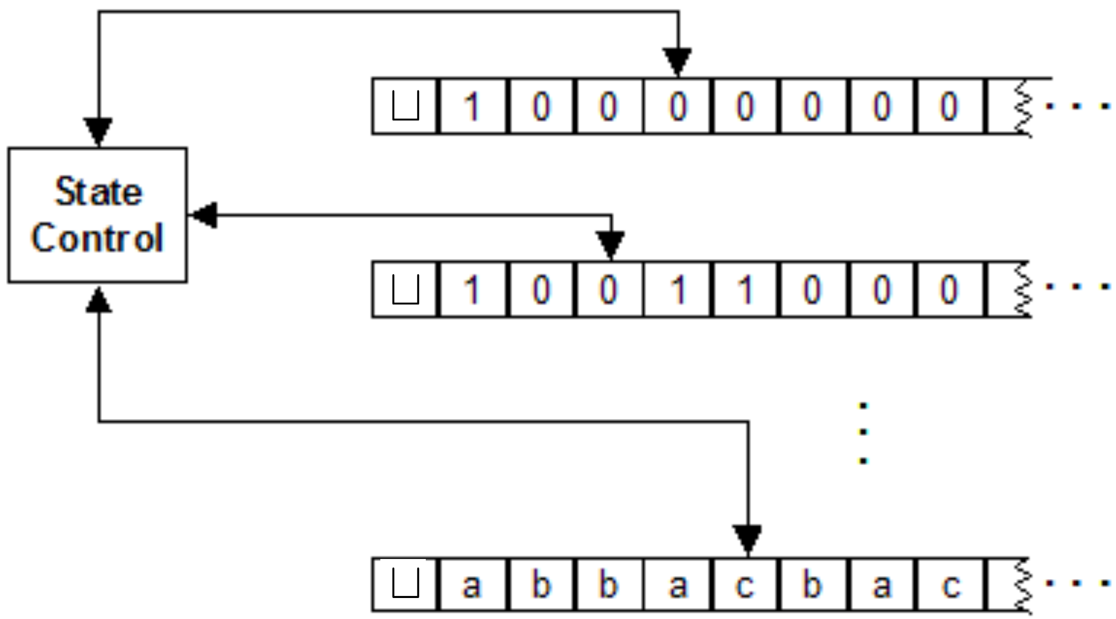

Multiple Tape/Head Turing Machines

Is it more powerful?

Every multitape TM has an equivalent single tape TM.

How the model can affect time

$$L = \{0^k 1^k : k \geq 0\}$$

How many steps does a single-tape TM take?

On input string w:

- Scan the input and **reject** if a 0 is found to the right of a 1.
- Repeat while both 0s and 1s remain on the tape:
  - Scan the tape, cross off a single 0 and a single 1.
- If 0s remain but no 1s remain **or** 1s remain but no 0s remain **reject**
- Else **accept**

Number of steps: $O(n^2)$ (n is the input length.)

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How many steps does a two-tape TM take?

On input string w:

- Scan the input and **reject** if a 0 is found to the right of a 1.
- Scan the 0s until the first 1, and copy the 0s to Tape 2.
- Scan the 1s. For each 1 read, cross off a 0 on Tape 2.
- If all 0s are crossed off before all 1s are read, **reject**.
- If all 0s are crossed off **accept**.
- Else **reject** (some 0s remain).

0	0	0	1	1	1
0	0	0			

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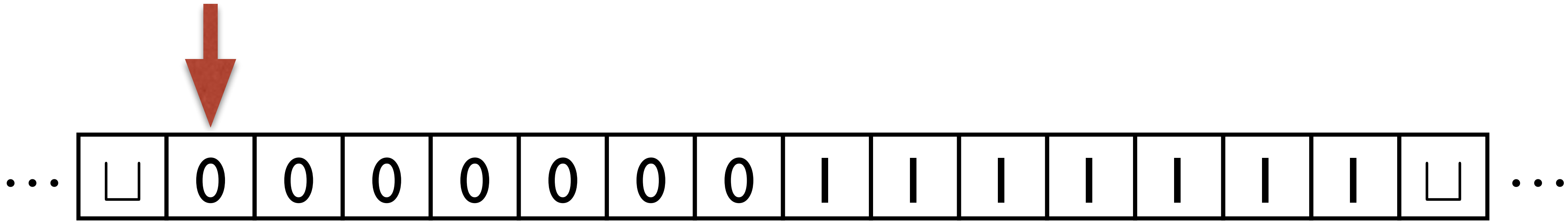

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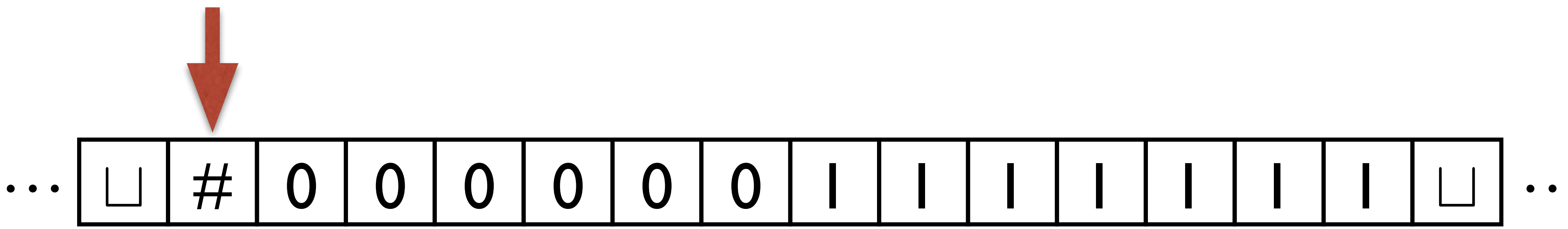

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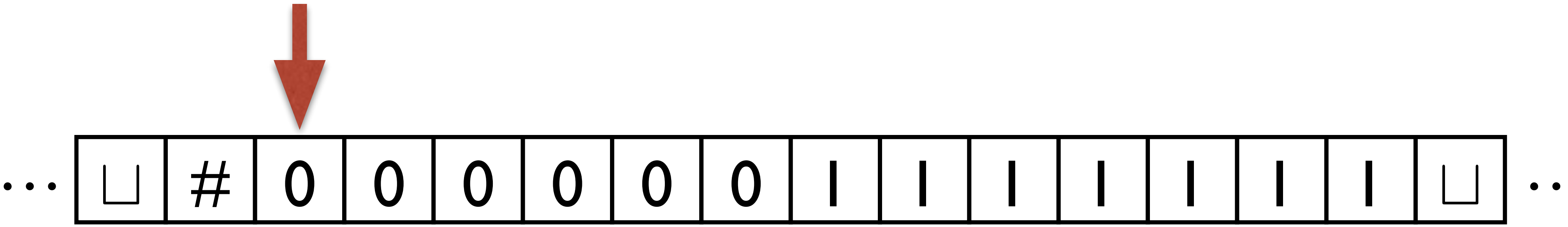

