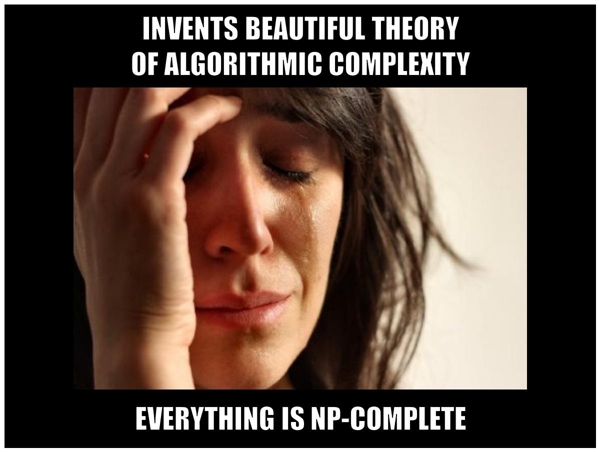
15-251: Great Theoretical Ideas in Computer Science
Lecture 15

Approximation Algorithms

SAT	given propositional formula F, is it satisfiable?
3SAT	same, but F is a 3CNF
Vertex-Cover	given G and k... are there k vertices which touch all edges?
Clique	are there k vertices all connected?
Max-Cut	is there a vertex 2-coloring with at least k "cut" edges?
Hamiltonian-Cycle	is there a cycle touching each vertex exactly once?

SAT	... is **NP-complete**
3SAT	... is **NP-complete**
Vertex-Cover	... is **NP-complete**
Clique	... is **NP-complete**
Max-Cut	... is **NP-complete**
Hamiltonian-Cycle	... is **NP-complete**



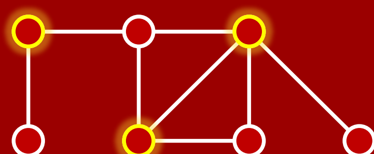
There is only one idea in this lecture:

Don't Give Up

Vertex-Cover

Given graph G = (V,E) and number k,
is there a size-k "vertex-cover" for G?

(S ⊆ V is a "vertex-cover" if it touches all edges.)



G has a vertex-cover of size 3.

Vertex-Cover

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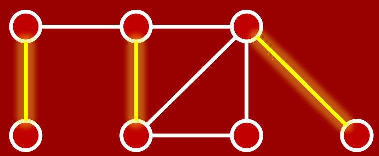


G has no vertex-cover of size 2.
(Because you need ≥ 1 vertex per yellow edge.)

Vertex-Cover

Given graph G = (V,E) and number k,
is there a size-k "vertex-cover" for G?

(S ⊆ V is a "vertex-cover" if it touches all edges.)

The Vertex-Cover problem is NP-complete. ☹

∴ assuming "P ≠ NP", there is **no** algorithm
running in **polynomial time**
which, for **all graphs** G,
finds the **minimum**-size vertex-cover.

Don't Give Up

Subexponential-time algorithms:

- Brute-force tries all 2^n subsets of n vertices.
- Maybe there's an $O(1.5^n)$-time algorithm.
- Or $O(1.1^n)$ time, or $O(2^{n^{.1}})$ time, or…
- Could be quite okay if n = 100, say.

As of 2010: there **is** an $O(1.28^n)$-time algorithm.

∴ assuming "P ≠ NP", there is **no** algorithm
running in **polynomial time**
which, for **all graphs** G,
finds the **minimum**-size vertex-cover.

Don't Give Up

Special cases:

Solvable in poly-time for…

- **tree** graphs,
- **bipartite** graphs,
- "**series-parallel**" graphs…

Perhaps for "graphs encountered in practice"?

∴ assuming "P ≠ NP", there is **no** algorithm
running in **polynomial time**
which, for **all graphs** G,
finds the **minimum**-size vertex-cover.

Don't Give Up

Approximation algorithms:

Try to find *pretty small* vertex-covers.

Still want polynomial time, and for **all** graphs.

∴ assuming "P ≠ NP", there is **no** algorithm
running in **polynomial time**
which, for **all graphs** G,
finds the **minimum**-size vertex-cover.

