15-251
Great Theoretical Ideas in Computer Science

Life After 15-251

April 28, 2015


FREEDOM!!


THE NEW FILM BY
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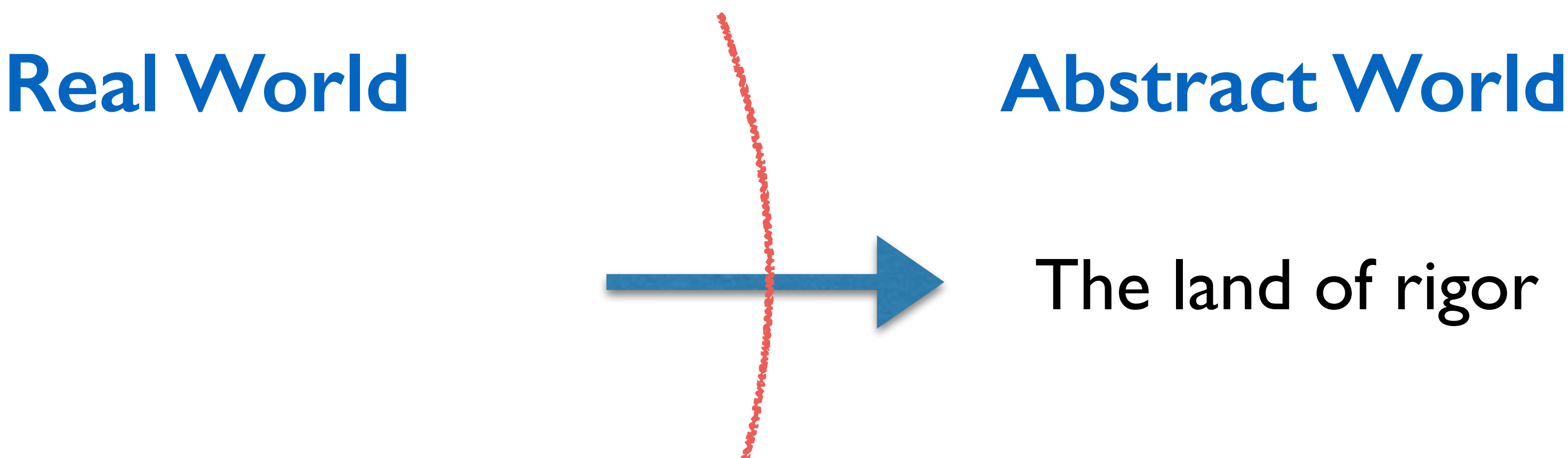
UNCHAINED

Goals (from lecture 1)

1. Learn about the theoretical foundations of computation
2. Learn the basic math topics, i.e. the language
3. Become better at reasoning abstractly and formally.
4. Become better problem solvers
5. Become better at expressing yourself clearly.



What we learned

- Formalization of mathematical proof
- Formalization of computation (DFAs, TMs)
- Computability
- Computational complexity
  (and some interesting algorithms)
- NP-completeness and the P vs NP question
- Approximation algorithms
- Randomization

What we learned

- Cryptography
- Markov Chains
- Quantum computation
- Communication complexity
- Computer science perspective on proofs

What we learned

- Infinite sets (countable and uncountable sets)
- Graph theory
- Probability theory
- Number theory
- Fields and polynomials
- Linear algebra

Some big open questions

Relative power of resources

Relative power of resources

Resources: time, space, randomness, non-determinism.

Does non-determinism help
with respect to time efficient computation?

$$\mathrm{P} = \mathrm{NP}?$$

Relative power of resources

Resources: time, space, randomness, non-determinism.

Does non-determinism help
with respect to space efficient computation?

$$\mathsf{L} = \mathsf{NL}?$$

Relative power of resources

Resources: time, space, randomness, non-determinism.

Is time equivalent to space
with respect to efficient computation?

$$\mathsf{P} = \mathsf{PSPACE}?$$

Note:

$$\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE}$$

Relative power of resources

Resources: time, space, randomness, non-determinism.

Does randomness give us more power
with respect to time efficient computation?

$$\mathsf{P} = \mathsf{BPP}?$$

Interesting connection to circuit complexity:

certain circuit complexity lower bounds $\implies$ $\mathsf{P} = \mathsf{BPP}$

$\mathsf{P} = \mathsf{BPP}$ $\implies$ certain circuit complexity lower bounds

Relative power of resources

Resources: time, space, randomness, non-determinism.

Does randomness give us more power
with respect to time efficient computation?

$$\mathrm{P} = \mathrm{BPP}?$$

A major related result:

$$\text{PRIMES} \in \mathrm{P}$$

Relative power of resources

Resources: time, space, randomness, non-determinism.

Does randomness give us more power with respect to space efficient computation?

$$\mathsf{L} = \mathsf{BPL}?$$

A major related result:

$$\mathrm{USTCONN} \in \mathsf{L}$$

Relative power of resources

$$\mathsf{P} = \mathsf{NP}?$$

$$\mathsf{L} = \mathsf{NL}?$$

$$\mathsf{P} = \mathsf{PSPACE}?$$

$$\mathsf{P} = \mathsf{BPP}?$$

Your lower bound please.