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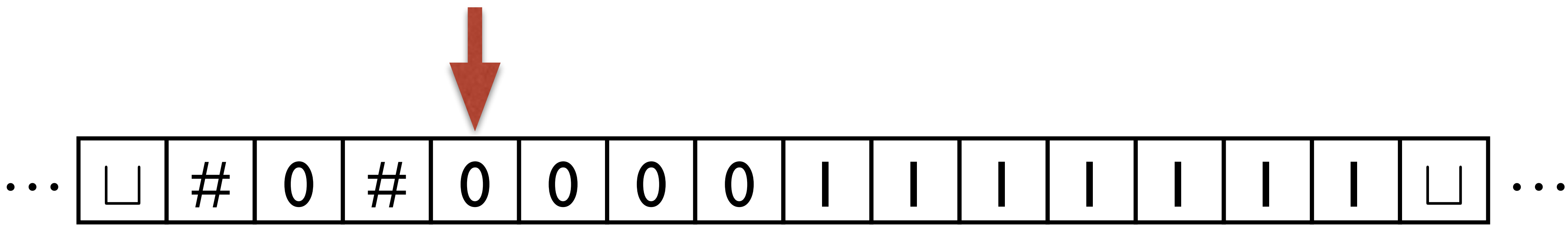

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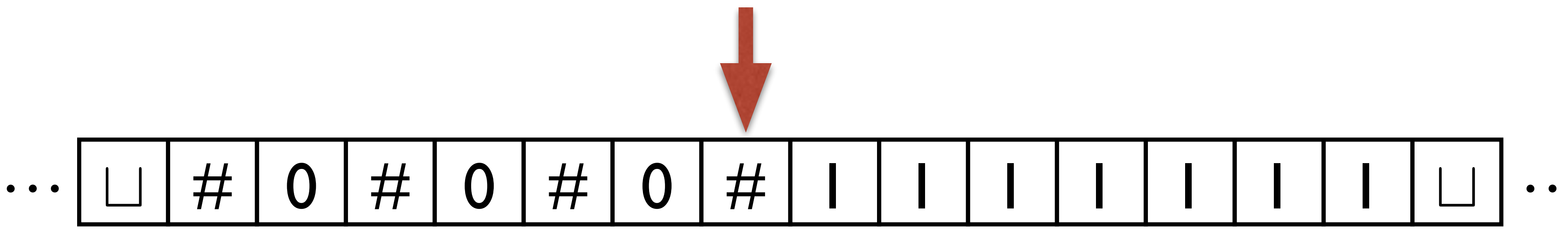

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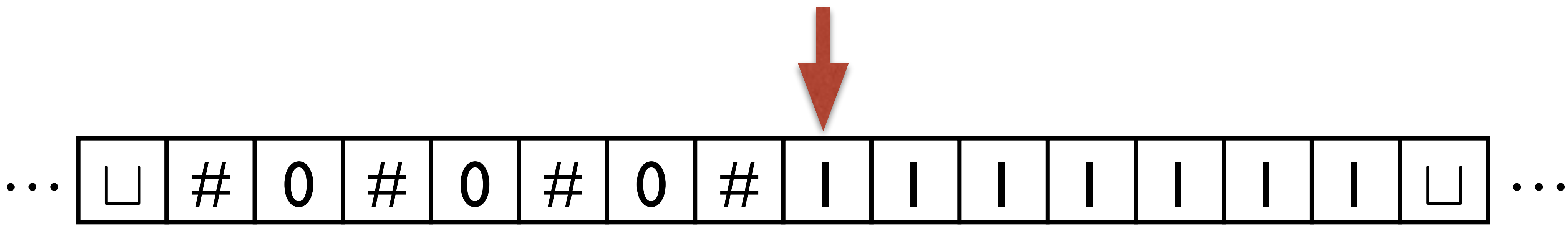

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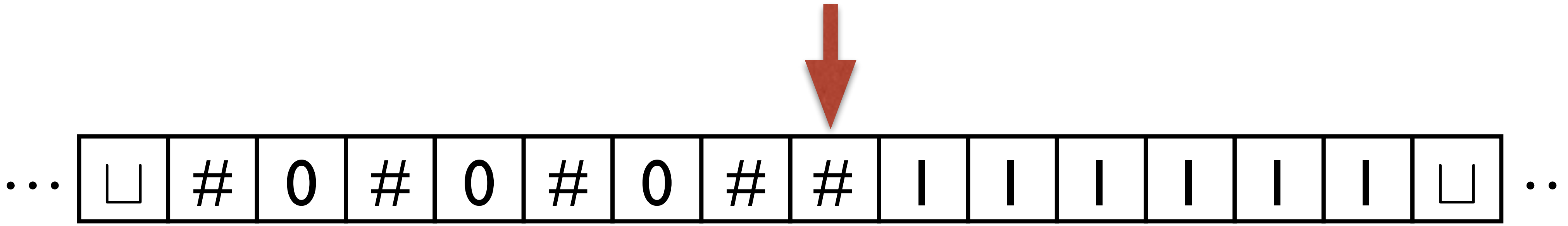

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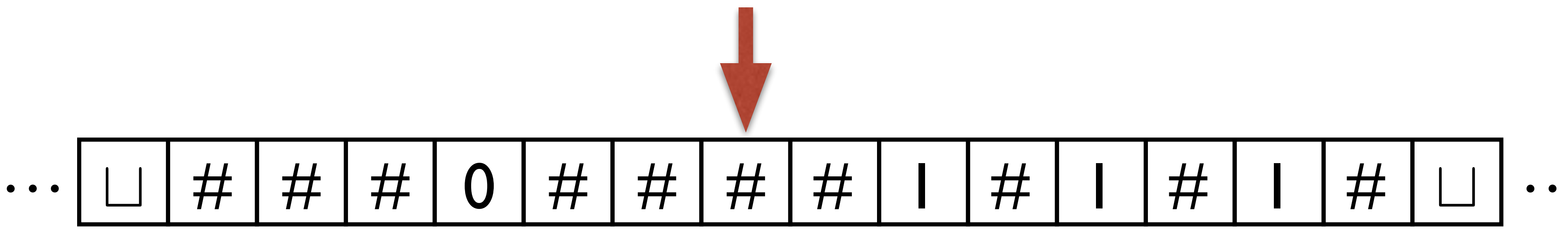

