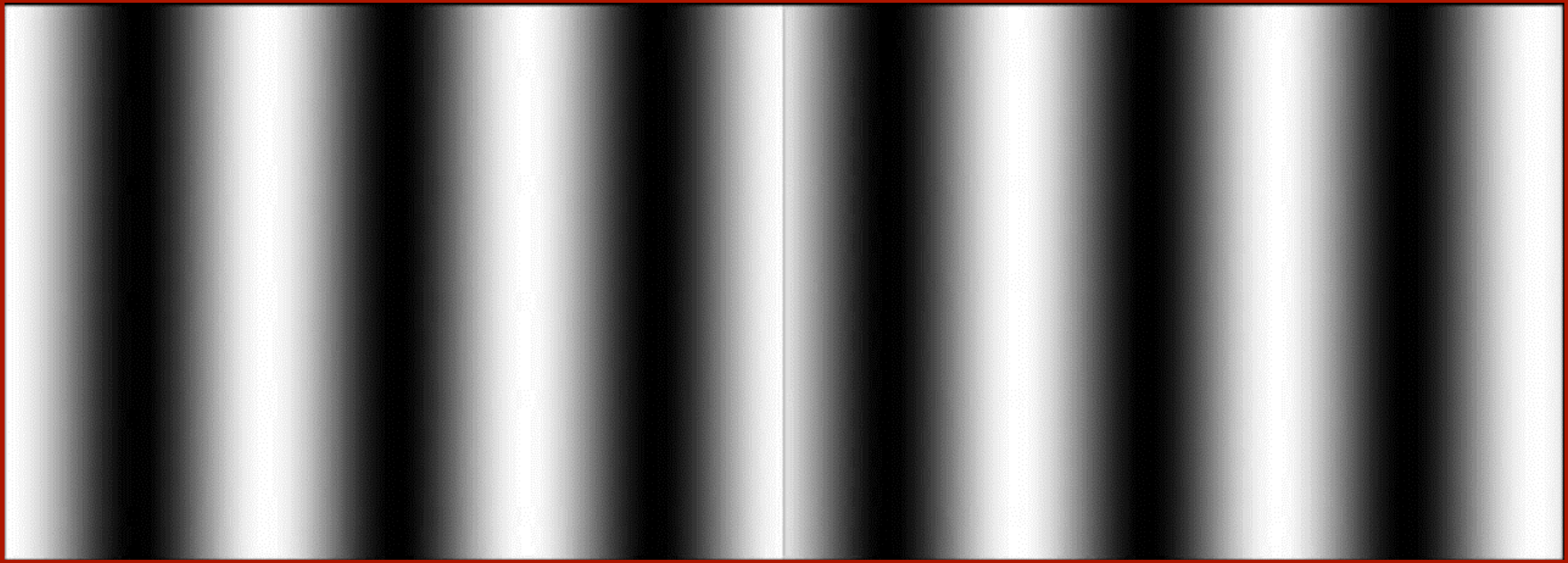
15-252: More Great Ideas in Theoretical Computer Science

Spring 2017

Fast Fourier Transform



Integer multiplication

Multiplying two n-bit integers A and B:

"Grade School" Method: $O(n^2)$ time.

Karatsuba's Algorithm: $O(n^{\log_2 3}) = O(n^{1.58\ldots})$

(what Python uses)

Generalizations thereof: $O(n^{1+\epsilon})$

Fürer 2007: circuits of size $n (\log n) 2^{O(\log^* n)}$

Schönhage−Strassen late '60s : $O(n)$ time (!!)

via **Fast Fourier Transform**

(in RAM model)

Volker Strassen

& Arnold Schönhage,

late ’60s

Ideas discussed on the homework…

1. Multiplying integers reduces to multiplying ***polynomials*** with integer coefficients.

2. Multiplying polynomials is easy in the "Values Representation".

3. With a **magic** set of interpolation points, going between "Coefficients Representation" and "Values Representation" is super-fast.

Goal

Multiplying two polynomials with degree < N

(and coefficients fitting in a “word”)

in O(N log N) time.

Implies O(n) time multiplication of n-bit integers.

Polynomial multiplication

Let $P(x)$ and $Q(x)$ be polynomials of degree $< N$.

Assumed in "Coefficients Representation",

$$P(x) = a_0 + a_1\, x + a_2\, x^2 + \cdots + a_{N-1}\, x^{N-1}$$
$$Q(x) = b_0 + b_1\, x + b_2\, x^2 + \cdots + b_{N-1}\, x^{N-1}$$

(where a_j's, b_k's are ints fitting in a word).

Let $R(x) = P(x) \cdot Q(x)$, of degree $< 2N$.

Task is to get $R(x)$ in Coefficients Representation.

Polynomial multiplication

Let $P(x)$ and $Q(x)$ be polynomials of degree $< N$.

Assumed in “Coefficients Representation”,

Let $R(x) = P(x) \cdot Q(x)$, of degree $< 2N$.

Task is to get $R(x)$ in Coefficients Representation.

If only everything were in “Values Representation” instead…

Polynomial multiplication

Let $P(x)$ and $Q(x)$ be polynomials of degree $< N$.

Assumed in "Coefficients Representation",

Let $R(x) = P(x) \cdot Q(x)$, of degree $< 2N$.

Task is to get $R(x)$ in Coefficients Representation.

If only we knew

$P(1), P(2), \ldots, P(2N)$,

$Q(1), Q(2), \ldots, Q(2N)$,

$R(1), R(2), \ldots, R(2N)$

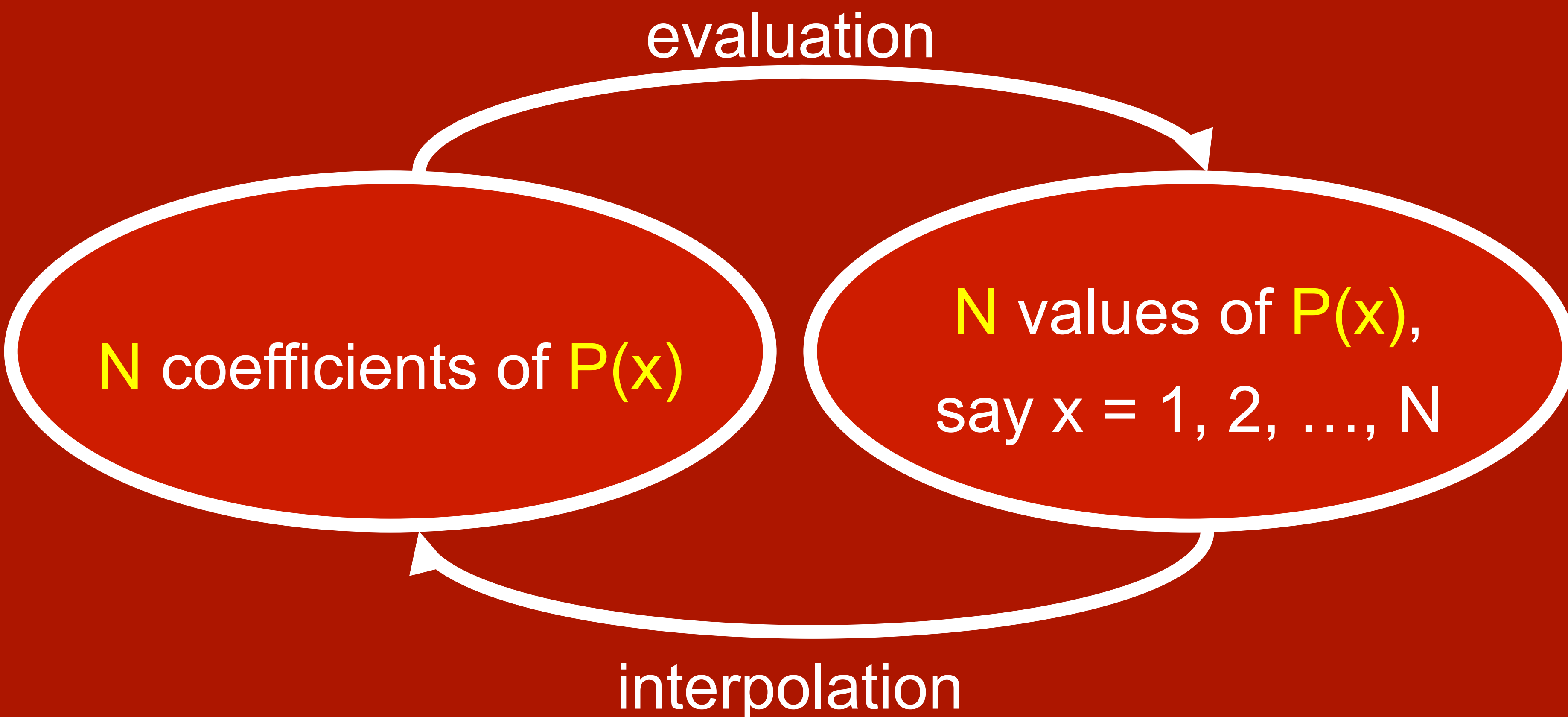
multiply in $O(N)$ time

uniquely determines $R(x)$ by interpolation

If we could somehow pass between

Coefficients Representation & Values Representation

in O(N log N) time, we'd be done.



