15-251: Great Theoretical Ideas in Computer Science
Lecture 18

# Probability 2

## Random Variables

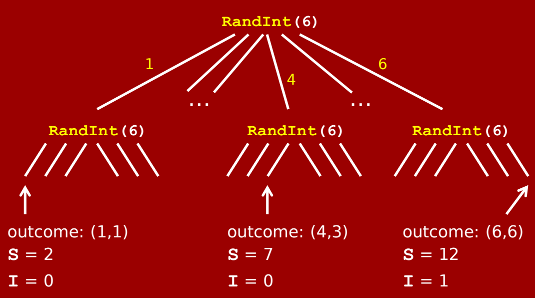
Definition 1:

A random variable is a variable,
in some randomized code.

Of type 'real number'.

(Better to say it's the variable's value at the end of an execution.)

```
S = RandInt(6) + RandInt(6)
if S == 12 then I = 1
          else I = 0
```




## Random Variables

Definition 2:

A random variable **X** assigns a real number to each outcome.

I.e., it is a *function* **X** : $\Omega \to \mathbb{R}$
from the sample space to the reals.

E.g., **S**( (1,1) ) = 2, **S**( (1,2) ) = 3,
… **S**( (6,6) ) = 12

## Random Variables: introducing them

Retroactively:

"Let **D** be the random variable given by subtracting the first roll from the second."

**D**( (1,1) ) = 0, …, **D**( (5, 3) ) = −2, etc.

## Random Variables: introducing them

In terms of other random variables:

"Let **Y** = $\mathbf{S}^2$+**D**." ⇒ **Y**( (5,3) ) = 62

"Suppose you win $30 on a roll of double-6, and you lose $1 otherwise. Let **W** be the random variable representing your winnings."

**W** = 31 · **I** − 1

## Random Variables: introducing them

Without bothering to give an "experiment":

"Let **X** be a Bernoulli(1/3) random variable."

"Let **T** be a random variable which is uniformly distributed (= each value equal probability) on the set {0,2,4,6,8}."

```
T = 2·(RandInt(5)−1)
```

## Random Variables to Events

E.g.: "Let A be the event that **S** ≥ 10."

A = { (4,6), (5,5), (5,6), (6,4), (6,5), (6,6) }

**Pr**[**S** ≥ 10] = 6/36 = 1/6

Shorthand notation for the **event** { ℓ : **S**(ℓ) ≥ 10 }.

## Events to Random Variables

Definition:

Let A be an event. The indicator of A is the random variable **X** which is 1 when A occurs and 0 when A doesn't occur.

$$\mathbf{X} : \Omega \to \mathbb{R} \qquad \mathbf{X}(\ell) = \begin{cases} 1 & \text{if } \ell \in A \\ 0 & \text{if } \ell \notin A \end{cases}$$

## Independence of Random Variables

Definition:

Random variables **X** and **Y** are independent if the events "X = u" and "Y = v" are independent for all u,v ∈ ℝ.

(And similarly for more than 2 random vbls.)

(And 'Principle of Independence' still holds.)

## Expectation

aka Expected Value

aka Mean

## Expectation

Intuitively, expectation of **X** is what its average value would be if you ran the code millions and millions of times.

Definition:

Let **X** be a random variable in experiment with sample space Ω. Its expectation is:

$$\mathbf{E}[\mathbf{X}] = \sum_{\ell \in \Omega} \mathbf{Pr}[\ell] \cdot \mathbf{X}(\ell)$$

## Expectation — examples

Let **R** be the roll of a standard die.

$$\mathbf{E}[\mathbf{R}] = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6$$

$= 3.5$

Question: What is **Pr**[**R** = 3.5]?

Answer: 0. Don't always expect the expected!