Circuit complexity

Circuit complexity

<u>Circuits</u>: a clean and simple definition of computation.

Just a composition of And, Or, Not gates.

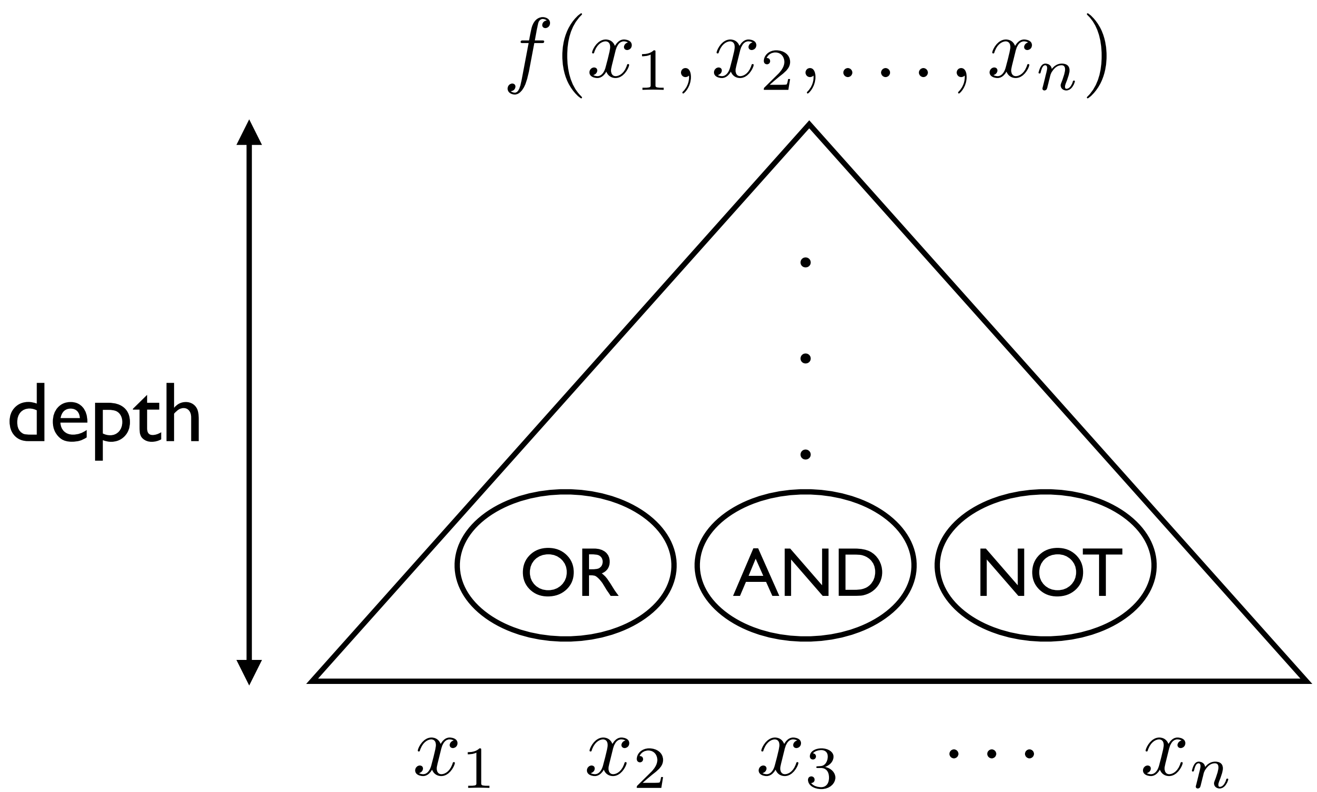
(TMs are a bit messy to work with.
Not as elegant.)

poly-time TM $\implies$ poly-size circuits

no poly-size circuits $\implies$ no poly-time TM

So let's show SAT cannot be computed with poly-size circuits.