Gavril's Approximation Algorithm

Easy Theorem (from 1976):

There is a **polynomial-time** algorithm that,
given **any** graph G = (V,E),
outputs a vertex-cover S ⊆ V such that

$$|S| \le 2|S^*|$$

where S^* is the **smallest** vertex-cover.

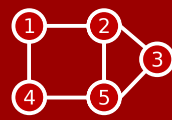
"A factor 2-approximation for Vertex-Cover."

Let's recall a similar situation from Lecture 10:

My favorite problem, **Max-Cut**.

Max-Cut

Input: A graph G=(V,E).

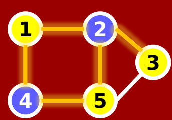


Output: A "**2-coloring**" of V: each vertex designated yellow or blue.

Goal: Have as many **cut** edges as possible. An edge is *cut* if its endpoints have different colors.

Max-Cut

Input: A graph G=(V,E).



Output: A "**2-coloring**" of V: each vertex designated yellow or blue.

Goal: Have as many **cut** edges as possible. An edge is *cut* if its endpoints have different colors.

Max-Cut

On one hand:
Finding the **MAX**-Cut is NP-hard.

On the other hand:
Polynomial-time "Local Search" algorithm guarantees cutting ≥ ½|E| edges.

In particular:
(# cut by Local Search) ≥ ½ (max # cuttable)

"A factor ½-approximation for Max-Cut."

Max-Cut

By the way:

Goemans and Williamson (1994) gave a polynomial-time

0.87856-approximation

for Max-Cut.

It is very beautiful, but pretty difficult!

A technicality: **Optimization vs. Decision**

NP defined to be a class of **decision problems**.
This is for technical convenience.
Usually have natural 'optimization' version.

3SAT	Given a 3-CNF formula, is it satisfiable?
Vertex-Cover	Given G and k, are there k vertices which touch all edges?
Clique	Given G and k, are there k vertices which are all mutually connected?
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3SAT	
Vertex-Cover	Given G, find the smallest S ⊆ V touching all edges.
Clique	Given G, find the largest clique (set of mutually connected vertices).
Max-Cut	Given G, find the largest number of edges 'cut' by some vertex 2-coloring.
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Vertex-Cover	Given G, find the smallest S ⊆ V touching all edges.
Clique	Given G, find the largest clique (set of mutually connected vertices).
Max-Cut	Given G, find the largest number of edges 'cut' by some vertex 2-coloring.
TSP	Given G with edge costs, find the cheapest cycle touching each vertex exactly once.

A technicality: **Optimization vs. Decision**

NP defined to be a class of **decision problems**.
This is for technical convenience.
Usually have natural 'optimization' version.

Technically, the 'optimization' versions can't be in NP, since they're not decision problems.

We often still say they are NP-hard.
This means: *if* you could solve them in poly-time, *then* you could solve any NP problem in poly-time.

Let's not worry about this boring technicality!

Not all NP-hard problems created equal!

3SAT, Vertex-Cover, Clique, Max-Cut, TSP, …

All of these problems are equally NP-hard.

(There's no poly-time algorithm to find the optimal solution unless P = NP.)

But from the point of view of finding ***approximately*** optimal solutions, there is an intricate, fascinating, and wide range of possibilities…

Today: A case study of approximation algorithms

1. A somewhat good approximation algorithm for Vertex-Cover.
2. A pretty good approximation algorithm for the "k-Coverage Problem".
3. Some very good approximation algorithms for TSP.

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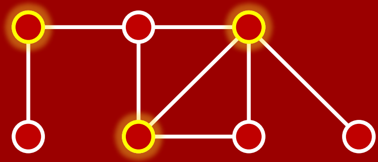
Vertex-Cover

Given graph G = (V,E) try to find the smallest “vertex-cover” for G.

(S ⊆ V is a “vertex-cover” if it touches all edges.)