## Expectation — examples

"Suppose you win $30 on a roll of double-6, and you lose $1 otherwise. Let **W** be the random variable representing your winnings."

$$\mathbf{E}[\mathbf{W}] = \frac{1}{36}\cdot(-1) + \frac{1}{36}\cdot(-1) + \cdots + \frac{1}{36}\cdot(-1) + \frac{1}{36}\cdot 30$$

$= -5/36 \approx -13.9$¢

## Expectation — examples

Let $\mathbf{R}_1$ = RandInt(6), $\mathbf{R}_2$ = RandInt(6),
$\mathbf{S} = \mathbf{R}_1 + \mathbf{R}_2$.

$$\mathbf{E}[\mathbf{S}] = \frac{1}{36}\cdot(1+1) + \frac{1}{36}\cdot(1+2) + \cdots + \frac{1}{36}\cdot(6+6)$$

= lots of arithmetic ☹

= 7 (eventually)

One of the top tricks in probability...

## Linearity of Expectation

Given an experiment,
let **X** and **Y** be any random variables.

Then $\mathbf{E}[\mathbf{X}+\mathbf{Y}] = \mathbf{E}[\mathbf{X}] + \mathbf{E}[\mathbf{Y}]$

**X** and **Y** do not have to be independent!!

## Linearity of Expectation

$$\mathbf{E}[\mathbf{X}+\mathbf{Y}] = \mathbf{E}[\mathbf{X}] + \mathbf{E}[\mathbf{Y}]$$

Proof: Let **Z** = **X**+**Y** (another random vbl).

Then
$$\begin{aligned}\mathbf{E}[\mathbf{Z}] &= \sum_{\ell\in\Omega} \mathbf{Pr}[\ell]\cdot\mathbf{Z}(\ell) \\ &= \sum_{\ell\in\Omega} \mathbf{Pr}[\ell]\cdot(\mathbf{X}(\ell)+\mathbf{Y}(\ell)) \\ &= \sum_{\ell\in\Omega} \mathbf{Pr}[\ell]\cdot\mathbf{X}(\ell) + \sum_{\ell\in\Omega} \mathbf{Pr}[\ell]\cdot\mathbf{Y}(\ell) \\ &= \mathbf{E}[\mathbf{X}] + \mathbf{E}[\mathbf{Y}]\end{aligned}$$

## Linearity of Expectation

**E**[**X**+**Y**] = **E**[**X**] + **E**[**Y**]

Also:

**E**[a**X**+b] = a**E**[**X**]+b for any $a,b \in \mathbb{R}$,

$\mathbf{E}[\mathbf{X}_1 + \cdots + \mathbf{X}_n] = \mathbf{E}[\mathbf{X}_1] + \cdots + \mathbf{E}[\mathbf{X}_n]$

## Linearity of Expectation example

Let $\mathbf{R}_1$ = RandInt(6), $\mathbf{R}_2$ = RandInt(6),
$\mathbf{S} = \mathbf{R}_1+\mathbf{R}_2$.

$\mathbf{E}[\mathbf{S}] = \mathbf{E}[\mathbf{R}_1] + \mathbf{E}[\mathbf{R}_2]$

= 3.5 + 3.5

= 7

## Expectation of an Indicator

Fact:

Let A be an event, let **X** be its indicator rand. vbl.

Then **E**[**X**] = **Pr**[A].

Proof:

$$\mathbf{E}[\mathbf{X}] = \sum_{\ell \in \Omega} \mathbf{Pr}[\ell] \cdot \mathbf{X}(\ell)$$
$$= \sum_{\ell \in A} \mathbf{Pr}[\ell] \cdot 1 + \sum_{\ell \notin A} \mathbf{Pr}[\ell] \cdot 0$$
$$= \sum_{\ell \in A} \mathbf{Pr}[\ell]$$
$$= \mathbf{Pr}[A]$$

## Linearity of Expectation
## +
## Indicators

## = best friends forever

## Linearity of Expectation + Indicators

There are 251 students in a class.

The TAs randomly permute their midterms before handing them back.

Let **X** be the number of students getting their own midterm back.

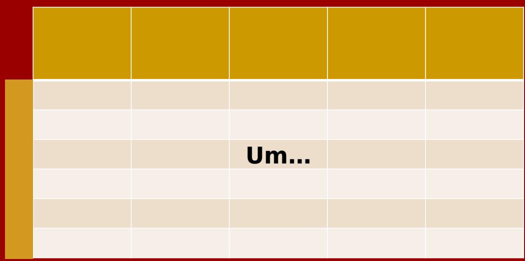
What is **E**[**X**]?