Circuit complexity

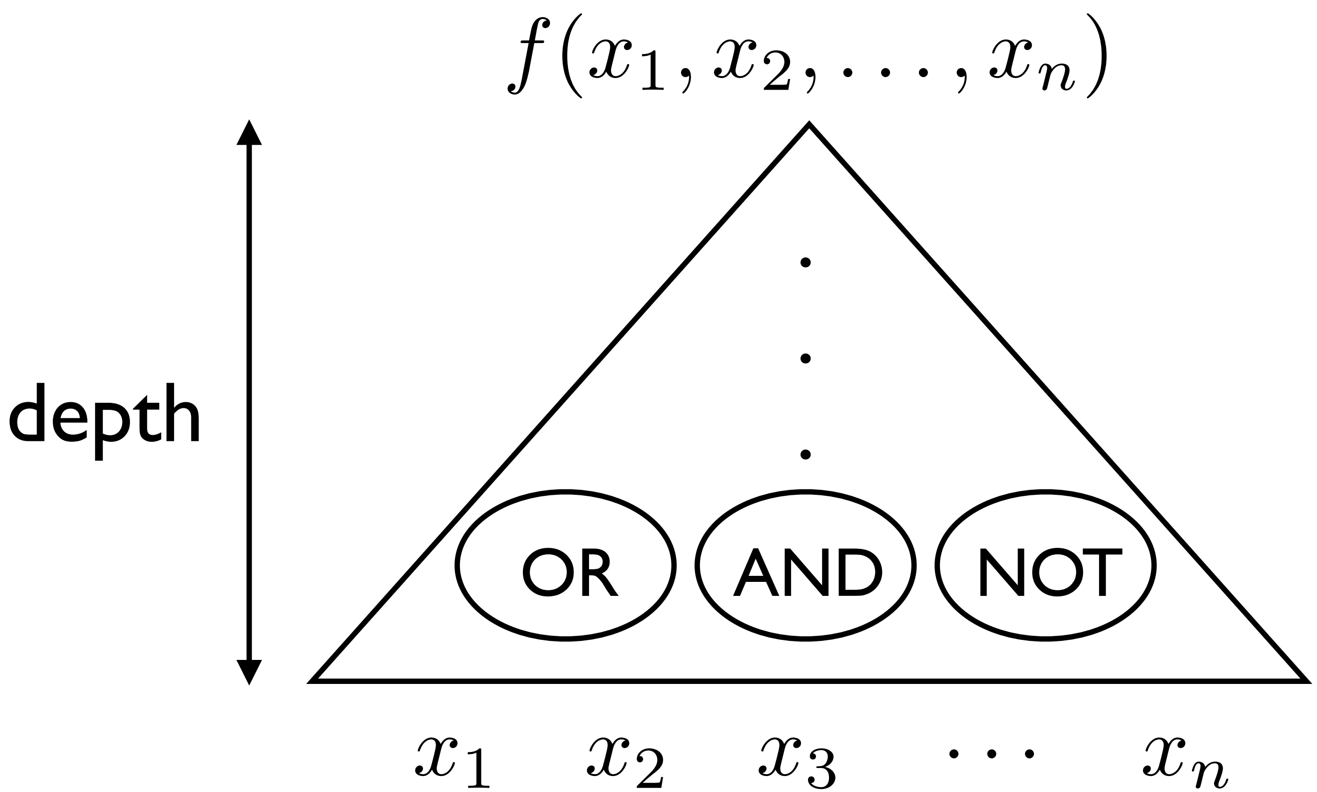


Let's restrict the circuit, make it less powerful.

What if we just allow constant depth?

Such circuits, in sub-exponential size, cannot compute parity function: $x_1 + x_2 + \cdots + x_n \pmod 2$