Unfortunately, these seem to take $O(N^2)$ time.

If we could somehow pass between

Coefficients Representation & Values Representation

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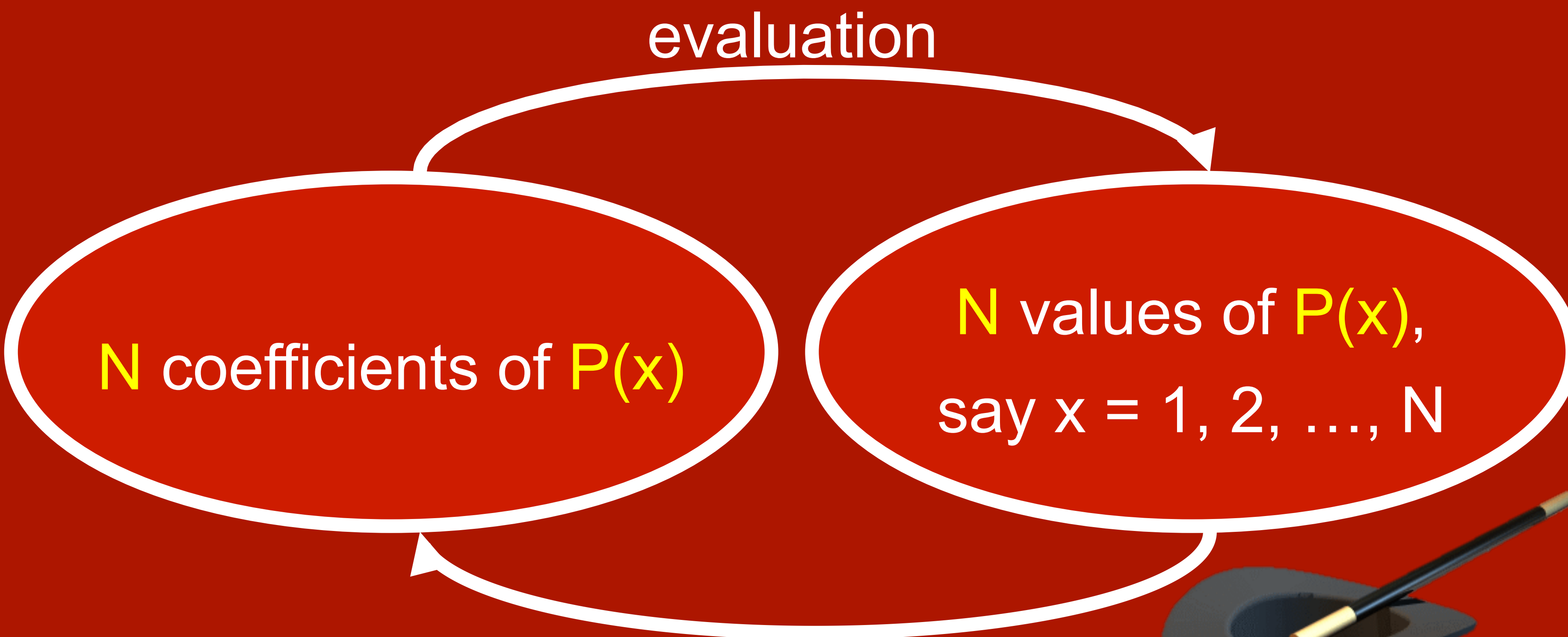

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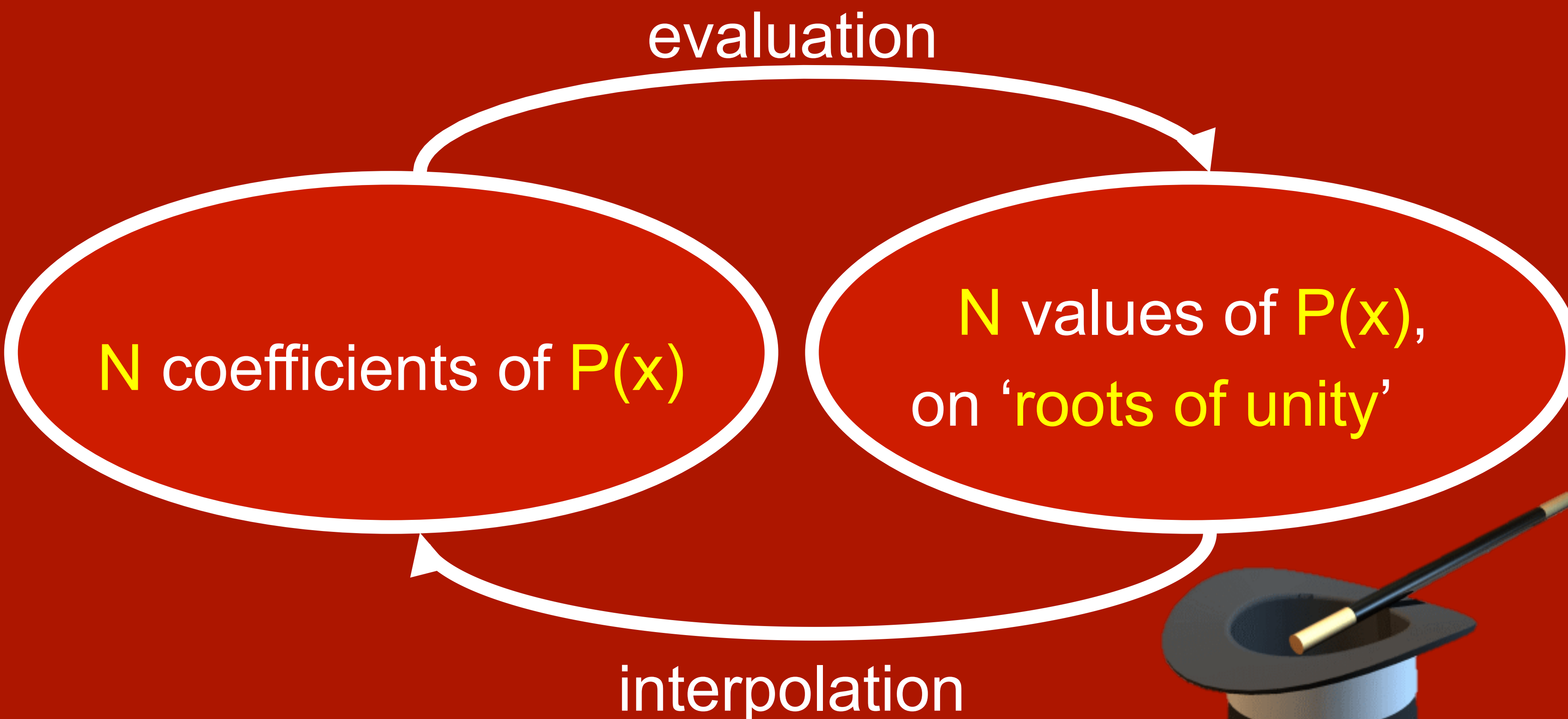


Voila! O(N log N) ops with "FFT".

Discrete Fourier Transform (& Inverse)

Let N be a power of 2.

$S_N = \{1, \omega_N^1, \omega_N^2, \omega_N^3, \ldots, \omega_N^{N-1}\}$ is the set of N "complex roots of unity" that I'll describe shortly.

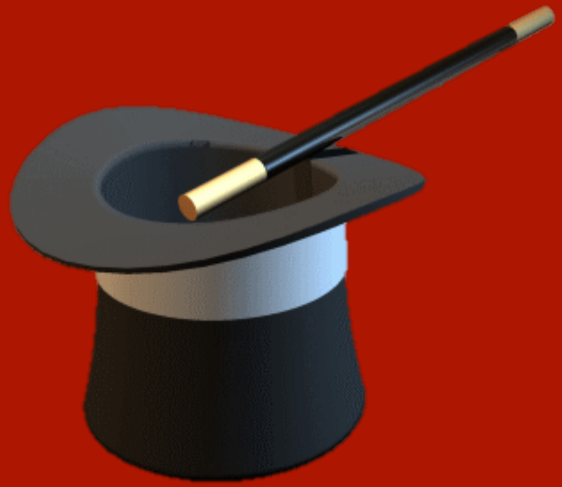
Let P(x) be a polynomial of degree N−1.