## Let's try 3 students first

| | Student 1 | Student 2 | Student 3 | Prob | # getting own midterm |
|---|---|---|---|---|---|
| Midterm they got | 1 | 2 | 3 | 1/6 | 3 |
| | 1 | 3 | 2 | 1/6 | 1 |
| | 2 | 1 | 3 | 1/6 | 1 |
| | 2 | 3 | 1 | 1/6 | 0 |
| | 3 | 1 | 2 | 1/6 | 0 |
| | 3 | 2 | 1 | 1/6 | 1 |

∴ **E**[**X**] = (1/6)(3+1+1+0+0+1) = 1

## Now let's do 251 students




## Now let's do 251 students

Let $A_i$ be the event that $i^{th}$ student gets own midterm.

Let $\mathbf{X}_i$ be the indicator of $A_i$.

Then $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n$

Thus $\mathbf{E}[\mathbf{X}] = \mathbf{E}[\mathbf{X}_1] + \mathbf{E}[\mathbf{X}_2] + \cdots + \mathbf{E}[\mathbf{X}_n]$
by linearity of expectation

$\mathbf{E}[\mathbf{X}_i] = \mathbf{Pr}[A_i]$, and $\mathbf{Pr}[A_i] = 1/251$ for each i.

$\therefore\ \mathbf{E}[\mathbf{X}] = 251 \cdot (1/251) = 1$

## A Formula for Expectation

$$\mathbf{E}[\mathbf{X}] = \sum_{u \in \text{range}(\mathbf{X})} \mathbf{Pr}[\mathbf{X} = u] \cdot u$$

Remarks:

- range($\mathbf{X}$) = the set of real numbers $\mathbf{X}$ may take on
- "$\mathbf{X} = u$" is an event
- some people (not us) take this as the *definition*

$$\mathbf{E}[\mathbf{X}] = \sum_{u \in \text{range}(\mathbf{X})} \mathbf{Pr}[\mathbf{X} = u] \cdot u$$

Proof by "counting two ways":

$$\begin{aligned}\mathbf{E}[\mathbf{X}] &= \sum_{\ell \in \Omega} \mathbf{Pr}[\ell] \cdot \mathbf{X}(\ell) \\ &= \sum_{u \in \text{range}(\mathbf{X})} \sum_{\ell : \mathbf{X}(\ell) = u} \mathbf{Pr}[\ell] \cdot \mathbf{X}(\ell) \\ &= \sum_{u \in \text{range}(\mathbf{X})} \sum_{\ell : \mathbf{X}(\ell) = u} \mathbf{Pr}[\ell] \cdot u \\ &= \sum_{u \in \text{range}(\mathbf{X})} u \cdot \sum_{\ell : \mathbf{X}(\ell) = u} \mathbf{Pr}[\ell] \\ &= \sum_{u \in \text{range}(\mathbf{X})} u \cdot \mathbf{Pr}[\mathbf{X} = u]\end{aligned}$$

## Example

Question: Let $\mathbf{X}$ be a uniformly random integer between 1 and 10. Let $\mathbf{Y} = \mathbf{X}$ mod 3.

What is $\mathbf{E}[\mathbf{Y}]$?

range($\mathbf{Y}$) = {0,1,2}

$\mathbf{E}[\mathbf{Y}] = \mathbf{Pr}[\mathbf{Y} = 0] \cdot 0 + \mathbf{Pr}[\mathbf{Y} = 1] \cdot 1 + \mathbf{Pr}[\mathbf{Y} = 2] \cdot 2$

$= \mathbf{Pr}[\mathbf{Y} = 1] + 2\mathbf{Pr}[\mathbf{Y} = 2]$

$= \mathbf{Pr}[\{1,4,7,10\}] + 2\mathbf{Pr}[\{2,5,8\}]$

$= 4/10 + 2(3/10) = 1$

## Example

Question: Let $\mathbf{X}$ be a uniformly random integer between 1 and 10. Let $\mathbf{Y} = \mathbf{X}$ mod 3.