Circuit complexity

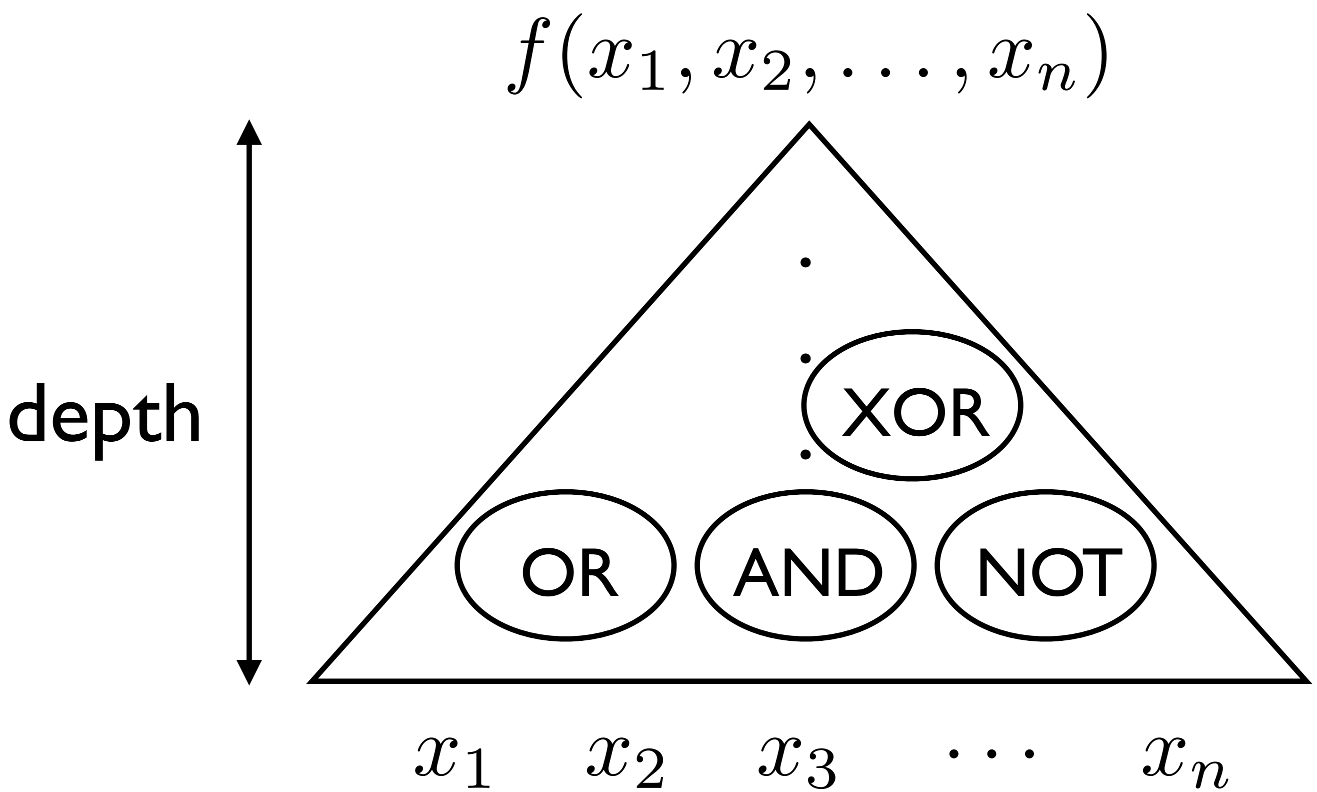


What if we just allow $O(\log n)$ depth?

parity **<u>can</u>** be computed in poly-size.

we can't prove lower bounds.

Circuit complexity



What if we just allow constant depth

but **add** parity gates to the circuit?

Circuit complexity

What if we just allow constant depth

but **add** parity gates to the circuit?

Such circuits, in polynomial size, cannot compute

$$\mathrm{mod}_3(x) = \begin{cases} 0 & \text{if } x_1 + x_2 + \cdots + x_n \equiv_3 0 \\ 1 & \text{otherwise} \end{cases}$$

Ok, let's add mod_3 gates to the circuit.

Or, instead of mod_2 and mod_3 gates,
just allow mod_6 gates.

Circuit complexity

Meanwhile…

Another restriction: remove NOT gates
(but no restriction on depth)

Alexander Razborov (1985):

Such poly-size circuits cannot compute CLIQUE.

We are so close to separating P and NP…

Circuit complexity

Alas…

Circuit complexity

<u>Current frontier in circuit complexity:</u>

Find a language in NP that cannot be computed by constant-depth, poly-size circuits with $\mathrm{and}, \mathrm{or}, \mathrm{not}, \mathrm{mod}_6$ gates.

In fact:

Find a language in NP that cannot be computed by **<u>depth 3</u>**, poly-size circuits with **<u>just</u>** mod_6 gates.

Circuit complexity

In fact:

Let's define a "generalized" mod6 gate.

For $A \subseteq \{0, 1, 2, 3, 4, 5\}$

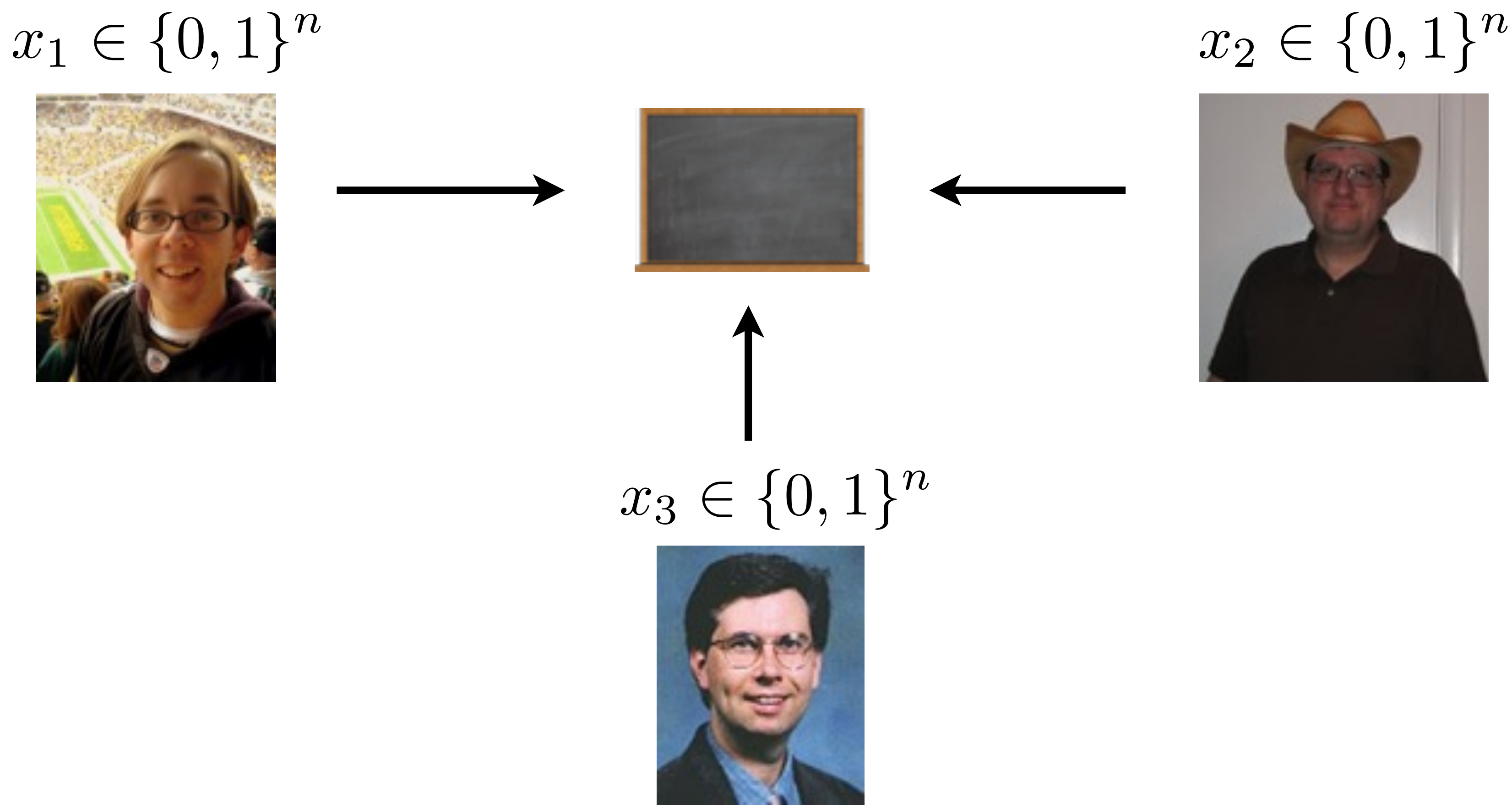
$$\mathrm{mod}_6^A(x) = \begin{cases} 1 & \text{if } x_1 + x_2 + \cdots + x_n \pmod 6 \in A \\ 0 & \text{otherwise} \end{cases}$$

