15-251: Great Theoretical Ideas In Computer Science

Recitation 6

Announcements

- Midterm 1 next **Wednesday, March 1**! It will be held in **DH 2210** from **6.30pm** to 9.30pm in place of the writing session. (Note the later end time.)
- Practice problems have been posted. We will be holding two **solution sessions** to go over these problems in place of the **Monday 7pm - 8.30pm** and **Tuesday 6pm - 7.30pm** early evening office hours at the regular location (**GHC 4301**).
- We will also be holding reviews for the following topics:
  - DFAs, (Ir)regularity, and (Un)countability: Saturday 12pm - 1.30pm at MM 103
  - Turing Machines and (Un)decidability: Sunday 12pm - 1.30pm at DH 1112
  - Computational Complexity: Sunday 4.30pm - 6pm at DH 1112
- Sunday office hours are cancelled in lieu of the above reviews. All other office hours will continue as per normal. Make good use of them, and good luck for the midterm!

Definitions

There are like a hundred of them. Sorry!

Primitive Problems

This question is about the Minimum Spanning Tree problem.

(a) Suppose an instance of the Minimum Spanning Tree problem is allowed to have negative costs for the edges. Explain whether the Jarník-Prim algorithm would work in this case as well.

(b) Consider the problem of computing the maximum spanning tree, i.e., a spanning tree that maximizes the sum of the edge costs. Explain whether the Jarník-Prim algorithm solves this problem if we modify it so that at each iteration, the algorithm chooses the edge between V' and $V \backslash V'$ with the maximum cost.

"Clearly" Correct

A connected graph with no cycles is a tree. Consider the following claim and its proof.

> **Claim:** Any graph with n vertices and $n-1$ edges is a tree.
> **Proof:** We prove the claim by induction. The claim is clearly true for $n = 1$ and $n = 2$. Now suppose the claim holds for $n = k$. We'll prove that it also holds for $n = k+1$. Let G be a graph with k vertices and $k-1$ edges. By the induction hypothesis, G is a tree (and therefore clearly connected). Add a new vertex v to G by connecting it with any other vertex in G. So we create a new graph G' with $k+1$ vertices and k edges. The new vertex we added is clearly a leaf, so it clearly does not create a cycle. Also, since G was connected, G' is clearly also connected. A connected graph with no cycles is a tree, so G' is also a tree. So the claim follows by induction.

Explain why the given proof is incorrect.

2 3 Proofs 4 You

Give two proofs (one using induction and another using a degree counting argument) for the following claim: A tree with $n \geq 2$ vertices has

$$2 + \sum_{\substack{v \in V \\ \deg(v) \geq 3}} (\deg(v) - 2) \text{ leaves.}$$

Counting Colors 1, 2, 3, ...

Let $G = (V, E)$ be an undirected graph. Let $k \in \mathbb{N}^+$. A k-*coloring* of V is just a map $\chi : V \to C$ where C is a set of cardinality k. (Usually the elements of C are called *colors*. If $k = 3$ then $\{$red, green, blue$\}$ is a popular choice. If k is large, we often just call the colors $1, 2, \ldots, k$.) A k-coloring is said to be *legal* for G if every edge in E is *bichromatic*, meaning that its two endpoints have different colors. (I.e., for all $\{u, v\} \in E$ it is required that $\chi(u) \neq \chi(v)$.) Finally, we say that G is k-*colorable* if it has a legal k-coloring.

(a) Suppose G has no cycles of length greater than 251. Prove that G is 251-colorable. Hint: DFS.

(b) Give an example to show that the above is tight, i.e., find a graph G with no cycles of length greater than 251 that is not 250-colorable.

(Extra) Long Walks

Suppose a graph G has minimum degree δ (so the vertex of lowest degree has degree δ). Show that G contains a path of length (at least) δ.

(Bonus) Graphitti

How many colors do you need to color the vertices of this graph?