What is $\mathbf{E}[\mathbf{Y}]$?

range($\mathbf{Y}$) = {0,1,2}

$\mathbf{E}[\mathbf{Y}] = \mathbf{Pr}[\mathbf{Y} = 0] \cdot 0 + \mathbf{Pr}[\mathbf{Y} = 1] \cdot 1 + \mathbf{Pr}[\mathbf{Y} = 2] \cdot 2$

Note: We didn't really care how $\mathbf{Y}$ was created.

We only needed $\mathbf{Pr}[\mathbf{Y}=u]$ for each $u \in$ range($\mathbf{Y}$).

## Probability Mass Functions

def: The **probability mass function** (PMF) of a random variable **X** is the function $p_X : \mathbb{R} \to \mathbb{R}$ defined by

$$p_X(u) = \mathbf{Pr}[X = u]$$

Properties:

- $p_X(u) \neq 0$ only for $u \in \text{range}(X)$
- $\sum_{u \in \text{range}(X)} p_X(u) = 1$
- $\mathbf{E}[X] = \sum_{u \in \text{range}(X)} p_X(u) \cdot u$

## Probability Mass Functions

The PMF of a random variable **X** captures most information you need to know about it.

(Exception: relationship to other rv's.)

Random variables sometimes just defined by a PMF, with no reference to an experiment.

E.g.: "Let **X** be a random variable with $p_X(1) = .2$, $p_X(2) = .5$, $p_X(3) = .3$"

---

E.g.: "Let **X** be a random variable with $p_X(1) = .2$, $p_X(2) = .5$, $p_X(3) = .3$"

Is this legit? Could you write code that generated such an **X**?

This is a legitimate PMF whenever:

- $p_X(u) \geq 0$ for all u
- $\sum_u p_X(u) = 1$

## Expectation Formula Generalized

$$\mathbf{E}[X] = \sum_{u \in \text{range}(X)} \mathbf{Pr}[X = u] \cdot u$$

and $\mathbf{E}[X^2] = \sum_{u \in \text{range}(X)} \mathbf{Pr}[X = u] \cdot u^2$

and $\mathbf{E}[\sin(X)] = \sum_{u \in \text{range}(X)} \mathbf{Pr}[X = u] \cdot \sin(u)$

etc.: $\mathbf{E}[f(X)] = \sum_{u \in \text{range}(X)} \mathbf{Pr}[X = u] \cdot f(u)$

---

## Expectation Formula Generalized

$$\mathbf{E}[f(X)] = \sum_{u \in \text{range}(X)} \mathbf{Pr}[X = u] \cdot f(u)$$

More generally:

$$\mathbf{E}[g(X, Y)] = \sum_{\substack{u \in \text{range}(X) \\ v \in \text{range}(Y)}} \mathbf{Pr}[X = u \cap Y = v] \cdot g(u, v)$$

## Example

```
X = RandInt(2)
Y = RandInt(X+1)
```

Question: What is **E[XY]**?

$$\begin{aligned}
= &\ \mathbf{Pr}[X=1 \cap Y=1] \cdot 1 \cdot 1 & = &\ (1/4) \cdot 1 \cdot 1 \\
+ &\ \mathbf{Pr}[X=1 \cap Y=2] \cdot 1 \cdot 2 & + &\ (1/4) \cdot 1 \cdot 2 \\
+ &\ \mathbf{Pr}[X=1 \cap Y=3] \cdot 1 \cdot 3 & + &\ (0) \cdot 1 \cdot 3 \\
+ &\ \mathbf{Pr}[X=2 \cap Y=1] \cdot 2 \cdot 1 & + &\ (1/6) \cdot 2 \cdot 1 \\
+ &\ \mathbf{Pr}[X=2 \cap Y=2] \cdot 2 \cdot 2 & + &\ (1/6) \cdot 2 \cdot 2 \\
+ &\ \mathbf{Pr}[X=2 \cap Y=3] \cdot 2 \cdot 3 & + &\ (1/6) \cdot 2 \cdot 3 \quad = 11/4
\end{aligned}$$

## Example

```
X = RandInt(2)
Y = RandInt(X+1)
```

$E[XY] = 11/4$

$E[X] = 3/2$

$E[Y] = 7/4$ (exercise)

Notice: $E[XY] \neq E[X]\,E[Y]$ in general!