P's coefficients $\xrightarrow[\text{evaluation}]{\mathbf{DFT}_N}$ P's values on S_N

P's values on S_N $\xrightarrow[\text{interpolation}]{\mathbf{IDFT}_N}$ P's coefficients

Fast Fourier Transform

A recursive algorithm for DFT_N and $IDFT_N$ that uses only $O(N \log N)$ arithmetic operations.



P's coefficients	**DFT_N** → evaluation	P's values on S_N
P's values on S_N	**$IDFT_N$** → interpolation	P's coefficients

Fast Fourier Transform

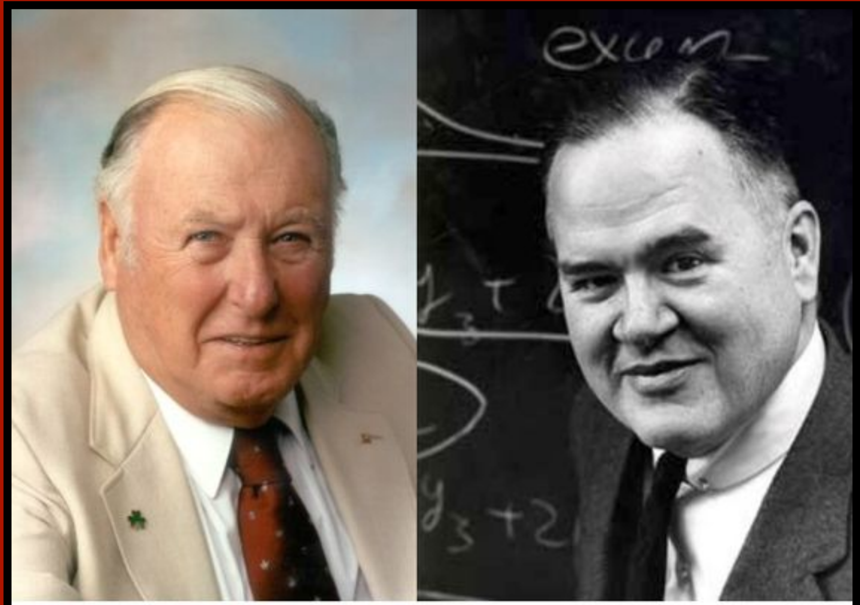
G. Strang, ’94: “*The most important numerical algorithm of our lifetime*.”

“Brigham [1974] says that [Richard] Garwin asked [John] Tukey to give him a rapid way to compute the Fourier transform during a meeting of the President's [Kennedy’s] Scientific Advisory Committee. Then Garwin went to the computing center at IBM Research in Yorktown Heights where [James] Cooley programmed the Fourier transform, because he had nothing better to do. After receiving many requests for the program, Cooley and Tukey published their paper in 1965.”

−A. Terras, ’99

Fast Fourier Transform

G. Strang, '94: "*The most important numerical algorithm of our lifetime.*"



James William Cooley　　John Wilder Tukey

1965

Fast Fourier Transform

G. Strang, ’94: “*The most important numerical algorithm of our lifetime*.”

“Heideman et al. [1984] note that [Carl Friedrich] Gauss discovered the fast Fourier transform in 1805 [two years before Fourier invented Fourier series!] while computing the eccentricity of the orbit of the asteroid Juno.”

−A. Terras, ’99

Fast Fourier Transform

G. Strang, ’94: “*The most important numerical algorithm of our lifetime*.”

OG, 1805

Multiplying polynomials with the FFT

Let P(x), Q(x) be polynomials of degree < N.

Want R(x) = P(x)·Q(x), which has degree < 2N.

1. Use DFT_{2N} to get P(w), Q(w) for all $w \in S_{2N}$ d
2. Multiply pairs, getting R(w) for all $w \in S_{2N}$ d
3. Use $IDFT_{2N}$ to get R's coefficients

P's coefficients —**DFT_N** (evaluation)→ P's values on S_N

P's values on S_N —**$IDFT_N$** (interpolation)→ P's coefficients

Multiplying polynomials with the FFT

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3. Use $IDFT_{2N}$ to get R's coefficients

Time:

1. O(N log N) arithmetic ops
2. O(N) arithmetic ops
3. O(N log N) arithmetic ops

O(N log N) arithmetic ops

Multiplying polynomials with the FFT

Can multiply two degree-N polynomials using O(N log N) arithmetic operations.

If the coefficients are ints fitting in a word, can multiply polynomials in O(N log N) time.

* Requires proving that you can compute the N^{th} roots of unity to O(log N) bits of precision in O(N log N) time, and that this precision is sufficient. This is fairly easy to prove, but also boring to prove.

Multiplying polynomials with the FFT

Can multiply two degree-N polynomials using O(N log N) arithmetic operations.

If the coefficients are ints fitting in a word, can multiply polynomials in O(N log N) time.

Implies O(n)-time multiplication of n-bit integers (in the Word RAM model).