Find a language in NP that cannot be computed by <u>**depth 2**</u>, poly-size circuits with <u>**just**</u> "generalized" mod6 gates.

Please solve this problem!

Circuit complexity and communication

"Number on the Forehead" (NOF) model



Number on the forehead: Player i sees all strings except x_i

Compute $F(x_1, x_2, x_3)$

$\mathbf{D}_k(F), \mathbf{R}_k^{\epsilon}(F)$

Circuit complexity and communication

Current frontier in circuit complexity:

Find a language in NP that cannot be computed by constant-depth, poly-size circuits with $\mathrm{and}, \mathrm{or}, \mathrm{not}, \mathrm{mod}_6$ gates.

Suffices to:

Find a function that cannot be computed efficiently in the NOF model with poly-log(n) many players.

The log *n* **Barrier**: No lower bounds when $k = \log n$

Holy Grail of
Communication Complexity

Circuit complexity

<u>Best known lower bound</u>

For circuits with AND, OR, NOT gates:

Best known lower bound for an "explicit" function is

$$5n - \text{peanuts}$$

Circuit complexity

Another interesting type of circuit:

Circuits with threshold gates.

For $w_0, w_1, w_2, \ldots, w_n \in \mathbb{Z}$

$$\mathrm{thr}_w(x) = \begin{cases} 1 & \text{if } w_1 x_1 + w_2 x_2 + \cdots + w_n x_n > w_0 \\ 0 & \text{otherwise} \end{cases}$$

Another major open problem:

Find a function that cannot be computed by poly-size, **dept-2** circuits composed of **only threshold gates**.

Circuit complexity

Why are circuit lower bounds so hard to prove?

1994

Steven Rudich
(CMU professor)

Alexander Razborov

Current techniques are unlikely to work!

"Natural Proof barrier"

Algorithms

Algorithms

Matrix Multiplication

1978: $O(n^{2.796})$	by Pan
1979: $O(n^{2.78})$	by Bini, Capovani, Romani, Lotti
1981: $O(n^{2.522})$	by Schönhage
1981: $O(n^{2.517})$	by Romani
1981: $O(n^{2.496})$	by Coppersmith, Winograd
1986: $O(n^{2.479})$	by Strassen
1990: $O(n^{2.376})$	by Coppersmith, Winograd
2010: $O(n^{2.374})$	by Andrew Stothers (PhD thesis)
2011: $O(n^{2.373})$	by Virginia Vassilevska Williams

Algorithms

Matrix Multiplication

2014: $O(n^{2.372})$ by François Le Gall

2014: Ambainis, Filmus, Le Gall

These techniques are not going to let you go below $O(n^{2.3})$

Can we go down to $O(n^2)$?

Graph Isomorphism

Given two n-vertex graphs, are they isomorphic?

One of few problems not known to be in
P nor NP-complete.

Best known algorithm: $2^{O(\sqrt{n \log n})}$

Algorithms

Factoring

Given a composite number, output a non-trivial factor.

One of few problems not known to be in
P nor NP-complete.