## But…

## If X and Y are independent then E[XY] = E[X] E[Y].

Proof:

$$E[XY] = \sum_{\substack{u \in range(X) \\ v \in range(Y)}} Pr[X = u \cap Y = v] \cdot uv$$

$$= \sum_{u,v} Pr[X = u] Pr[Y = v] \cdot uv \quad \text{(independence!)}$$

$$= \sum_{u,v} p_X(u) u \cdot p_Y(v) v$$

$$= \left(\sum_u p_X(u) u\right) \cdot \left(\sum_v p_Y(v) v\right)$$

$$= E[X] E[Y]$$

## Your two favorite kinds of random variables

## Binomial Random Variables

Let $n \in \mathbb{N}^+$ and $0 < p < 1$.

$X \sim$ Binomial(n,p) means

$$X = X_1 + X_2 + \cdots + X_n$$

where $X_i$'s are Bernoulli(p) (and independent).

## Binomial Random Variables

$X \sim$ Binomial(n,p)

What is range(X)? $\{0, 1, 2, \ldots, n\}$

What is $p_X(u) = Pr[X = u]$?

There are $2^n$ outcomes for $X_i$'s; e.g., $00101\cdots1$

$X = u$ when outcome has $u$ 1's and $n-u$ 0's

Such an outcome has probability $p^u(1-p)^{n-u}$

# of such outcomes is $\binom{n}{u}$

$$\therefore\ p_X(u) = \binom{n}{u} p^u (1-p)^{n-u}, \quad u = 0, 1, 2, \ldots, n$$

## Binomial Random Variables

$\mathbf{X} \sim \text{Binomial}(n,p)$

$$p_{\mathbf{X}}(u) = \binom{n}{u} p^u (1-p)^{n-u}, \quad u = 0, 1, 2, \ldots, n$$

Check: $\sum_{u=0}^{n} \binom{n}{u} p^u (1-p)^{n-u} = (p + (1-p))^n \;\; = 1$

("Binomial Theorem")

$$\mathbf{E}[\mathbf{X}] = \sum_{u=0}^{n} \binom{n}{u} p^u (1-p)^{n-u} \cdot u \;\; = ??$$

## Binomial Random Variables

$\mathbf{X} \sim \text{Binomial}(n,p)$

$$p_{\mathbf{X}}(u) = \binom{n}{u} p^u (1-p)^{n-u}, \quad u = 0, 1, 2, \ldots, n$$

Check: $\sum_{u=0}^{n} \binom{n}{u} p^u (1-p)^{n-u} = (p + (1-p))^n \;\; = 1$

("Binomial Theorem")

$$\mathbf{E}[\mathbf{X}] = \mathbf{E}[\mathbf{X}_1] + \cdots + \mathbf{E}[\mathbf{X}_n] = np$$

(linearity of expectation)

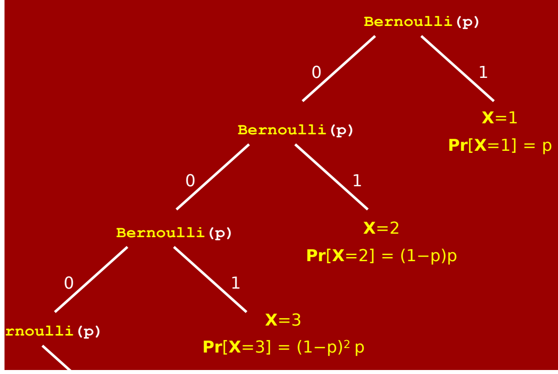
## Geometric Random Variables

Let $0 < p < 1$.

$\mathbf{X} \sim \text{Geometric}(p)$ means we keep doing "p-biased coin flips" until we get Heads; then $\mathbf{X}$ is the number of flips it took.

```
X = 1
while Bernoulli(p) == 0
    X = X+1
```

## Geometric Random Variables




## Geometric Random Variables

$\mathbf{X} \sim \text{Geometric}(p)$

What is range($\mathbf{X}$)? $\{1, 2, 3, 4, \ldots\}$


Super-math-nerds: if you care about technicalities, ask.