The Discrete Fourier Transform

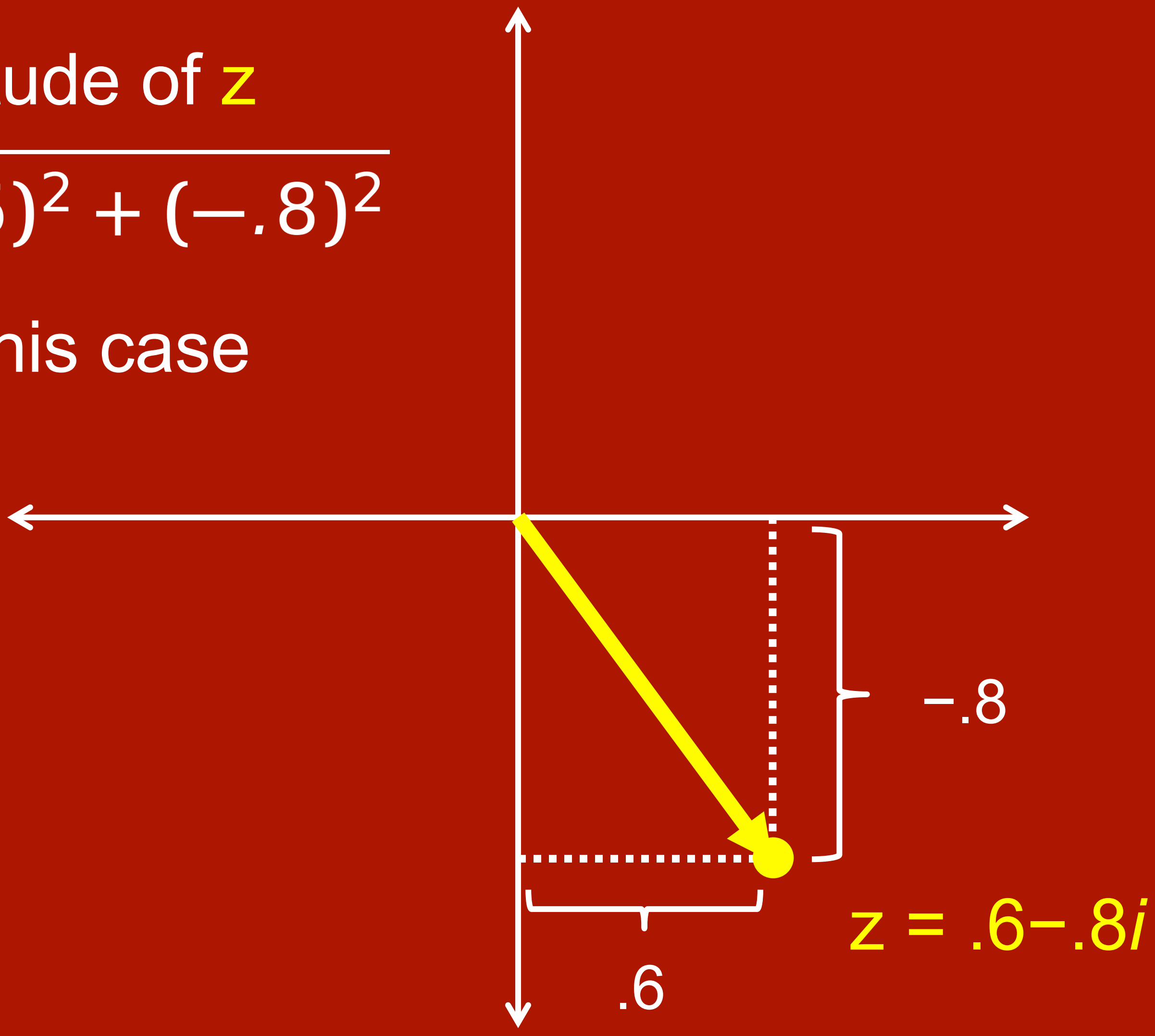
& The Fast Fourier Transform

The complex numbers ℂ

$|z|$ = magnitude of z

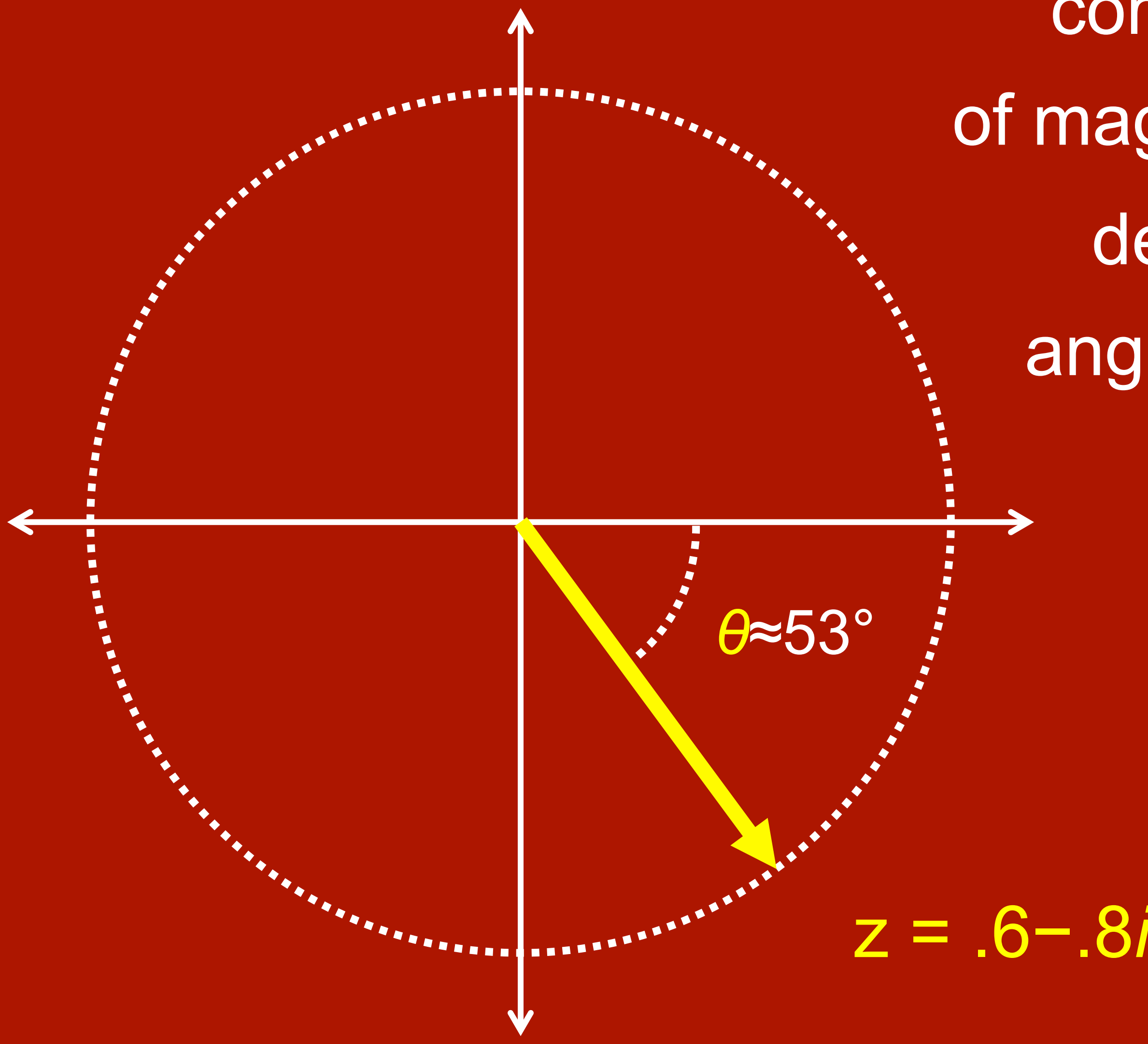
$$= \sqrt{(.6)^2 + (-.8)^2}$$

= 1, in this case



The complex numbers ℂ

complex #'s of magnitude 1 defined by angle θ from x-axis



Key Rule:

Multiplication by z = rotation by θ.