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How the model can affect time

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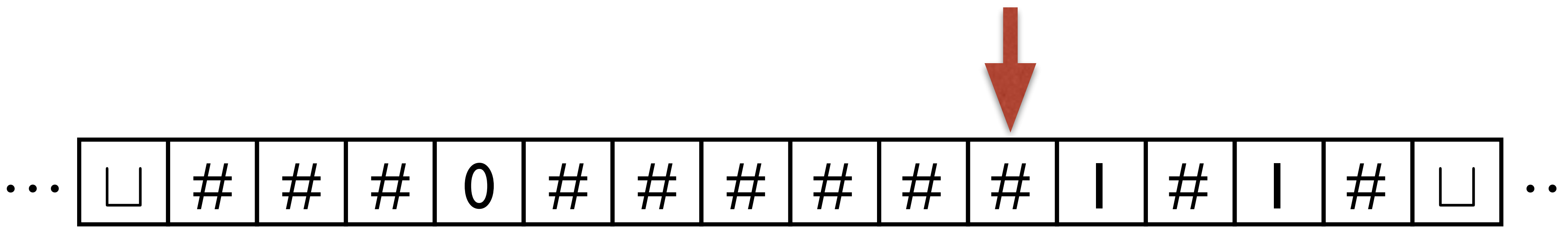

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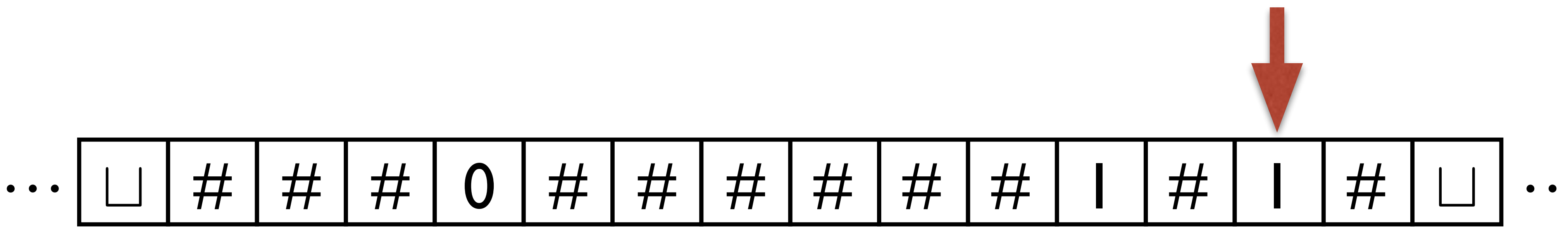

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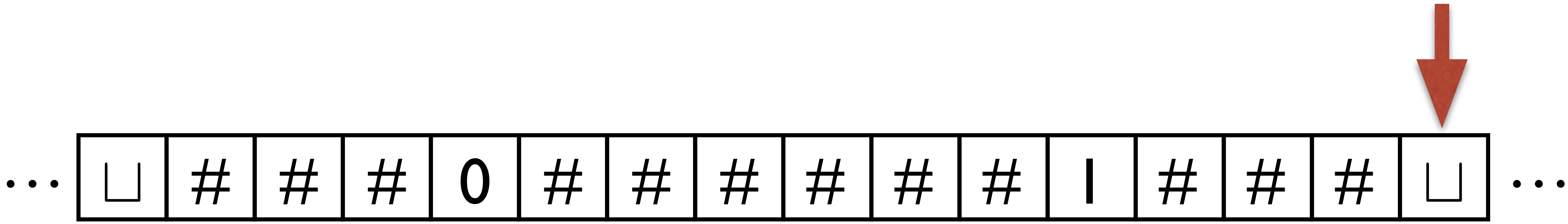

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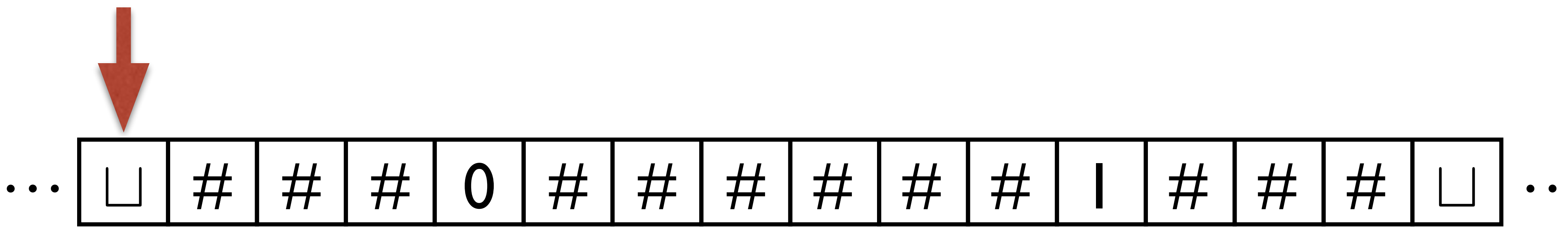

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- If no 0s and no 1s remain **accept**.
- Else **reject**.

Number of steps: $O(n \log n)$

How the model can affect time

$$L = \{0^k 1^k : k \geq 0\}$$

Why is it correct?

- Scan the tape. If (# of 1s + # of 0s) is odd, **reject**.
- Scan the tape. Cross off every other 0 starting with first 0.
  Cross off every other 1 starting with first 1.

(# of 1s + # of 0s) is odd if and only if

(# of 1s) and (# of 0s) have different parities.

Sequence of parities of (# of 1s) —>
binary representation of (# of 1s) in the input

Sequence of parities of (# of 0s) —>
binary representation of (# of 0s) in the input

How the model can affect time

$$L = \{0^k 1^k : k \geq 0\}$$

Can we do better?

$O(n \log n)$ is the best for 1-tape TMs.

$O(n)$ is the best for 2-tape TMs.