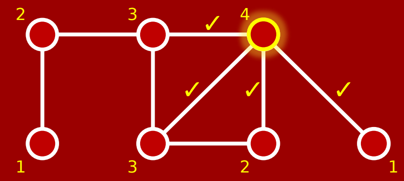


A possible Vertex-Cover algorithm

Simplest heuristic you might think of:

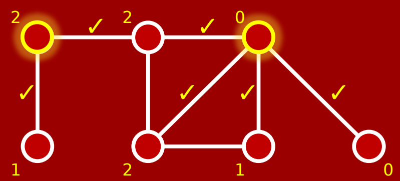
GreedyVC(G)
  S := ∅
  while **not** all edges marked as “covered”
    find v∈V touching most unmarked edges
    S := S ∪ {v}
    mark all edges v touches

GreedyVC example



GreedyVC example

(Break ties arbitrarily.)



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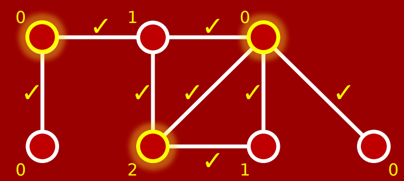

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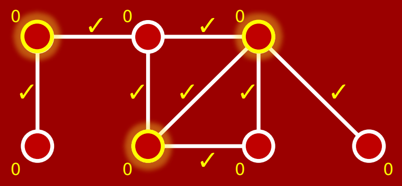


Done. Vertex-cover size 3 (optimal) ☺.

GreedyVC analysis

Correctness:

✓ Always outputs a **valid** vertex-cover.

Running time:

✓ Polynomial time (good enough).

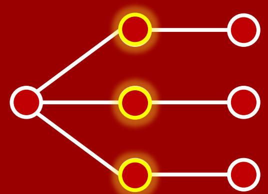
Solution quality:

This is the interesting question.

There must be some graph G where it doesn't find the **smallest** vertex-cover.

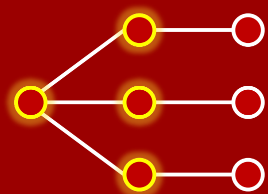
Because otherwise... P = NP!

A bad graph for GreedyVC



Smallest? 3

A bad graph for GreedyVC

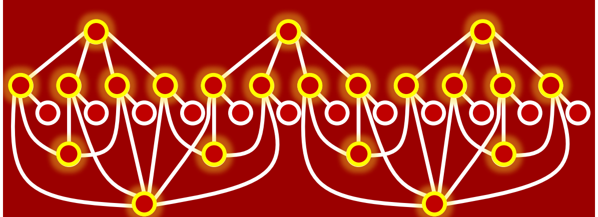


Smallest? 3

GreedyVC? 4

So GreedyVC is **not** even a 1.33-approximation. (Because 1.33 < 4/3.)

A worse graph for GreedyVC



Smallest? 12

GreedyVC? 21

So GreedyVC is **not** even a 1.74-approximation. (Because 1.74 < 21/12.)

Even worse graph for GreedyVC

Well... it's a good homework problem.

We know GreedyVC is **not** a 1.74-approximation.

Fact: GreedyVC is **not** a 2.08-approximation.

Fact: GreedyVC is **not** a 3.14-approximation.

Fact: GreedyVC is **not** a 42-approximation.

Fact: GreedyVC is **not** a 999-approximation.

Greed is Bad (for Vertex-Cover)

Theorem: ∀C, GreedyVC is **not** a C-approximation.

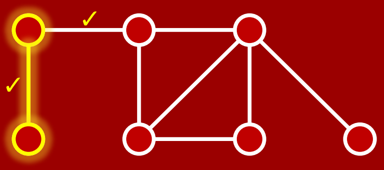
In other words:
For any constant C,
there is a graph G such that

|GreedyVC(G)| > C · |Min-Vertex-Cover(G)|.