The complex numbers ℂ

complex #'s
of magnitude 1
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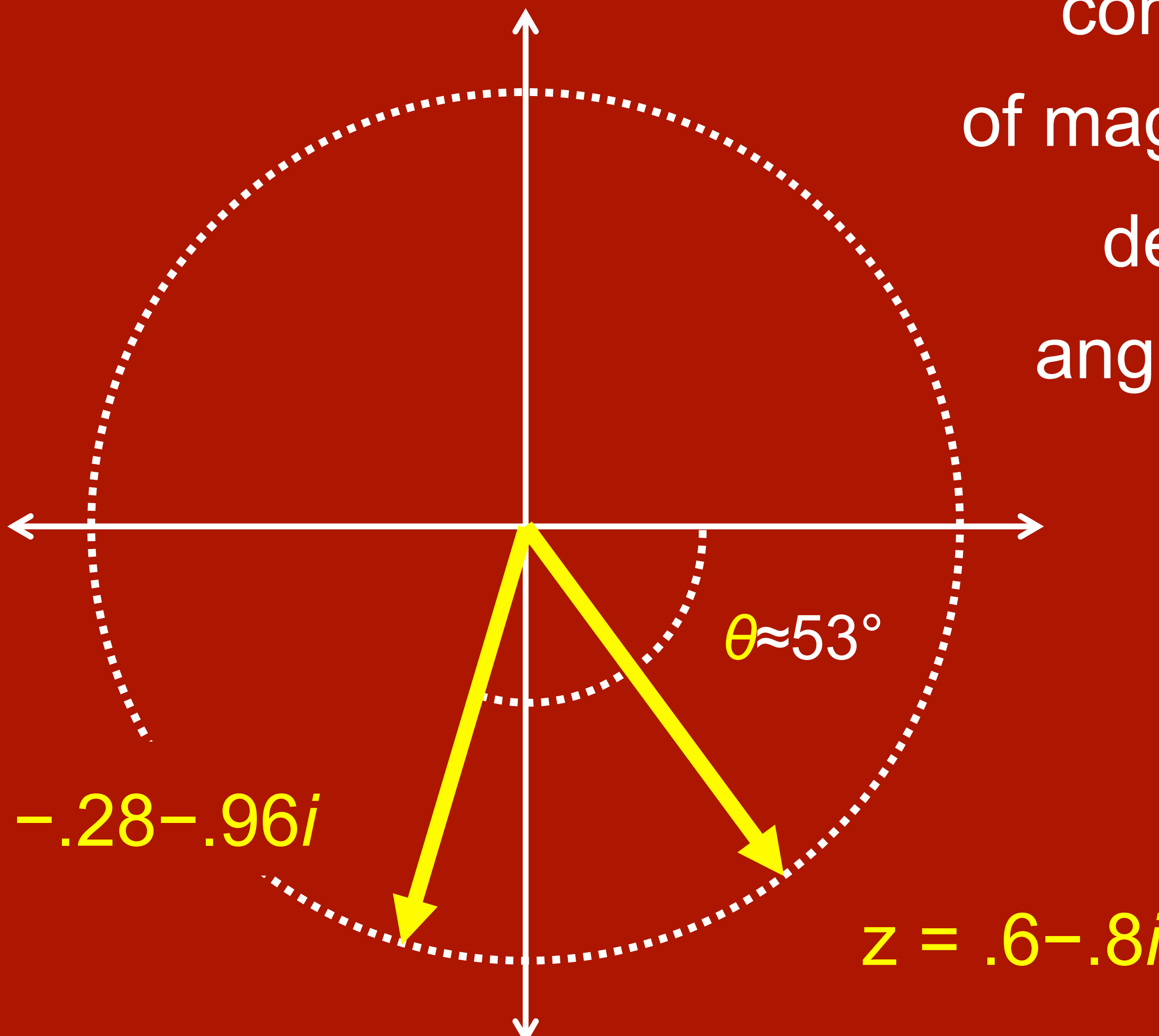
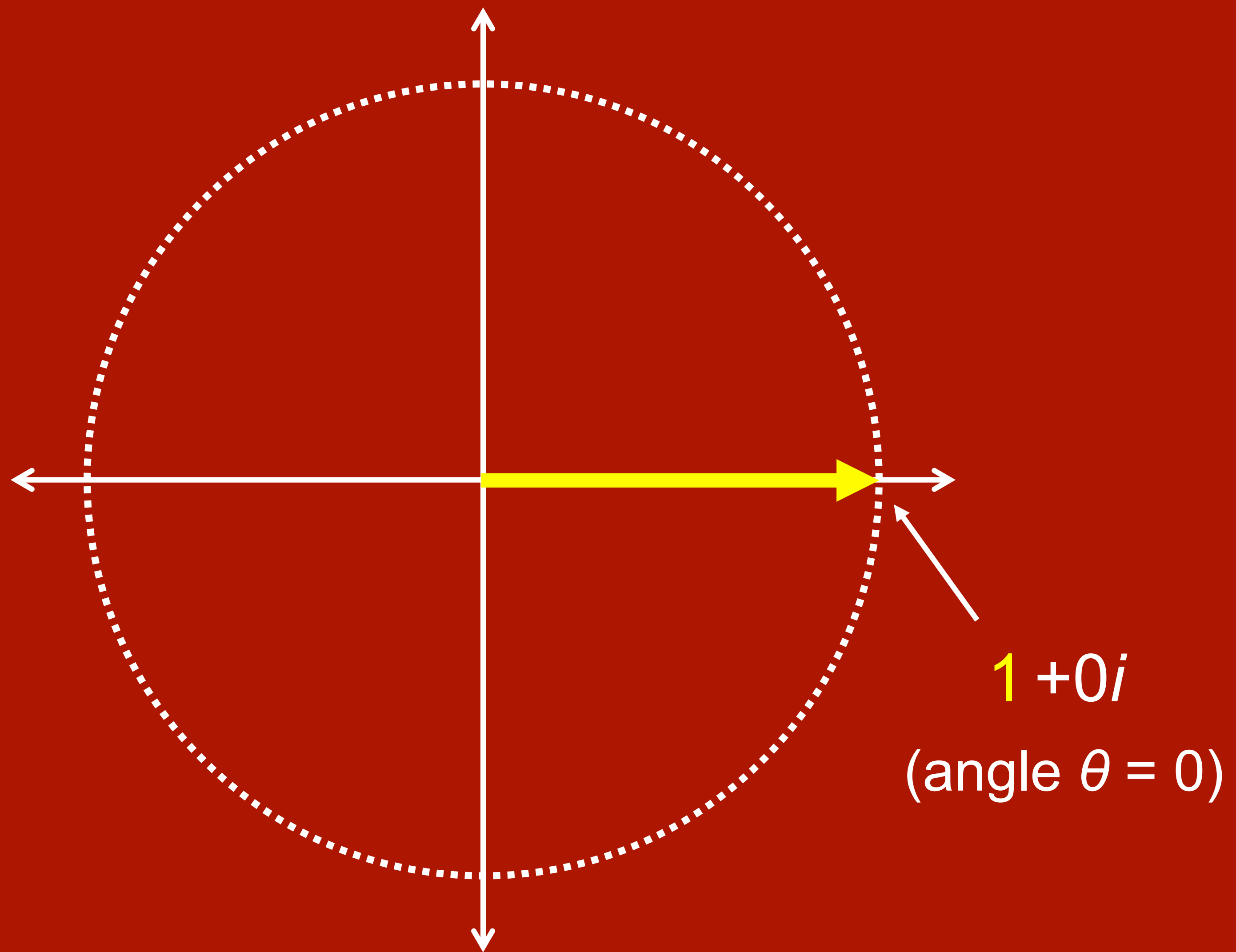


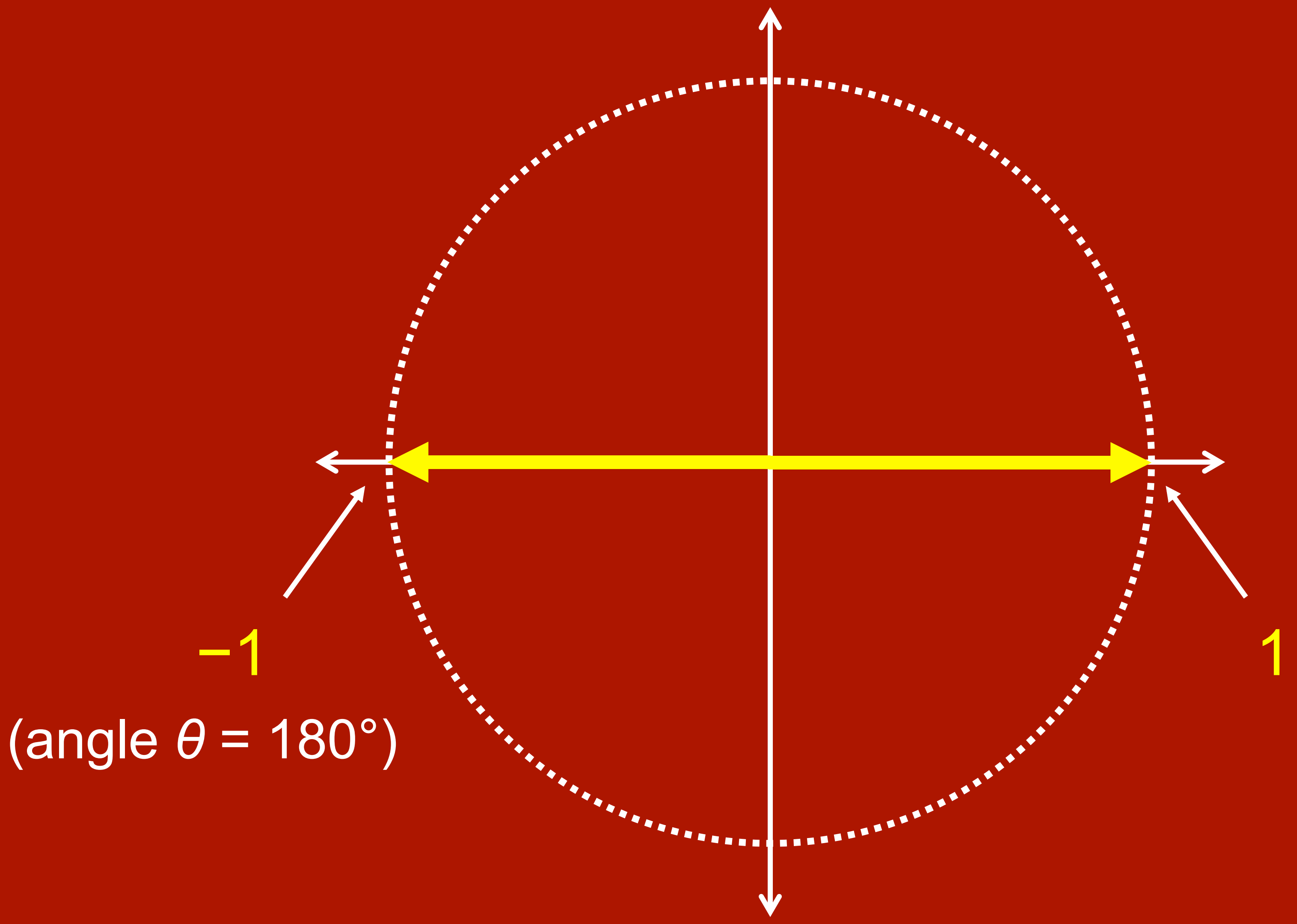
Key Rule:

Multiplication by z = rotation by θ.