How the model can affect time

$$L = \{0^k 1^k : k \geq 0\}$$

A function in Python:

# of steps

```
def twoFingers(s):
    lo = 0                              1
    hi = len(s)-1                       1
    while (lo < hi):                    1
        if (s[lo] != 0 or s[hi] != 1):  3? 4? 5?
            return False                1
        lo += 1                         1
        hi -= 1                         1
    return True                         1
```

Seems like $O(n)$

How the model can affect time

$$L = \{0^k 1^k : k \geq 0\}$$

hi -= 1

Initially hi = n-1 (the length of the input - 1)

How many bits to store hi ? $\sim \log_2 n$

What if n is a power of 2 ?

$$\begin{array}{rl} \text{hi} = & 1\ 0\ 0\ 0\ 0\ 0\ \dots\ 0 \\ \text{hi} = & 0\ 1\ 1\ 1\ 1\ 1\ \dots\ 1 \end{array}$$

-1

$\sim \log_2 n$ steps

How the model can affect time

$$L = \{0^k 1^k : k \geq 0\}$$

A function in Python: **# of steps**

```
def twoFingers(s):            1
    lo = 0                    1
    hi = len(s)-1             1
    while (lo < hi):          3? 4? 5?
        if (s[lo] != 0 or s[hi] != 1):   1
            return False      1
        lo += 1               log n ?
        hi -= 1               1
    return True
```

$O(n \log n)$?

How the model can affect time

$$L = \{0^k 1^k : k \geq 0\}$$

if (s[lo] != 0 or s[hi] != 1):

Initially lo = 0, hi = n-1

Does it take n steps to go from s[0] to s[n-1] ?

How the model can affect time

$$L = \{0^k 1^k : k \geq 0\}$$

A function in Python:

# of steps

```
def twoFingers(s):              
    lo = 0                      ........ 1
    hi = len(s)-1               ........ 1
    while (lo < hi):            ........ 1
        if (s[lo] != 0 or s[hi] != 1):  ........ n ??
            return False        ........ 1
        lo += 1                 ........ 1
        hi -= 1                 ........ log n ?
    return True                 ........ 1
```

$O(n^2)$?

How the model can affect time

SO

Number of steps (running time)
depends on the particular model you choose.

Which one is the best model?

No such thing.

1. Be clear about what the model is!
2. All reasonable deterministic models
   are polynomially equivalent.

How the model can affect time

Which model does this correspond to ?

```
def twoFingers(s):
    lo = 0
    hi = len(s)-1
    while (lo < hi):
        if (s[lo] != 0 or s[hi] != 1):
            return False
        lo += 1
        hi -= 1
    return True
```

`lo = 0`	1
`hi = len(s)-1`	1
`while (lo < hi):`	1
`if (s[lo] != 0 or s[hi] != 1):`	3? 4? 5?
`return False`	1
`lo += 1`	1
`hi -= 1`	1
`return True`	1

$O(n)$

How the model can affect time

The Random-Access Machine (RAM) model

Good combination of reality/simplicity.

+ , - , / , *, <, >, etc. e.g. 245*12894 take 1 unit time

memory access e.g. A[94] takes 1 unit time

Note:

Good model when, say, you work with int data type.

Not a good model if you are working with 1000000-digit numbers.

Which model are we going to use?

Defining time complexity of an algorithm
and
Intrinsic complexity of a problem

Defining running time

Recall:

A computational problem *P* is just a function

$$P : \Sigma^* \to \Sigma^*$$

that maps instances to solutions.

If P is of the form $P : \Sigma^* \to \{0, 1\}$
it is called a decision problem.

An algorithm solves *P* if it outputs the correct solution on every instance.

Defining running time

With a specific computational model in mind:

Definition:

The **running time** of an algorithm *A* is a function

$$T_A : \mathbb{N}^+ \to \mathbb{N}^+$$

defined by

worst-case

$$T_A(n) = \max_{\substack{\text{instances } I \\ \text{of size } n}} \{\# \text{ steps } A \text{ takes on } I\}$$

We drop the subscript *A*, and write $T(n)$ when *A* is clear.

n always denotes the input length.

Why worst-case?

We are not dogmatic about it.

Can study "average-case" (random inputs)

Can try to look at "typical" instances.

Can do "smoothed analysis".

…

BUT worst-case analysis has its advantages:

- An ironclad guarantee.
- Matches our worst-case notion of an alg. solving a prob.
- Hard to define "typical" instances.
- Random instances are often not representative.
- Often much easier to analyze.

Defining intrinsic complexity

With a specific computational model in mind:

The **<u>intrinsic complexity</u>** (with respect to running time) of a problem

$$P : \Sigma^* \to \Sigma^*$$

is defined by

$$\min_{\substack{\text{algorithms } A \\ \text{that solve } P}} T_A(\cdot) \longleftarrow ???$$

How do you compare functions?

$$n^2 \leq 100n \quad ?$$

The CS way to compare functions:

$O(\cdot)$	$\Omega(\cdot)$	$\Theta(\cdot)$
$\leq$	$\geq$	$=$

Big Oh

Our notation for $\leq$ when comparing functions.

The idea is that these functions represent computational complexity (e.g. time complexity)

We want to use the right level of abstraction!

"Sweet spot"

- coarse enough to suppress details like programming language, compiler, architecture,…
- sharp enough to make comparisons between different algorithmic approaches.

Big Oh

$$8n^2 - 3n + 84$$

Analogous to "too many significant digits".

$$O(n^2)$$

- We don't care about constant factors.

    (even a change in alphabet size leads to constant factor difference)

    What if the running time is $10^{20}n^2$?

- We don't care about small values of n.

    (the only interesting instances are the big ones)