What is $p_{\mathbf{X}}(u) = \mathbf{Pr}[\mathbf{X} = u]$? $(1-p)^{u-1} p$

Check: (sum of geometric series)

$$\sum_{u=1}^{\infty} (1-p)^{u-1} p = p \cdot \sum_{u=0}^{\infty} (1-p)^u = p \cdot \frac{1}{1-(1-p)} = \frac{p}{p} = 1$$

## Geometric Random Variables

$\mathbf{X} \sim \text{Geometric}(p)$

What is $\mathbf{E}[\mathbf{X}]$?

Average number of p-biased coin flips until you get Heads: you might guess $1/p$.

You'll see the proof in recitation.

## The Coupon Collector

There are n different kinds of coupons.

On each day, you get a random coupon. (You may get duplicates.)

Let **X** be the # of days till you have them all.

What is **E**[**X**]?

## The Coupon Collector

Let **X** be the # of days till you have them all.

What is **E**[**X**]?

Key idea: Let $\mathbf{X}_i$ be # of days it took you to go from i−1 to i coupons.

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n$$

$$\therefore\ \mathbf{E}[\mathbf{X}] = \mathbf{E}[\mathbf{X}_1] + \mathbf{E}[\mathbf{X}_2] + \cdots + \mathbf{E}[\mathbf{X}_n]$$

So we need to figure out $\mathbf{E}[\mathbf{X}_i]$.

## The Coupon Collector

Key idea: Let $\mathbf{X}_i$ be # of days it took you to go from i−1 to i coupons.

When sitting on i−1 distinct coupons, each day you have probability $\frac{n-(i-1)}{n}$ of getting a new one.

$\therefore\ \mathbf{X}_i \sim \text{Geometric}(\frac{n-(i-1)}{n})$  $\therefore\ \mathbf{E}[\mathbf{X}_i] = \frac{n}{n-(i-1)}$

for example,

$$\mathbf{E}[\mathbf{X}_1] = \frac{n}{n} = 1,\quad \mathbf{E}[\mathbf{X}_2] = \frac{n}{n-1},\quad \cdots,\quad \mathbf{E}[\mathbf{X}_n] = \frac{n}{1} = n$$

## The Coupon Collector

$$\therefore\ \mathbf{E}[\mathbf{X}] = \mathbf{E}[\mathbf{X}_1] + \mathbf{E}[\mathbf{X}_2] + \cdots + \mathbf{E}[\mathbf{X}_n]$$

$$= \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}$$

$$= n\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$$= n \cdot H_n \qquad \approx n \ln n$$

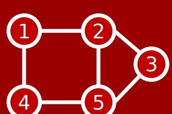
where $H_n$ = “the nth harmonic number”

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx \ln n$$

(see Lecture 8)

## My favorite problem: Max-Cut

**Input:** A graph G=(V,E).

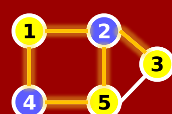



**Output:** A “**2-coloring**” of V: each vertex designated yellow or blue.

**Goal:** Have as many **cut** edges as possible. An edge is *cut* if its endpoints have different colors.

## My favorite problem: Max-Cut

**Input:** A graph G=(V,E).




**Output:** A “**2-coloring**” of V: each vertex designated yellow or blue.

**Goal:** Have as many **cut** edges as possible. An edge is *cut* if its endpoints have different colors.

## My favorite problem: Max-Cut

On one hand:
Finding the **MAX**-Cut is NP-hard.

On the other hand:
Polynomial-time “Local Search” algorithm guarantees cutting ≥ ½ m out of m edges.

There’s another, **super-duper-simple** O(n)-time algorithm with a similar guarantee.

So simple, it doesn’t even really look at the input!

## My favorite problem: Max-Cut

Idea: Try a **random** 2-coloring!

```
for i = 1...n
    color[i] = RandInt(2)
```

Let **X** be the random variable giving the number of edges cut.

What is **E**[**X**]?

## Indicators + Linearity to the rescue

```
for i = 1...n
    color[i] = RandInt(2)
```

Let **X** be the number of edges cut.