Unity

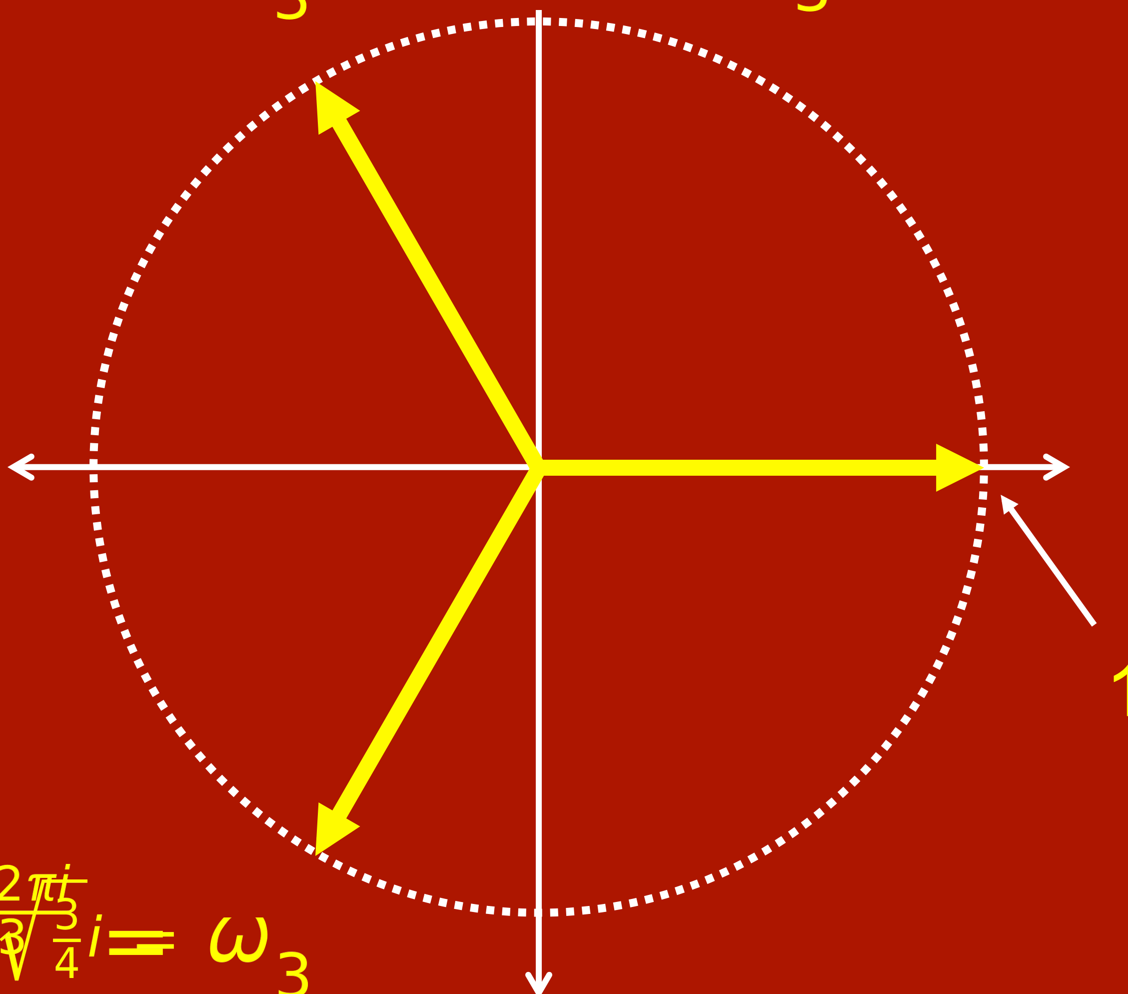


Square Roots of Unity



Cube Roots of Unity

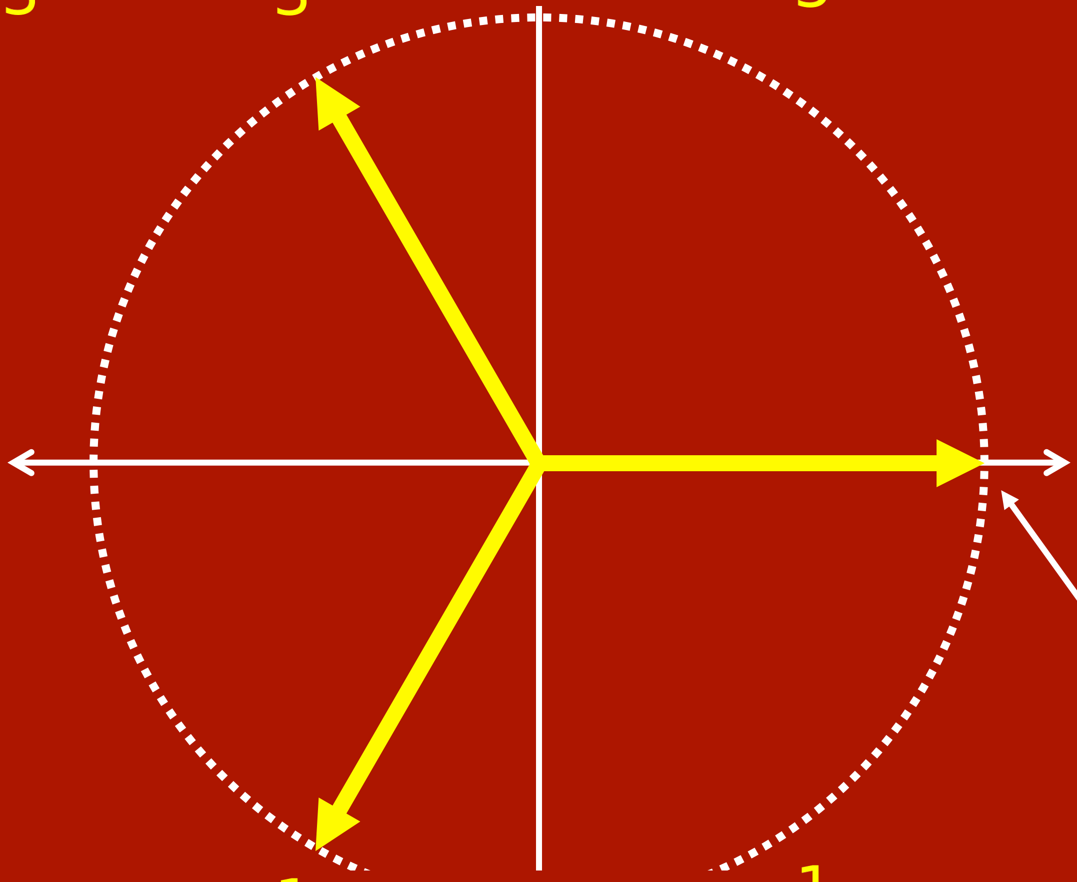
ω_3^2 = rotation by $\frac{2}{3}$ of a circle



