There are many different kinds of “Fourier Transforms”. The DFT we saw in class is a kind of “physics-y” one, transforming discrete sequences into discrete sequences. In this homework you will see a more “computer science-y” one. We’ll call it the DWT.¹

Let \( N = 2^n \) for some positive integer \( n \). Define a real \( N \times N \) matrix \( \text{DWT}_N \) as follows. We think of the rows/columns of \( \text{DWT}_N \) as being indexed by \( n \)-bit Boolean strings. Now for \( x, y \in \{0, 1\}^n \), the \([x, y] \) entry of \( \text{DWT}_N \) is defined by

\[
\text{DWT}_N[x, y] = (-1)^{x \cdot y},
\]

where \( x \cdot y \) denotes the “dot-product mod 2” of \( x \) and \( y \); i.e., \( \sum_{i=1}^{n} x_i y_i \pmod{2} \).

It is also sometimes convenient to define the scaled matrix \( \tilde{\text{DWT}}_N = \frac{1}{\sqrt{N}} \text{DWT}_N \).

1. (No points, do not turn in.) Explicitly write \( \text{DWT}_2 \) and \( \text{DWT}_4 \).

2. Identify the inverse matrix of \( \text{DWT}_N \), call it \( \text{IDWT}_N \), and prove it’s the inverse.

3. Prove that the matrix \( \tilde{\text{DWT}}_N \) “preserves vector lengths”. That is, for any vector \( \vec{a} \in \mathbb{R}^N \), if we write \( \vec{b} = \text{DWT}_N \cdot \vec{a} \), then \( \|\vec{a}\| = \|\vec{b}\| \), where \( \|\cdot\| \) is the usual Euclidean length of the vector (defined by \( \|\vec{c}\|^2 = \sum_{x \in \{0, 1\}^n} c_x^2 \)).

4. Describe and analyze an algorithm for computing \( \text{DWT}_N \cdot \vec{a} \), for an input vector \( \vec{a} \in \mathbb{R}^N \). Your algorithm should use \( O(N \log N) \) arithmetic operations. (You can assume adding/subtracting real numbers takes “1 step”.)

5. Let \( G \) be the \( N \times N \) matrix given by \( \text{IDWT}_N \cdot F \cdot \text{DWT}_N \), where \( F \) is the \( N \times N \) matrix (with rows/columns indexed by \( n \)-bit strings) defined by

\[
F[x, y] = \begin{cases} 
1 & \text{if } x = y = 000 \ldots 0 \\
-1 & \text{if } x = y \text{ but they’re not equal to } 000 \ldots 0 \\
0 & \text{if } x \neq y.
\end{cases}
\]

Prove that \( G \) acts on vectors \( \vec{a} \in \mathbb{R}^N \) by “flipping over the average”. That is, given \( \vec{a} \in \mathbb{R}^N \), if \( \mu = \text{avg}_{x \in \{0, 1\}^n} \{a_x\} \), and \( \vec{b} = G \cdot \vec{a} \), then \( b_x \) is equal to “the value of \( a_x \) reflected across \( \mu \) on the real line”. Prove also that \( G \) “preserves vector lengths”.

¹For math nerds/lovers: We might identify the complex vector space \( \mathbb{C}^N \) with the vector space of functions \( f : \mathbb{Z}_N \to \mathbb{C} \), where \( \mathbb{Z}_N \) is the group of integer mod \( N \), and the identification is to just view a vector as the “truth table” (list of values) of the function. The DFT we saw in class can be thought of as a change-of-basis matrix from the “standard basis” of \( \mathbb{C}^N \) to the orthonormal basis \( \{\rho_0, \rho_1, \ldots, \rho_{N-1}\} \), where \( \rho_j(k) = \omega_{N}^{jk} \), the multiplication \( j \cdot k \) being mod \( N \). In this problem, we are motivated by (real-valued) Boolean functions \( f : \mathbb{Z}_2^n \to \mathbb{R} \); equivalently, real vectors (truth tables) of dimension \( N = 2^n \). The DWT in this problem can again be thought of as a change-of-basis matrix from the standard basis to the orthonormal basis of “Boolean” functions \( \{\chi_y\}_{y \in \{0, 1\}^n} \), where \( \chi_y(x) = (-1)^{x \cdot y} \), with \( x \cdot y \) being the dot-product in \( \mathbb{Z}_2^n \). Note that \(-1 = \omega_{2} \).