For each of the m edges e,
let $B_e$ be the event that it’s cut,
let $X_e$ be the indicator random vbl. for $B_e$.

$$X = \sum_{\text{edges } e} X_e$$

## Indicators + Linearity to the rescue

For each of the m edges e,
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$$X = \sum_{\text{edges } e} X_e$$

$$\therefore\ \mathbf{E}[X] = \sum_e \mathbf{E}[X_e] = \sum_e \mathbf{Pr}[B_e]$$

**Pr**$[B_e]$ = 1/2 (cut if colors 1,2 or 2,1)

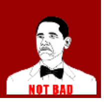
## Indicators + Linearity to the rescue

For each of the m edges e,
let $B_e$ be the event that it’s cut,
let $X_e$ be the indicator random vbl. for $B_e$.

$$X = \sum_{\text{edges } e} X_e$$

$$\therefore\ \mathbf{E}[X] = \sum_e \mathbf{E}[X_e] = \sum_e \mathbf{Pr}[B_e]$$

**Pr**$[B_e]$ = 1/2 $\quad \therefore\ \mathbf{E}[X] = ½\, m$




## Max-3SAT

Let S be a “CNF formula”, each clause having exactly 3 literals (with distinct variables).

e.g. $S = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_5) \wedge (x_1 \vee x_4 \vee \neg x_5) \wedge \cdots$

Given that S is satisfiable,
it’s **NP-hard** to find a satisfying assignment. ☹

Lecture 15: *Don’t give up!*
**Max-3SAT** asks: try to find a truth assignment satisfying as many of the clauses as you can.

## Max-3SAT

Dumb(?) idea: Try a **random** truth assignment!

Let **X** be the number of clauses satisfied.

What is E[**X**]? Just as with Max-Cut…

$\mathbf{E}[X] = \mathbf{E}[\mathbf{X}_1] + \mathbf{E}[\mathbf{X}_2] + \cdots + \mathbf{E}[\mathbf{X}_m]$
where $\mathbf{X}_j$ is indicator that jth clause satisfied.

$\mathbf{E}[\mathbf{X}_j]$ = **Pr**[jth clause satisfied] = 7/8

e.g. S = $(x_1 \vee x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_5) \wedge (x_1 \vee x_4 \vee \neg x_5) \wedge \cdots$

## Max-3SAT

Dumb(?) idea: Try a **random** truth assignment!

Let **X** be the number of clauses satisfied.

What is E[**X**]? Just as with Max-Cut…

$\mathbf{E}[X] = \mathbf{E}[\mathbf{X}_1] + \mathbf{E}[\mathbf{X}_2] + \cdots + \mathbf{E}[\mathbf{X}_m]$
where $\mathbf{X}_j$ is indicator that jth clause satisfied.

$\mathbf{E}[\mathbf{X}_j]$ = **Pr**[jth clause satisfied] = 7/8

∴ we satisfy (7/8)m clauses (87.5%) in expectation.

## Max-3SAT

Let S be a "CNF formula", each clause having exactly 3 literals (with distinct variables).

Given that S is satisfiable,
it's **NP-hard** to find a satisfying assignment. ☹

A super-duper-simple algorithm will satisfy 87.5% of the clauses (in expectation).

Can we do better?

**No !!**

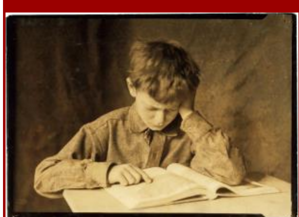
## Max-3SAT

Johan Håstad

**Theorem:**
Given a *satisfiable* 3CNF formula S, it's **NP-hard** to find a truth assignment satisfying ≥ 87.5001% of the clauses (or any fraction > 7/8).

Thus the trivial randomized algorithm is the **best** poly-time approximation algorithm for Max-3SAT!
(Assuming P ≠ NP.)

## Study Guide



Definitions:
- Random variables
- Independence of rv's
- Indicators
- Expectation
- Linearity of expectation
- PMFs
- Binomials, Geometrics

Solving problems:
- Linearity+Indicators
- Computing expectations
- Coupon Collector
- Max-Cut and Max-3Sat randomized algs.