Big Oh

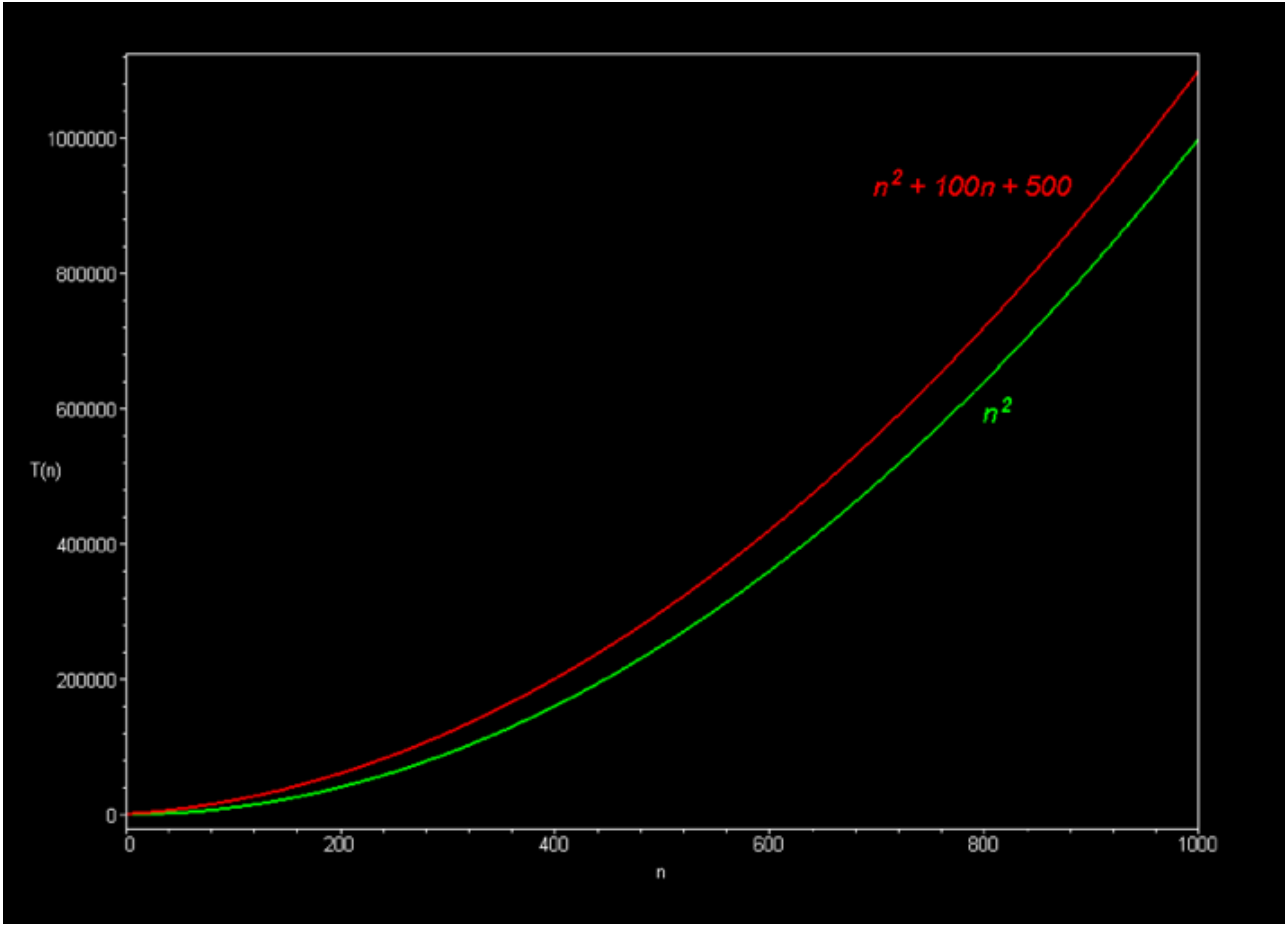
Informal: An upper bound that suppresses constant factors and ignores small n.

Suppressing constant factors means suppressing lower order additive terms.

$$n^2 + 100n + 500 \text{ is } O(n^2)$$

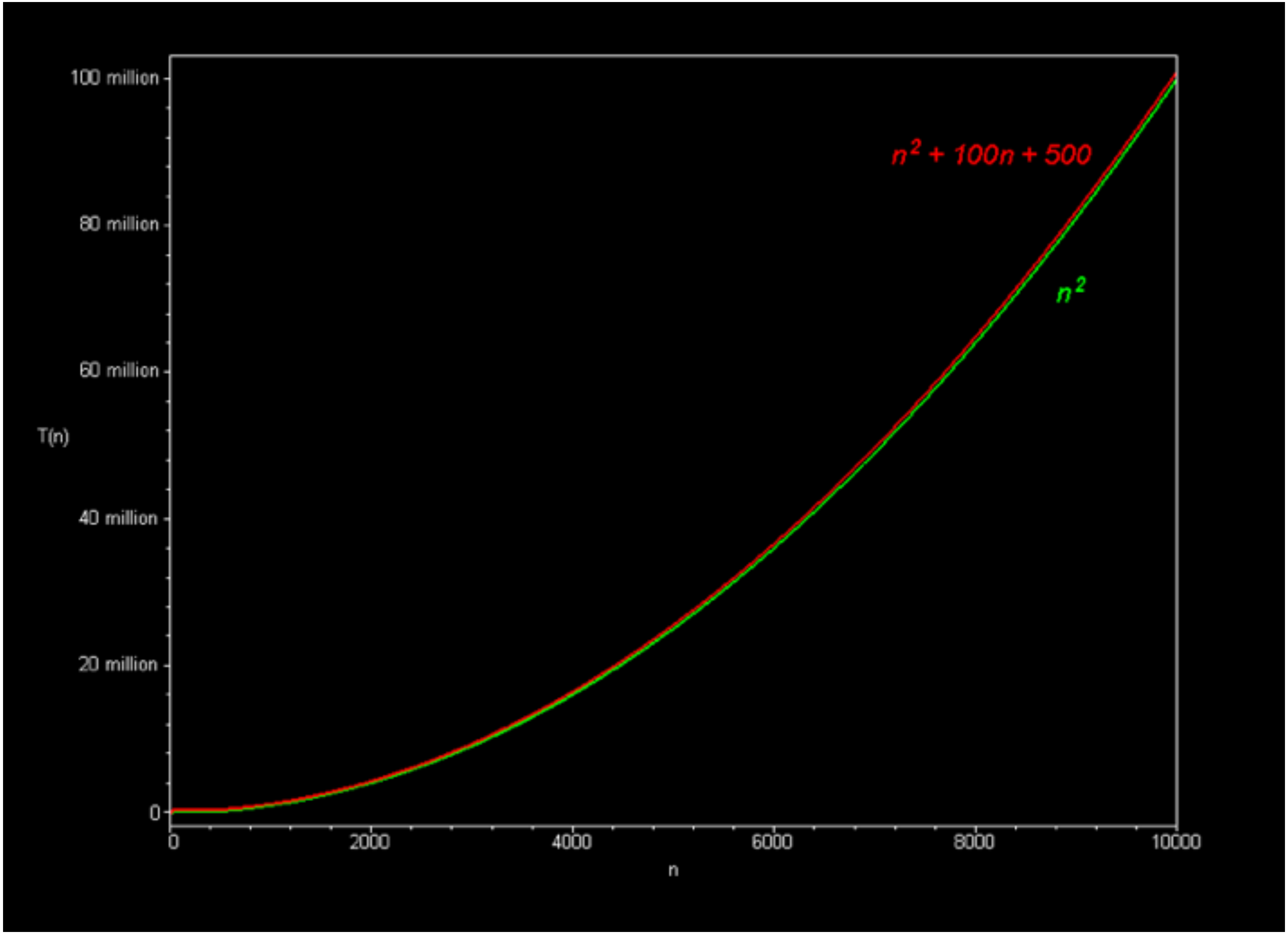
$$601n^2 = n^2 + 100n^2 + 500n^2 > n^2 + 100n + 500$$

Big Oh



1000000
800000
600000
400000
200000
0
T(n)
0
200
400
600
800
1000
n
$n^2 + 100n + 500$
n^2

Big Oh



100 million
80 million
60 million
40 million
20 million
0
T(n)
0
2000
4000
6000
8000
10000
n
$n^2 + 100n + 500$
n^2

Big Oh

Informal: An upper bound that suppresses constant factors and ignores small n.