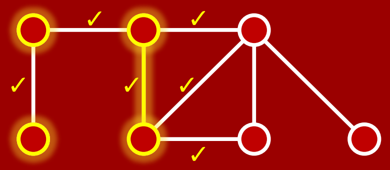
Gavril to the rescue

GavrilVC(G)
S := ∅
while **not** all edges marked as "covered"
let {v,w} be any unmarked edge
S := S ∪ {v,w} ❢
mark all edges v,w touch

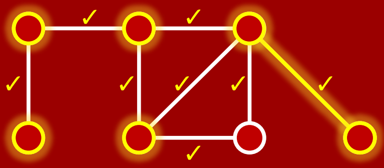
GavrilVC example



GavrilVC example



GavrilVC example



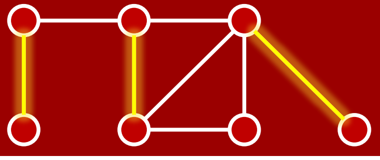
Smallest: 3

GavrilVC: 6

So GavrilVC is **at best** a 2-approximation.

Theorem:
GavrilVC is a **2**-approximation for Vertex-Cover.

Proof:
Say GavrilVC(G) does T iterations. So its |S| = <u>2T</u>.
Say it picked edges $e_1, e_2, \ldots, e_T \in E$.
Key claim: $\{e_1, e_2, \ldots, e_T\}$ is a <u>matching</u>.
Because… when e_j is picked, it's unmarked,
so its endpoints are not among $e_1, \ldots, e_{j-1}$.
So **any** vertex-cover must have ≥ 1 vertex from each e_j.



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Because… when e_j is picked, it's unmarked,
so its endpoints are not among $e_1, \ldots, e_{j-1}$.
So **any** vertex-cover must have ≥ 1 vertex from each e_j.
Including the **minimum** vertex-cover S^*, whatever it is.
Thus $|S^*| \geq T$.
So for Gavril's final vertex-cover S,

$$|S| = 2T \leq 2|S^*|.$$ ∎

Today: A case study of approximation algorithms

1. A 2-approximation algorithm for Vertex-Cover.
2. A pretty good approximation algorithm for the "k-Coverage Problem".
3. Some very good approximation algorithms for TSP.

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"k-Coverage" problem

"Pokémon-Coverage" problem

Let's say you have some Pokémon,

and some trainers, each having a subset of Pokémon.

Given k, choose a team of k trainers to maximize the # of distinct Pokémon.

"Pokémon-Coverage" problem

This problem is NP-hard. ☹

Approximation algorithm?

We could try to be greedy again…

```
GreedyCoverage()
  for i = 1…k
  add to the team the trainer bringing in the
    most new Pokémon, given the team so far
```

Example with k=3:

Optimum: 27

GreedyCoverage: 21

So Greedy is **at best** a 77.7%-approximation.

Greed is Pretty Good (for k-Coverage)

Theorem:

GreedyCoverage is a **63%**-approximation for k-Coverage.

More precisely, $1-1/e$

where $e \approx 2.718281828...$

Proof: (Don't read if you don't want to.)

Let P^* be the Pokémon covered by the best k trainers.
Define $r_i = |P^*| - \#$ Pokémon covered after i steps of Greedy.
We'll prove by induction that $r_i \le (1-1/k)^i \cdot |P^*|$.
The base case i=0 is clear, as $r_0 = |P^*|$.
For the inductive step, suppose Greedy enters its ith step.
At this point, the number of uncovered Pokémon in P^* must be $\ge r_{i-1}$.
We know there are some k trainers covering all these Pokémon.
Thus one of these trainers must cover at least r_{i-1}/k of them.
Therefore the trainer chosen in Greedy's ith step will cover $\ge r_{i-1}/k$ Pokémon.
Thus $r_i \le r_{i-1} - r_{i-1}/k = (1-1/k) \cdot r_{i-1} \le (1-1/k) \cdot (1-1/k)^i \cdot |P^*|$ by induction.
Thus we have completed the inductive proof that $r_i \le (1-1/k)^i \cdot |P^*|$.
Therefore the Greedy algorithm terminates with $r_k \le (1-1/k)^k \cdot |P^*|$.
Since $1-1/k \le e^{-1/k}$ (Taylor expansion), we get $r_k \le e^{-1} \cdot |P^*|$.
Thus Greedy covers at least $|P^*| - e^{-1} \cdot |P^*| = (1-1/e) \cdot |P^*|$ Pokémon.
This completes the proof that Greedy is a $(1-1/e)$-approximation algorithm. $\square$