$-\sqrt{\frac{1}{4}} - \sqrt{\frac{3}{4}}\,i = e^{\frac{-2\pi i}{3}} = \omega_3$

Cube Roots of Unity

$\omega_3^{-1} = \omega_3^2$ = rotation by $\frac{2}{3}$ of a circle

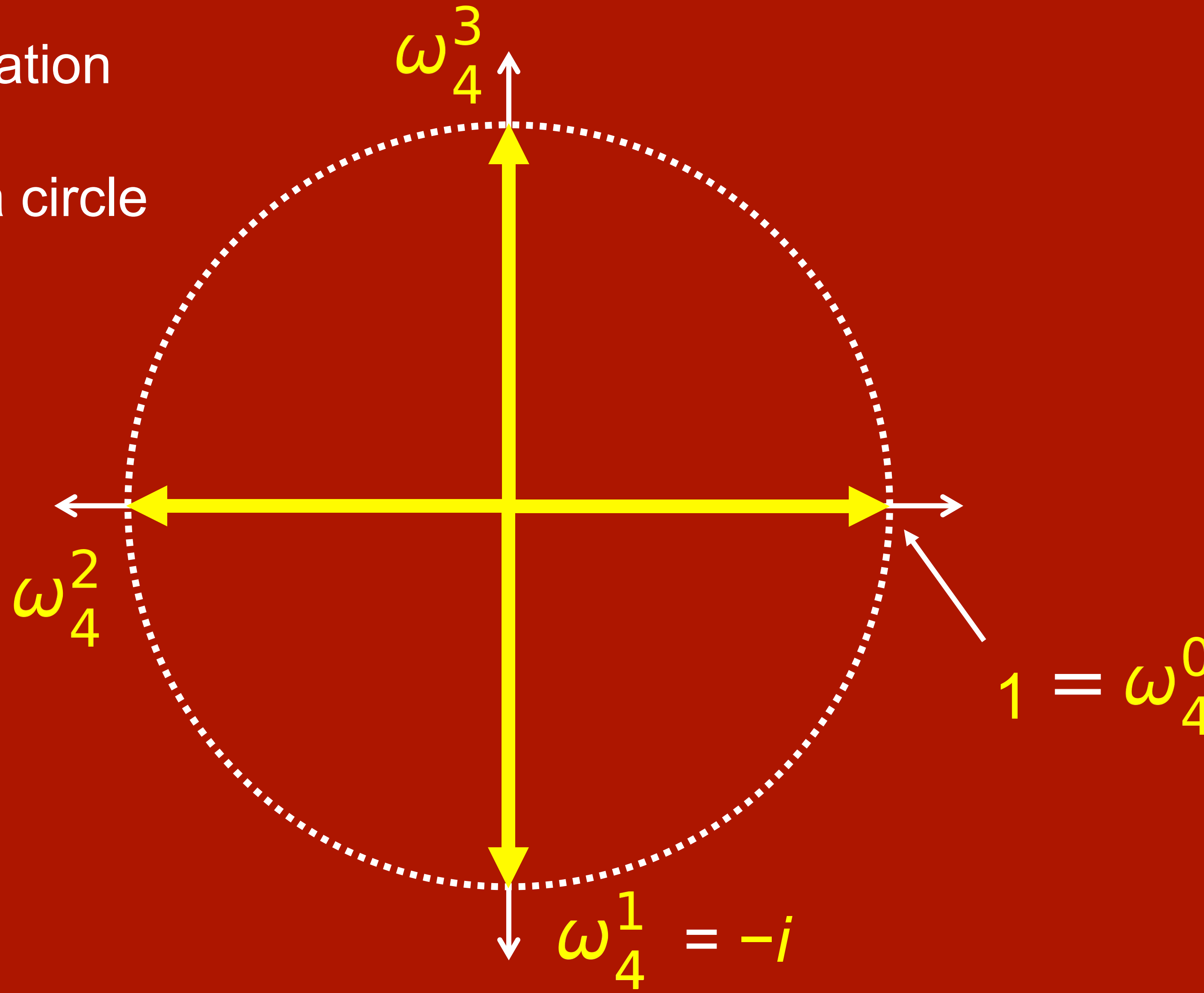


$1 = \omega_3^3 = \omega_3^0$

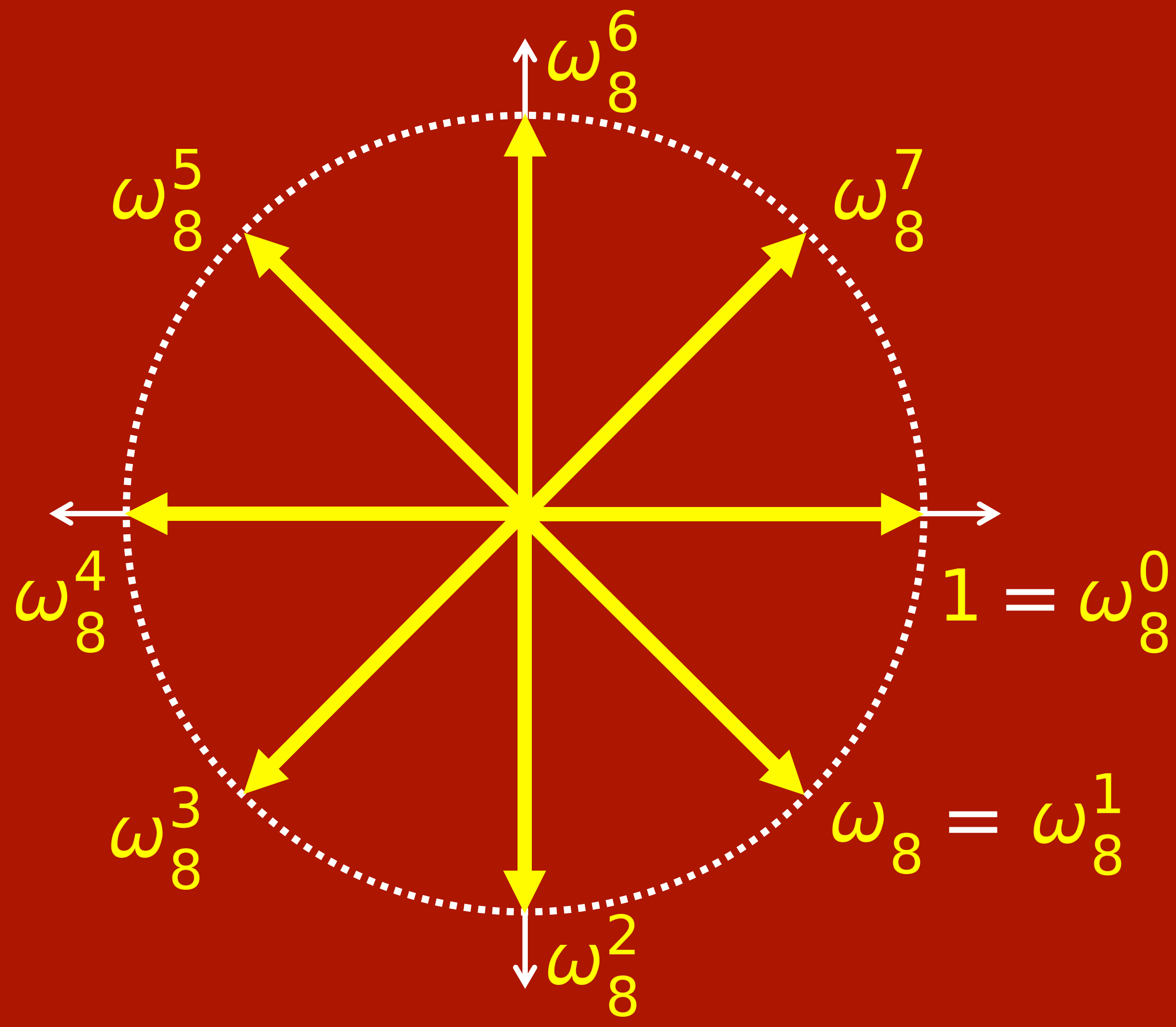
ω_3^1 = rotation by $\frac{1}{3}$ of a circle

4^{th} Roots of Unity

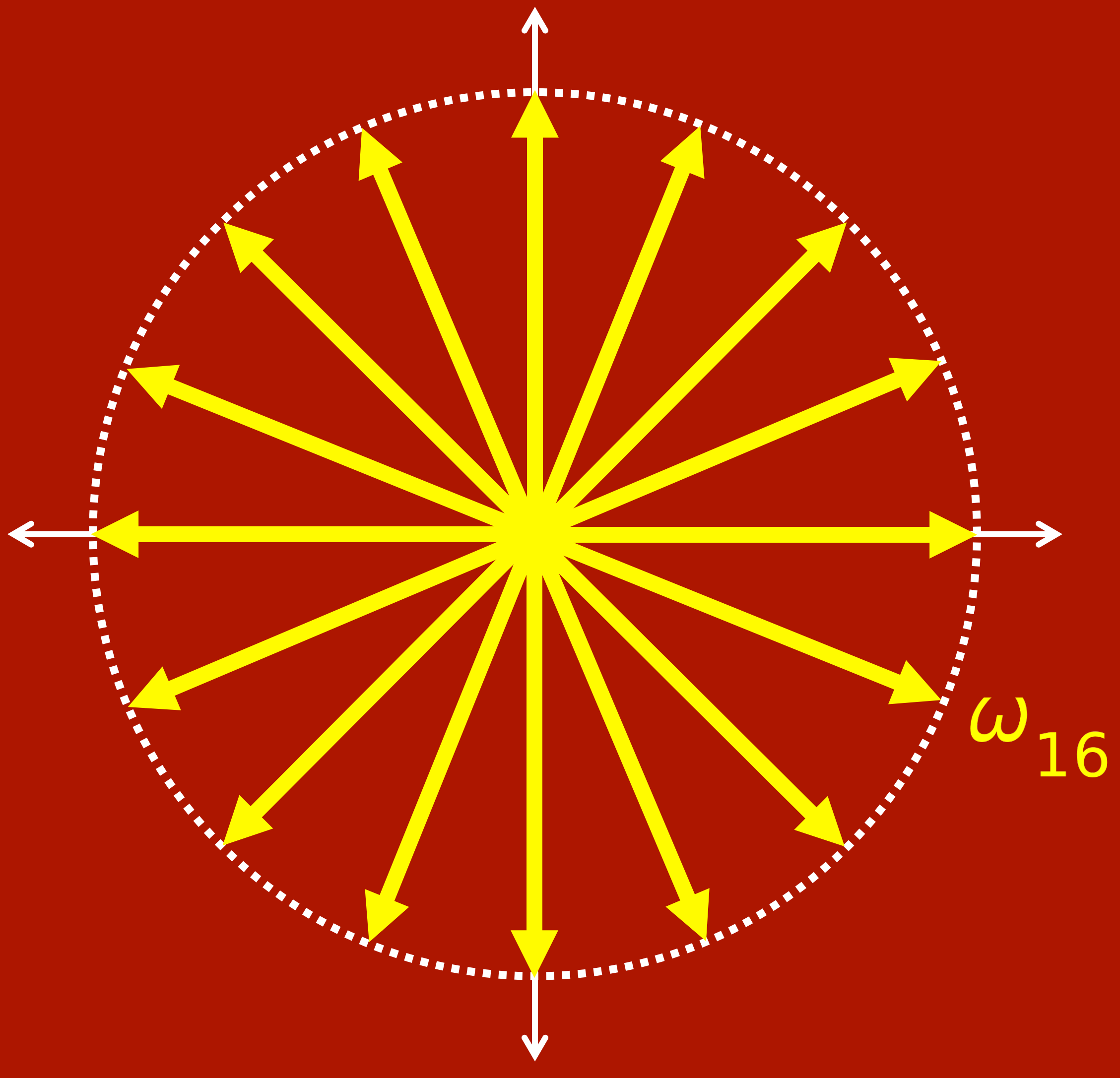
ω_4^j = rotation by $\frac{j}{4}$ of a circle



8^{th} Roots of Unity



16^{th} Roots of Unity



Discrete Fourier Transform (& Inverse)

Let N be a power of 2.