For $f, g : \mathbb{N}^+ \to \mathbb{R}^+$

$f(n) = O(g(n))$ roughly means

$f(n) \leq g(n)$ up to a constant factor and ignoring small n.

Formal Definition:

For $f, g : \mathbb{N}^+ \to \mathbb{R}^+$, we say $f(n) = O(g(n))$ if there exists constants $C, n_0 > 0$ such that

$$f(n) \leq Cg(n) \qquad \text{for all } n \geq n_0.$$

(C and n_0 cannot depend on n.)

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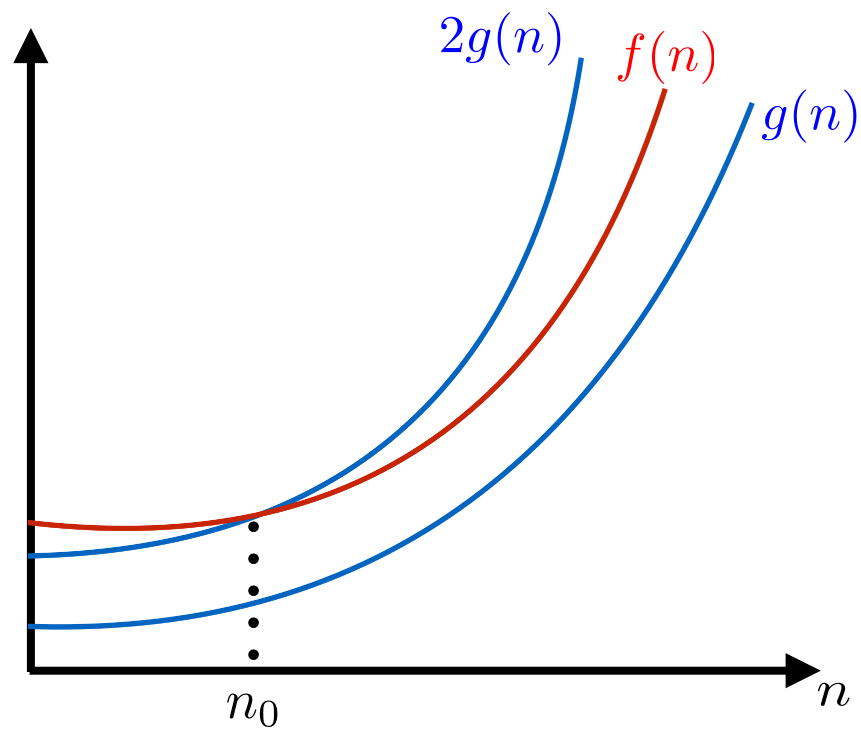

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(C and n_0 cannot depend on n.)

Example:

$$f(n) = 3n^2 + 10n + 30 \qquad g(n) = n^2$$

$$f(n) = O(g(n))$$

Take $C = 4, \quad n_0 = 13$

$$3n^2 + 10n + 30 \leq 4n^2 \quad \text{when} \quad n \geq 13$$

Big Oh

Example:

$$f(n) = 3n^2 + 10n + 30 \qquad\qquad g(n) = n^2$$

$$f(n) = O(g(n))$$

Take $C = 4, \quad n_0 = 13$

$$3n^2 + 10n + 30 \ \leq \ 4n^2 \quad \text{when} \quad n \geq 13$$

Proving f(n) is O(g(n)) is like a game:

You pick constants C, n_0

Adversary picks $n \geq n_0$

You win if $f(n) \leq Cg(n)$

You need to make sure you always win.

Big Oh

$1000n$ is $O(n)$

$0.0000001n$ is $O(n)$

$0.1n^2 + 10^{20}n + 10^{10000}$ is $O(n^2)$

n is $O(2^n)$

$0.0000001n^2$ <u>is not</u> $O(n)$

$n \log n$ <u>is not</u> $O(n)$

$\log_9 n$ is $O(\log n)$

$$\boxed{\log_b(n) = \frac{\log_k(n)}{\log_k(b)}}$$

constant

10^{10} is $O(1)$

<u>**Note on notation:**</u>

People usually write $4n^2 + 2n = O(n^2)$

Better notation would be $4n^2 + 2n \in O(n^2)$

Run time scaling

Running-time:			Ratio:
$c \cdot n$	double the input $\longrightarrow$	$c \cdot 2n$	2
$c \cdot n^2$	double the input $\longrightarrow$	$c \cdot (2n)^2$	4
$c \cdot n^3$	double the input $\longrightarrow$	$c \cdot (2n)^3$	8
$c \cdot n^k$	double the input $\longrightarrow$	$c \cdot (2n)^k$	2^k (constant)
$c \cdot 2^n$	double the input $\longrightarrow$	$c \cdot 2^{2n}$	2^n