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TSP

(Traveling Salesperson Problem)

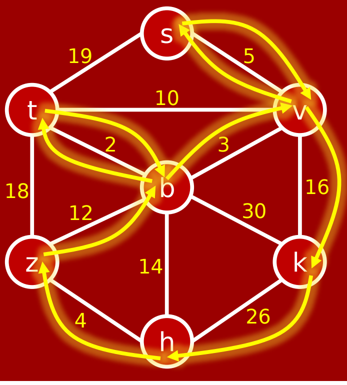
Many variants. Most common is "Metric-TSP":

Input: A graph G=(V,E) with edge costs.

Output: A "tour": i.e., a walk that visits each vertex **at least** once, and starts and ends at the same vertex.

Goal: Minimize total cost of tour.

TSP example



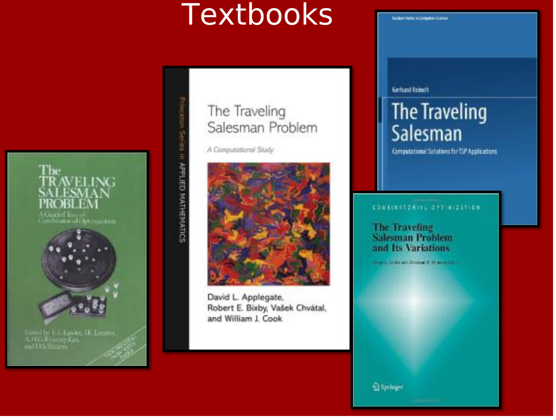
Cheapest tour:

3
\+ 5
\+ 5
\+ 16
\+ 26
\+ 4
\+ 12
\+ 2
\+ 2
= **71**

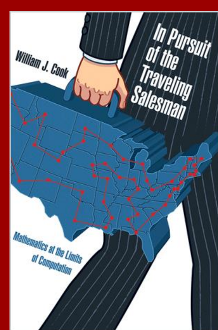
TSP is probably the most famous NP-complete problem.

It has inspired many things…

Textbooks

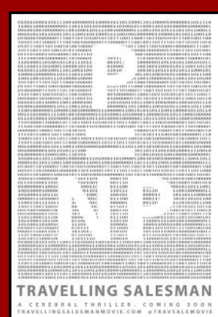


“Popular” books



Museum exhibits

Movies



(LPT: do not watch this movie)

’60s sitcom-themed household-goods conglomerate ad/contests

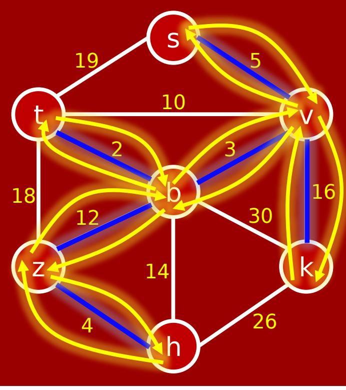
People genuinely want to solve large instances.

Applications in:
- Schoolbus routing
- Moving farm equipment
- Package delivery
- Space interferometer scheduling.
- Circuit board drilling
- Genome sequencing
- …

Basic Approximation Algorithm: The MST Heuristic

Given G with edge costs…
1. Compute an **MST** T for G, rooted at any s ∊ V.
2. Visit the vertices via **DFS** from s.

MST Heuristic example



Step 1: MST
Step 2: DFS

Valid tour? ✓

Poly-time? ✓

Cost?

2 × MST Cost

(84 in this case)

MST Heuristic

Theorem: MST Heuristic is factor-2 approximation.

Key Claim: Optimal TSP cost ≥ MST Cost always.

This implies the Theorem, since
MST Heuristic Cost = **2** × MST Cost.

Proof of Claim:

Take all edges in optimal TSP solution.
They form a connected graph on all |V| vertices.
Take any spanning tree from within these edges.
Its cost is at least the MST Cost.
Therefore the original TSP tour’s cost is ≥ MST Cost. □

Can we do better?