Best known algorithm: roughly $2^{O(n^{1/3})}$

There is a poly-time quantum algorithm.

Finding an n-bit prime

Given n, output a prime number with at least n digits.

Find a poly(n) time deterministic algorithm.

poly(n) time randomized algorithm exists.

Quantum computation

Quantum computation

The only difference between a probabilistic classical world and the equations of the quantum world is that somehow or other it appears as if the probabilities would have to go negative.

-Richard Feynman

Quantum computation

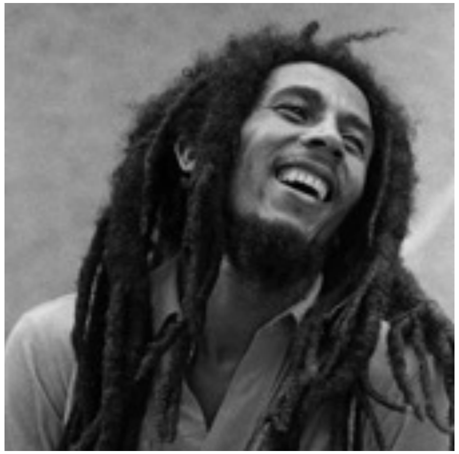
BQP = quantum analog of BPP

$$\mathsf{BQP} = \mathsf{BPP}?$$

$$\mathsf{BQP} = \mathsf{NP}?$$

Quantum computation

The parity game



randomly picks

$$x, y \in \{0, 1\}$$

$$\xleftarrow{\quad x \quad} \qquad\qquad \xrightarrow{\quad y \quad}$$

$$\xrightarrow{\quad a \quad} \qquad\qquad \xleftarrow{\quad b \quad}$$

They win if:

$$a \oplus b = x \wedge y$$

Quantum computation

The parity game

With best classical strategy:

They win with probability 0.75

With best quantum strategy:

They win with probability ~0.854

"Quantum entanglement enables two separated parties to exhibit classically impossible correlations."

Open question: Why is the best strategy ~0.854?

Quantum computation

The parity game

Open question: Why is the best strategy ~0.854?

Wim van Dam (2005):

If we could achieve success probability 1,
then $\mathbf{D}(F) \leq 2$ for any F.

Brassard et al. (2005):

If we could achieve success probability ~0.908 or more,
then $\mathbf{R}^{\epsilon}(F) \leq 2$ for any F.

What about success probability $0.854 \leq p \leq 0.908$?

How are we going to tackle these tough questions?

Tackling math problems

(SOLO)

Andrew Wiles

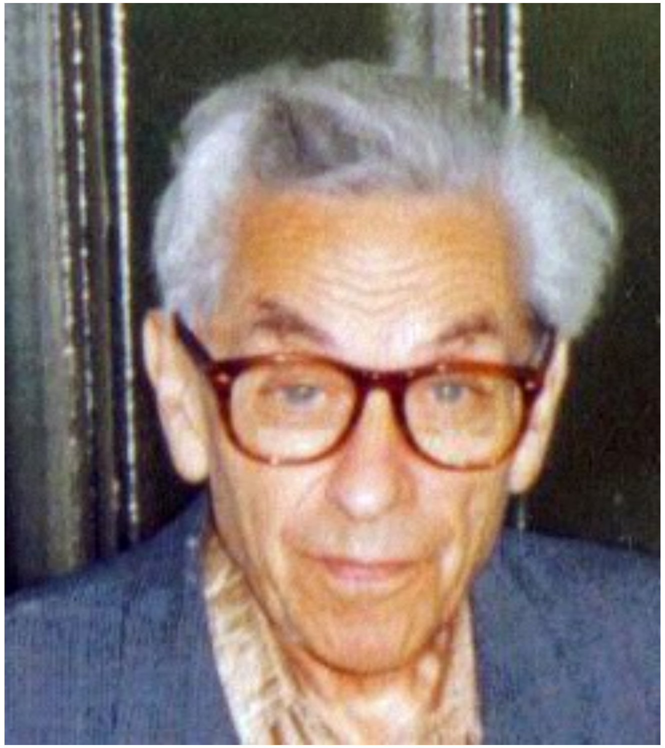
Proved Fermat's Last Theorem
1995

(was open for 358 years)

Spent 7 years on it in secrecy.