Big Oh

Common Big Oh classes and their names

Constant: $O(1)$

Logarithmic: $O(\log n)$

Square-root: $O(\sqrt{n}) = O(n^{0.5})$

Linear: $O(n)$

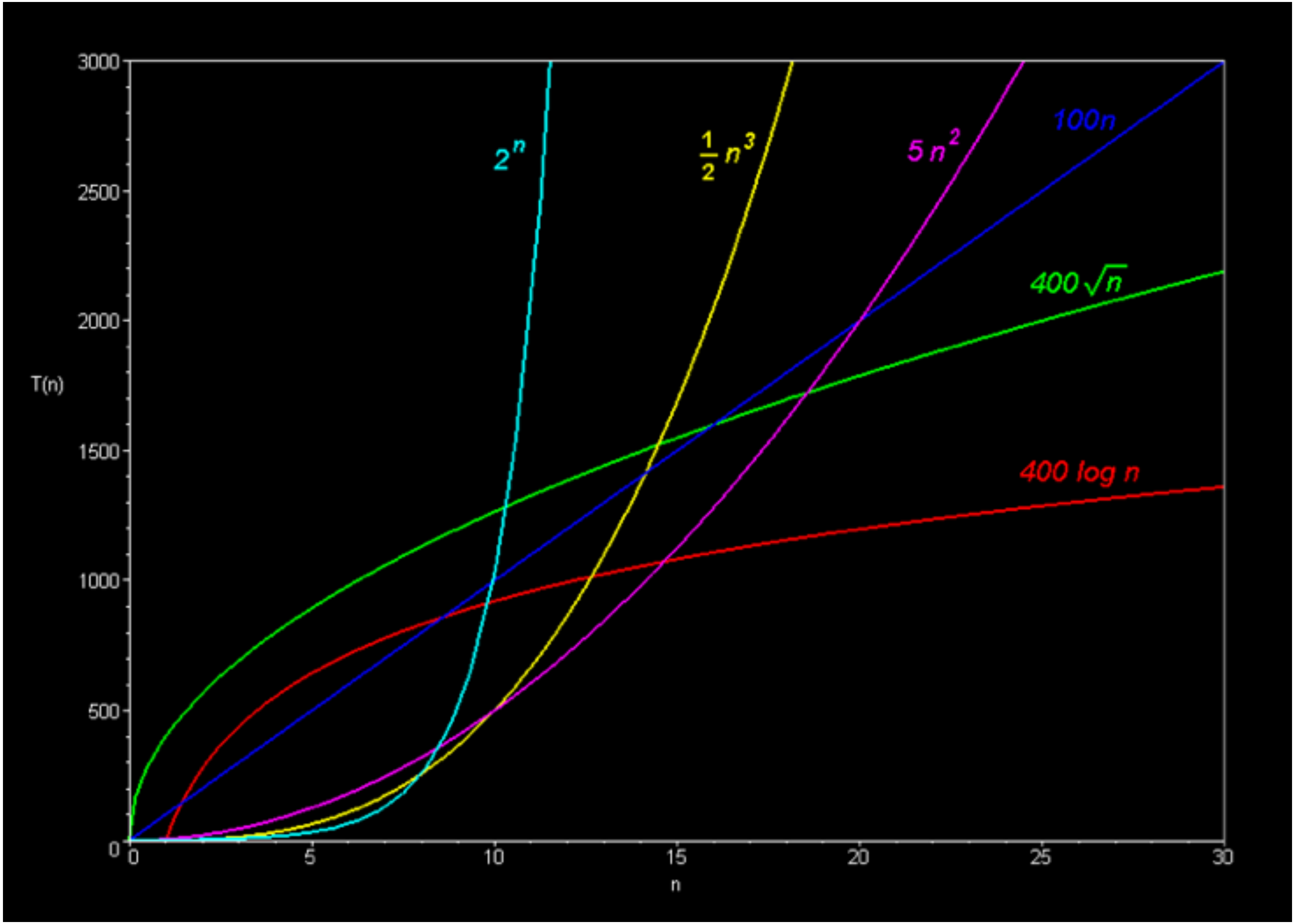
Loglinear: $O(n \log n)$

Quadratic: $O(n^2)$

Polynomial: $O(n^k)$

Exponential: $O(k^n)$

Big Oh



3000
2500
2000
1500
1000
500
0
T(n)
0
5
10
15
20
25
30
n
2^n
$\frac{1}{2}n^3$
$5n^2$
$100n$
$400\sqrt{n}$
$400 \log n$

n vs log n

How much smaller is log n compared to n ?

n	log n
2	1
8	3
128	7
1024	10
1,048,576	20
1,073,741,824	30
1,152,921,504,606,846,976	60

~ 1 quintillion

n vs 2^n

How much smaller is n compared to 2^n ?

2^n	n
2	1
8	3
128	7
1024	10
1,048,576	20
1,073,741,824	30
1,152,921,504,606,846,976	60

Exponential running time

If your algorithm has exponential running time
e.g. $\sim 2^n$

No hope of being practical.

Exponential running time: Example

Given a list of integers, determine if there is a subset of the integers that sum to 0.

4	-3	-2	7	99	5	1

Exponential running time: Example

Given a list of integers, determine if there is a subset of the integers that sum to 0.

4	-3	-2	7	99	5	1

Exhaustive Search (Brute Force Search):

Try every possible subset and see if it sums to 0.

Number of subsets is 2^n

So running time is at least 2^n

Big Oh

$$\log n <<< \sqrt{n} << n < n \log n << n^2 << n^3 <<< 2^n <<< 3^n$$

The theoretical divide between efficient and inefficient:

If it is not $O(n^k)$ for some constant k

(if it does not have polynomial complexity)

then it is inefficient.

Some exotic functions

1	n	2^n
$\log^* n$	$n \log n$	3^n
$\log \log n$	n^2	$n!$
$\log n$	n^3	n^n
$\sqrt{n}$	$n^{O(1)}$	2^{2^n}
$n / \log n$	$n^{\log n}$	$\underbrace{2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}}_{n \text{ times}}$

<u>Fastest algorithm for multiplication:</u>

$$n \cdot (\log n) \cdot 2^{O(\log^* n)}$$

Big Omega

If $O(\cdot)$ is like $\leq$

$\Omega(\cdot)$ is like $\geq$

$O(\cdot)$

Informal: An upper bound that suppresses constant factors and ignores small n.

$\Omega(\cdot)$

Informal: A lower bound that suppresses constant factors and ignores small n.

Big Omega

$\Omega(\cdot)$

Informal: A lower bound that suppresses constant factors and ignores small n.

Formal Definition:

For $f, g : \mathbb{N}^+ \to \mathbb{R}^+$, we say $f(n) = \Omega(g(n))$ if there exists constants c , $n_0 > 0$ such that

$$f(n) \geq cg(n) \qquad \text{for all } n \geq n_0.$$

(c and n_0 cannot depend on n.)