Nicos Christofides, Tepper faculty, 1976:

There is a polynomial-time, factor **1.5**-approximation algorithm for (Metric) TSP.

Proof is not **too** hard. Ingredients:
- MST Heuristic
- Enjoyable (Eulerian) graphs
- Cheapest Perfect Matching algorithm

Even better in a special case

In the important special case "Euclidean-TSP",
vertices are points in $\mathbb{R}^2$,
costs are just the straight-line distances.

This special case is still NP-hard.

Theorem (Arora, Mitchell, 1998):
For Euclidean-TSP, there is a polynomial-time factor 1.3 approximation algorithm.


Even better in a special case

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Theorem (Arora, Mitchell, 1998):
For Euclidean-TSP, there is a polynomial-time factor 1.1 approximation algorithm.


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Theorem (Arora, Mitchell, 1998):
For Euclidean-TSP, there is a polynomial-time factor 1.01 approximation algorithm.


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For Euclidean-TSP, there is a polynomial-time factor 1.001 approximation algorithm.


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Even better in a special case

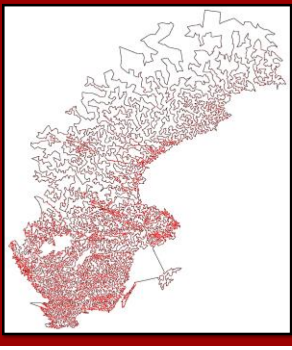
In the important special case "Euclidean-TSP",
vertices are points in $\mathbb{R}^2$,
costs are just the straight-line distances.

This special case is still NP-hard.

Theorem (Arora, Mitchell, 1998):
For Euclidean-TSP, there is a polynomial-time factor $1+\varepsilon$ approximation algorithm, for any $\varepsilon > 0$.


(Running time is like $O(n (\log n)^{1/\varepsilon})$.)

Euclidean-TSP:
NP-hard, but not **that** hard



n > 10,000
is feasible

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3. A 1.5-approximation algorithm for Metric-TSP.
4. A (1+ε)-approximation alg. for Euclidean-TSP.

Can we do better?

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Can we do better?

What more do you want?!

4. A (1+ε)-approximation alg. for Euclidean-TSP.

Can we do better?

3. A 1.5-approximation algorithm for Metric-TSP.

On one hand:
No improvement in the last 39 years.

On the other hand:
Researchers **strongly** believe we
can improve the factor of 1.5.

Lots of progress on special cases and
related problems in the last 5 years.

I predict an improvement within next 10 years.

Can we do better?

2. A 63% (1−1/e) approximation algorithm
   for the “k-Coverage Problem”.

We cannot do better. (Unless P=NP.)

Theorem: For **any** β > 1−1/e, it is NP-hard
to factor β-approximate k-Coverage.

Proved in 1998 by Feige,
building on many prior works.
Proof length of reduction: ≈ 100 pages.

Can we do better?

1. A 2-approximation algorithm for Vertex-Cover.

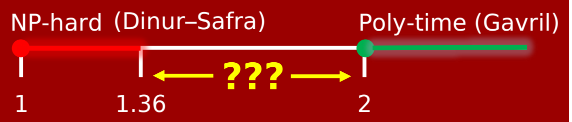
We have no idea if we can do better.

Theorem (Dinur & Safra, 2002, Annals of Math.):
For any $\beta > 10\sqrt{5} - 21 \approx 1.36$,
it is NP-hard to β-approximate Vertex-Cover.

Probably my favorite research paper of all time.

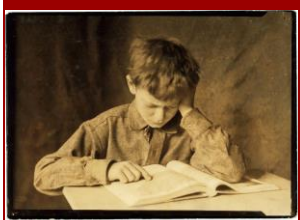
Approximating Vertex-Cover

Approximation Factor



Between 1.36 & 2: totally unknown.
Raging controversy.
I'll tell you about it another day.

Study Guide



Definitions:

Approximation algorithm.

The idea of "greedy" algorithms.

Algorithms and analysis:

Gavril algorithm for Vertex-Cover.

MST Heuristic for TSP.