Tackling math problems

(GROUP)



Paul Erdős

1913-1996

More than 500 collaborators

<u>Erdős number:</u>

degree of separation from Erdős

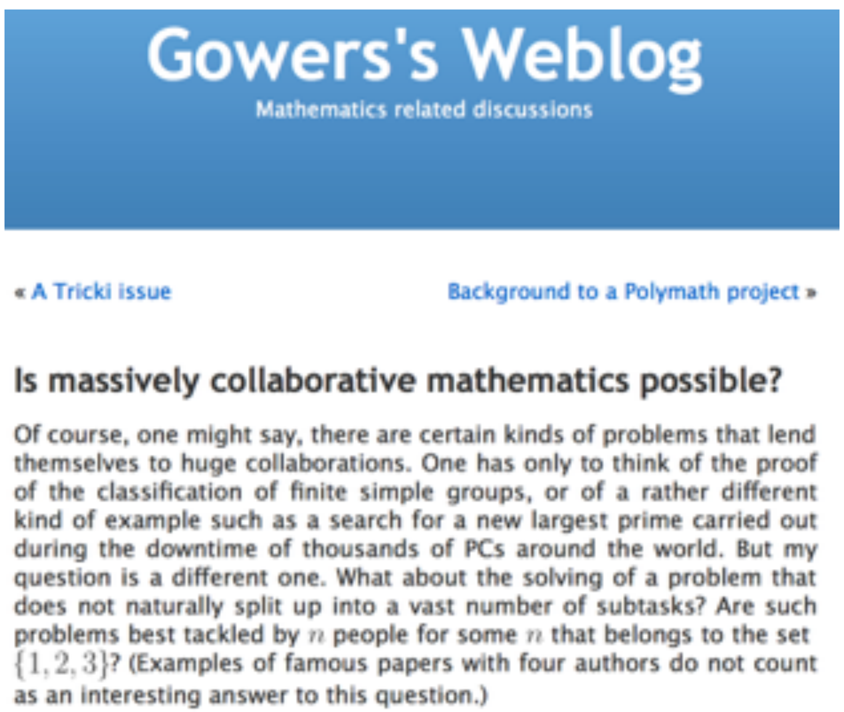
(he referred to children as "epsilons")

Tackling math problems

(OPEN)

Polymath projects:

Massively collaborative online mathematical projects



Gowers's Weblog

Mathematics related discussions

« A Tricki issue | Background to a Polymath project »

Is massively collaborative mathematics possible?

Of course, one might say, there are certain kinds of problems that lend themselves to huge collaborations. One has only to think of the proof of the classification of finite simple groups, or of a rather different kind of example such as a search for a new largest prime carried out during the downtime of thousands of PCs around the world. But my question is a different one. What about the solving of a problem that does not naturally split up into a vast number of subtasks? Are such problems best tackled by n people for some n that belongs to the set $\{1, 2, 3\}$? (Examples of famous papers with four authors do not count as an interesting answer to this question.)

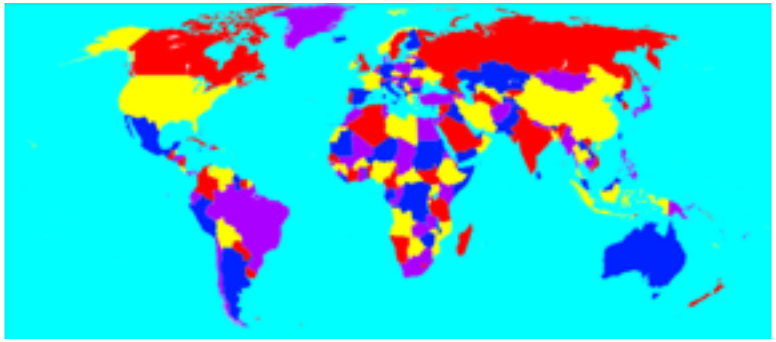
…

Timothy Gowers

Tackling math problems

(COMP)

4-Color Theorem



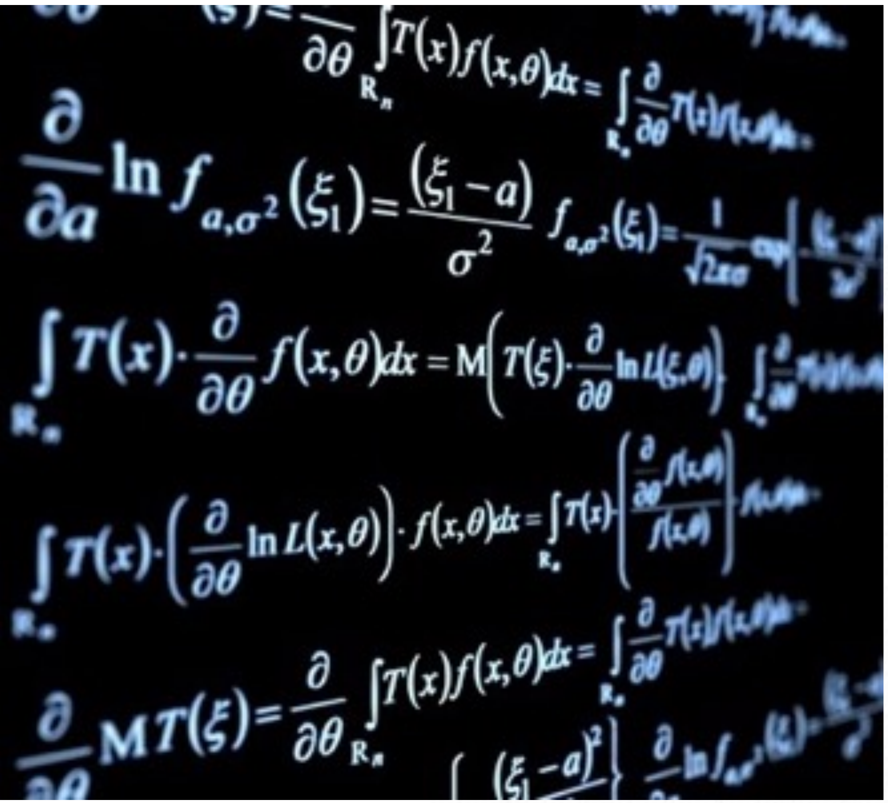
Reduce the problem to checking ~2000 cases.