Big Omega

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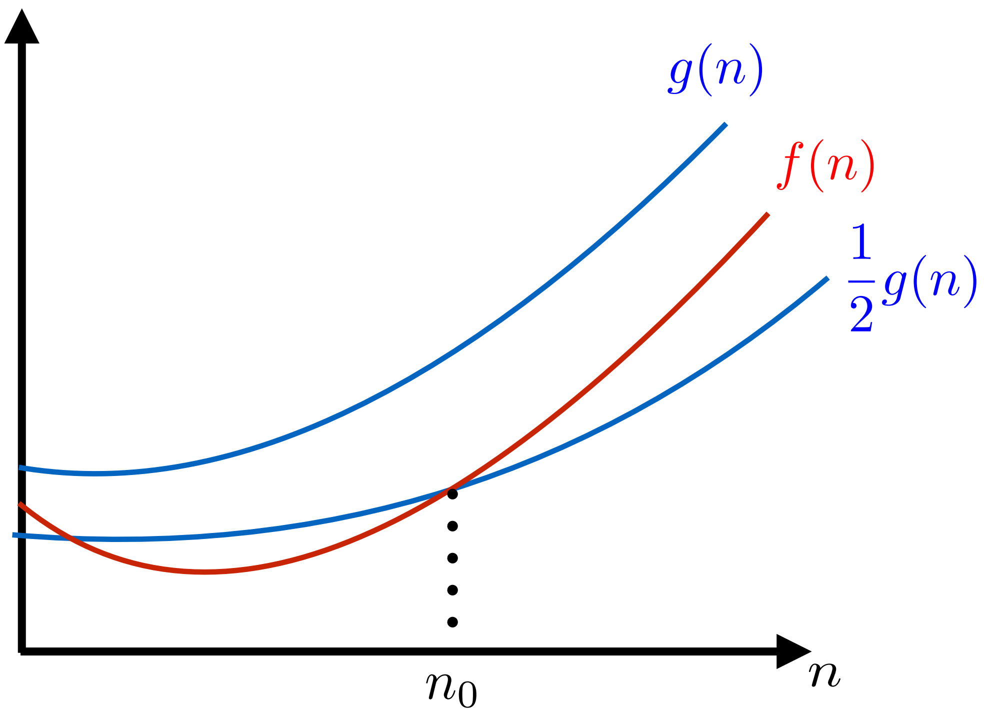

Big Omega

Some Examples:

$$10^{-10}n^4 \text{ is } \Omega(n^3)$$

$$0.001n^2 - 10^{10}n - 10^{30} \text{ is } \Omega(n^2)$$

$$n^{0.0001} \text{ is } \Omega(\log n)$$

$$n^{1.0001} \text{ is } \Omega(n \log n)$$

Theta

If $O(\cdot)$ is like $\leq$

and $\Omega(\cdot)$ is like $\geq$

$\Theta(\cdot)$ is like $=$

Theta

Formal Definition:

For $f, g : \mathbb{N}^+ \to \mathbb{R}^+$, we say $f(n) = \Theta(g(n))$ if

$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Equivalently:

There exists constants c, C, n_0 such that

$cg(n) \leq f(n) \leq Cg(n)$ for all $n \geq n_0$.

Back to intrinsic complexity

Defining intrinsic complexity

With a specific computational model and resource in mind:

The **intrinsic complexity** of a problem is the complexity of the most efficient algorithm solving it.

Intrinsic complexity

If you give an algorithm that solves a problem

→ upper bound on the intrinsic complexity

How to show a lower bound on the intrinsic complexity?

Argue against **all** possible algorithms that solves the problem.

The dream: Get a matching upper and lower bound.

Example

$$L = \{0^k 1^k : k \geq 0\}$$

```
def twoFingers(s):
    lo = 1
    hi = len(s)
    while (lo < hi):
        if (s[lo] != 0 or s[hi] != 1):
            return False
        lo += 1
        hi -= 1
    return True
```

In the RAM model:

$$O(n)$$

Could there be
a faster algorithm?

e.g. $O(n/\log n)$

Example

$$L = \{0^k 1^k : k \geq 0\}$$

Fact: Any algorithm that decides L must use $\geq n$ steps.

Proof:

Suppose there is an algorithm A that decides *L* in $< n$ steps.

Let I be instance (input) $a^k b^k$

When A runs on input I, there must be some index j such that A never reads $I[j]$.

Let I' be the same as I, but with j'th coordinate reversed. (I' is a NO instance)

When A runs on I', it has the same behavior as it does on I

But then A does not decide *L*. *Contradiction*. $\square$

Example

$$L = \{0^k 1^k : k \geq 0\}$$

Fact: Any algorithm that decides L must use $\geq n$ steps.

This shows the intrinsic complexity of *L* is $\Omega(n)$.

But we also know the intrinsic complexity of *L* is $O(n)$.

The dream achieved. Intrinsic complexity is $\Theta(n)$.

Representation of the input

How you represent the input matters

Technically, how the input is represented/encoded should be part of the problem description.

If it is not specified, input length is the number of bits needed to represent the input.

You should be careful about this!

How you represent the input matters

Multiplication Problem

Input: 2 numbers s and t

Output: the product of s and t

Obvious algorithm: Add s to itself t times.

How is the input represented?

s = 111...1 (s many 1s) $n = s + t$

t = 111...1 (t many 1s)

Running time: $O(st)$ $O(n^2)$

How you represent the input matters

Multiplication Problem

Input: 2 numbers s and t

Output: the product of s and t

<u>Obvious algorithm</u>: Add s to itself t times.

How is the input represented?

s = binary representation of s $\quad n \approx \log_2 s + \log_2 t$

t = binary representation of t

We'll do s+s in binary, so it is $O(\log s)$ steps.

Running time: $O(t \log s)$ $\quad O(2^n)$

How you represent the input matters

Multiplication Problem

Input: 2 numbers s and t

Output: the product of s and t

Obvious algorithm: Add s to itself t times.

This algorithm actually sucks!

When dealing with problems with integer inputs:

we want to be able to deal with numbers with say a million binary digits.

So numbers of magnitude 2^{10^6}.

How you represent the input matters

Multiplication Problem

Input: 2 numbers s and t

Output: the product of s and t

<u>Obvious algorithm</u>: Add s to itself t times.

Is there a more efficient algorithm?

$$
\begin{array}{cccccccc}
 & & & & 5 & 6 & 7 & 8 \\
\times & & & & 1 & 2 & 3 & 4 \\
\hline
 & & & 2 & 2 & 7 & 1 & 2 \\
 & & 1 & 7 & 0 & 3 & 4 & \\
 & 1 & 1 & 3 & 5 & 6 & & \\
+ & 5 & 6 & 7 & 8 & & & \\
\hline
7 & 0 & 0 & 6 & 6 & 5 & 2 &
\end{array}
$$

$O(n^2)$

Can we do better?

Strong Church Turing Thesis

Church Turing Thesis

Church-Turing Thesis:

The intuitive notion of "computable" is captured by functions computable by a Turing Machine.

Physical Church-Turing Thesis:

Any computational problem that can be solved by a physical device, can be solved by a Turing Machine.

Strong Church-Turing Thesis:

The intuitive notion of "efficiently computable" is captured by functions efficiently computable by a TM.

Strong Church Turing Thesis

Experience suggests it is true for all *deterministic* models.

First main challenger in 1970s:

Randomized computation.

In light of research from 1980s, we believe
SCCT holds even with randomized computation.

Second main challenger in 1980s:

Quantum computation.

In light of research from 1990s, we believe
SCCT is not true!

Challenge all ideas!