CMU 15-252 **Spring 2017**

Homework 7 — Due April 7 in class

There are many different kinds of "Fourier Transforms". The DFT we saw in class is a kind of "physics-y" one, transforming discrete sequences into discrete sequences. In this homework you will see a more "computer science-y" one. We'll call it the DWT.[1]

Let $N = 2^n$ for some positive integer n. Define a real $N \times N$ matrix DWT_N as follows. We think of the rows/columns of DWT_N as being indexed by n-bit Boolean strings. Now for $x, y \in \{0,1\}^n$, the $[x, y]$ entry of DWT_N is defined by

$$\mathrm{DWT}_N[x, y] = (-1)^{x \cdot y},$$

where $x \cdot y$ denotes the "dot-product mod 2" of x and y; i.e., $\sum_{i=1}^{n} x_i y_i \pmod 2$.

It is also sometimes convenient to define the scaled matrix $\widetilde{\mathrm{DWT}}_N = \frac{1}{\sqrt{N}}\mathrm{DWT}_N$.

1. (No points, do not turn in.) Explicitly write DWT_2 and DWT_4.
2. Identify the inverse matrix of DWT_N, call it IDWT_N, and prove it's the inverse.
3. Prove that the matrix $\widetilde{\mathrm{DWT}}_N$ "preserves vector lengths". That is, for any vector $\vec{a} \in \mathbb{R}^N$, if we write $\vec{b} = \widetilde{\mathrm{DWT}}_N \cdot \vec{a}$, then $\|\vec{a}\| = \|\vec{b}\|$, where $\|\cdot\|$ is the usual Euclidean length of the vector (defined by $\|\vec{c}\|^2 = \sum_{x \in \{0,1\}^n} c_x^2$).
4. Describe and analyze an algorithm for computing $\mathrm{DWT}_N \cdot \vec{a}$, for an input vector $\vec{a} \in \mathbb{R}^N$. Your algorithm should use $O(N \log N)$ arithmetic operations. (You can assume adding/subtracting real numbers takes "1 step".)
5. Let G be the $N \times N$ matrix given by $\mathrm{IDWT}_N \cdot F \cdot \mathrm{DWT}_N$, where F is the $N \times N$ matrix (with rows/columns indexed by n-bit strings) defined by

$$F[x, y] = \begin{cases} 1 & \text{if } x = y = 000 \cdots 0 \\ -1 & \text{if } x = y \text{ but they're not equal to } 000 \cdots 0 \\ 0 & \text{if } x \neq y. \end{cases}$$

   Prove that G acts on vectors $\vec{a} \in \mathbb{R}^N$ by "flipping over the average". That is, given $\vec{a} \in \mathbb{R}^N$, if $\mu = \mathrm{avg}_{x \in \{0,1\}^n}\{a_x\}$, and $\vec{b} = G \cdot \vec{a}$, then b_x is equal to "the value of a_x reflected across μ on the real line". Prove also that G "preserves vector lengths".

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[1]For math nerds/lovers: We might identify the complex vector space $\mathbb{C}^N$ with the vector space of functions $f : \mathbb{Z}_N \to \mathbb{C}$, where $\mathbb{Z}_N$ is the group of integer mod N, and the identification is to just view a vector as the "truth table" (list of values) of the function. The DFT we saw in class can be thought of as a change-of-basis matrix from the "standard basis" of $\mathbb{C}^N$ to the orthonormal basis $\{\rho_0, \rho_1, \ldots, \rho_{N-1}\}$, where $\rho_j(k) = \omega_N^{j \cdot k}$, the multiplication $j \cdot k$ being mod N. In this problem, we are motivated by (real-valued) Boolean functions $f : \mathbb{Z}_2^n \to \mathbb{R}$; equivalently, real vectors (truth tables) of dimension $N = 2^n$. The DWT in this problem can again be thought of as a change-of-basis matrix from the standard basis to the orthonormal basis of "Boolean" functions $\{\chi_y\}_{y \in \{0,1\}^n}$, where $\chi_y(x) = (-1)^{x \cdot y}$, with $x \cdot y$ being the dot-product in $\mathbb{Z}_2^n$. Note that $-1 = \omega_2$:)