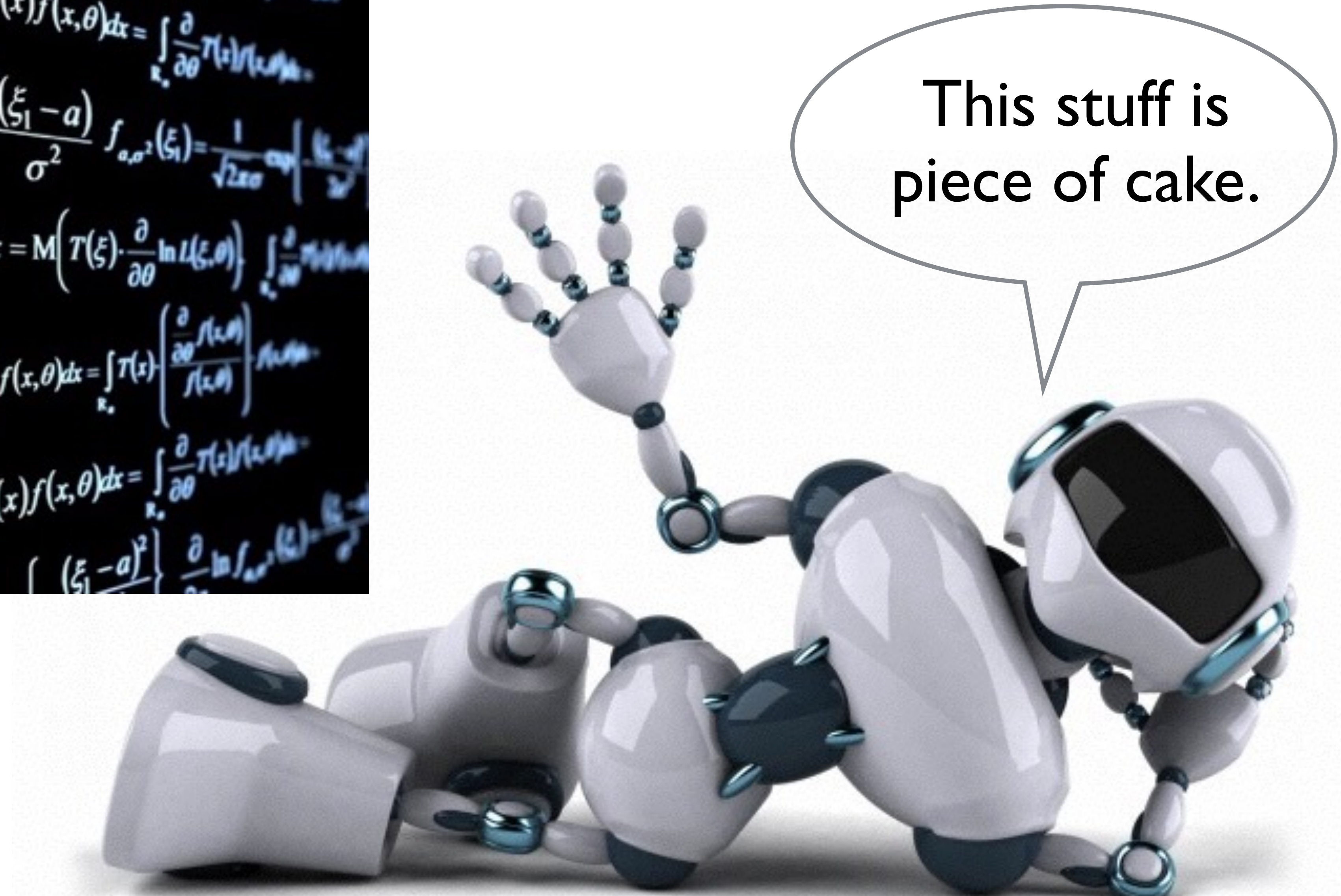
Let the machine check those cases.

Can expect more meaningful interactions between humans and computers in the future.

Tackling math problems

(SOLO FOR COMP)





Whatever the case may be, we need your help to make progress.

David Hilbert, 1900

The Problems of Mathematics

"Who among us would not be happy to lift the veil behind which is hidden the future; to gaze at the coming developments of our science and at the secrets of its development in the centuries to come? What will be the ends toward which the spirit of future generations of mathematicians will tend? What methods, what new facts will the new century reveal in the vast and rich field of mathematical thought?"