Let $S_N = \{1, \omega_N^1, \omega_N^2, \omega_N^3, \ldots, \omega_N^{N-1}\}$

Let $P(x)$ be a polynomial of degree $N-1$.

P's coefficients $\xrightarrow[\text{evaluation}]{\textbf{DFT}_N}$ P's values on S_N

P's values on S_N $\xrightarrow[\text{interpolation}]{\textbf{IDFT}_N}$ P's coefficients

Discrete Fourier Transform (& Inverse)

Let N be 8, and let $\omega = \omega_8$

Let $S_8 = \{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$

Let P(x) be a polynomial of degree 7.

P's coefficients $\xrightarrow[\text{evaluation}]{\textbf{DFT}_8}$ P's values on S_8

P's values on S_8 $\xrightarrow[\text{interpolation}]{\textbf{IDFT}_8}$ P's coefficients

Evaluation at $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$

Say $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ 1 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

Since $\omega^8 = 1$, we can reduce all exponents mod 8.

Evaluation at $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$

Say $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

DFT_8

$\mathrm{DFT}_8[j,k] = \omega^{jk \bmod 8}$ $(0 \le j, k < 7)$

$\ldots^4, \omega^5, \omega^6, \omega^7\}$

Multiplication modulo 8 table

•	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

$\ldots x^6 + a_7 x^7.$

$$\cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

$DFT_8[j,k] = \omega^{jk \bmod 8}$ $(0 \le j, k < 7)$

Evaluation at $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$

Say $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

$$DFT_8[j,k] = \omega^{jk \bmod 8} \quad (0 \le j, k < 7)$$

Evaluation at $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$

Say $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$.

$$DFT_8 \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

Interpolation?

Say $P(x) = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7$.

Given $P(1)$, $P(\omega)$, …, $P(\omega^7)$, how to get a_0, a_1, …, a_7?

$$\mathrm{DFT}_8 \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

Interpolation?

Say $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$.

Given $P(1)$, $P(\omega)$, …, $P(\omega^7)$, how to get a_0, a_1, …, a_7?

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \mathrm{DFT}_8^{-1} \cdot \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

also known as

IDFT_8

P's coefficients $\xrightarrow[\text{evaluation}]{\mathbf{DFT_N}}$ P's values on S_N

P's values on S_N $\xrightarrow[\text{interpolation}]{\mathbf{IDFT_N}}$ P's coefficients

also known as

$IDFT_8$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = DFT_8^{-1} \cdot \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

Interpolation?

Say $P(x) = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7$.

Given $P(1), P(\omega), \ldots, P(\omega^7)$, how to get $a_0, a_1, \ldots, a_7$?

also known as

IDFT_8

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \mathrm{DFT}_8^{-1} \cdot \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

DFT versus IDFT

Question:

We know what matrix DFT_8 is.

What is its inverse matrix, $IDFT_8$?

Answer:

It's extremely similar to DFT_8.

IDFT$_8$ DFT$_8$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$\text{DFT}_N[j,k] = \omega^{jk \bmod N}$$

($0 \le j, k < N$, $\omega = \omega_N$ is N^{th} root of unity)

$$\text{IDFT}_N[j,k] = \frac{1}{N}\omega^{-jk \bmod N}$$

$$\text{IDFT}_8 \qquad\qquad\qquad\qquad \text{DFT}_8$$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

Proof illustration.

We'll show the product =

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{IDFT}_8 \qquad\qquad \mathrm{DFT}_8$$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

Proof by picture.

We'll show the product =

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

IDFT$_8$ DFT$_8$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

Proof by picture.

We'll show the product =

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

IDFT$_8$ DFT$_8$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

Proof by picture.

We'll show the product =

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

IDFT_8 $\qquad\qquad$ DFT_8

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

Proof by picture.

We'll show the product =

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{IDFT}_8 \qquad\qquad\qquad \text{DFT}_8$$

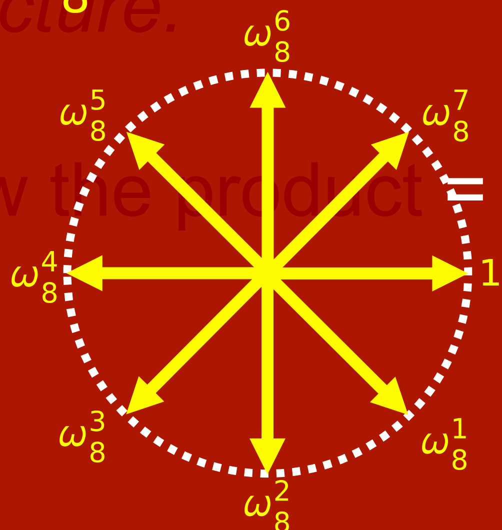
$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$\frac{1+\omega^1+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6+\omega^7}{8}$$

Proof by picture.

average
is 0

We'll show the product =



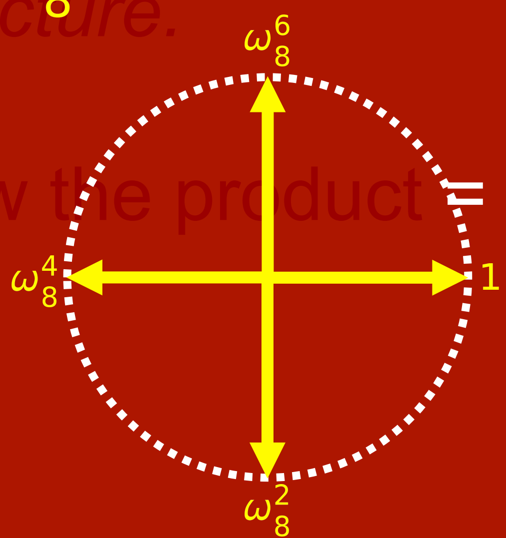
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix}}_{\text{IDFT}_8} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}}_{\text{DFT}_8}$$

$$\frac{1+\omega^2+\omega^4+\omega^6+1+\omega^2+\omega^4+\omega^6}{8}$$

Proof by picture.

average is 0



We'll show the product =

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

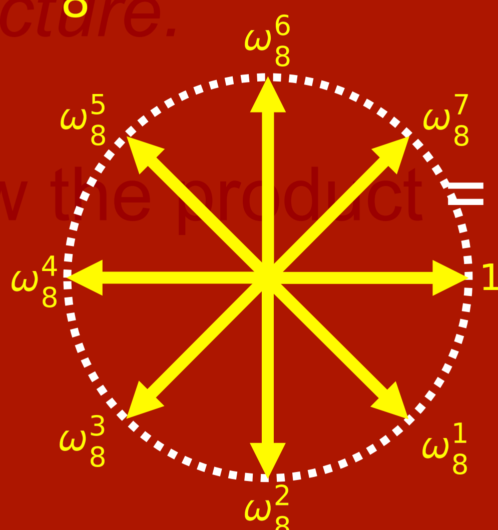
$$\frac{1}{8}\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix}}_{\text{IDFT}_8}
\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}}_{\text{DFT}_8}$$

$$\frac{1+\omega^3+\omega^6+\omega^1+\omega^4+\omega^7+\omega^2+\omega^5}{8}$$

Proof by picture.

average is 0

We'll show the product =



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{IDFT}_8 \qquad\qquad\qquad\qquad \text{DFT}_8$$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

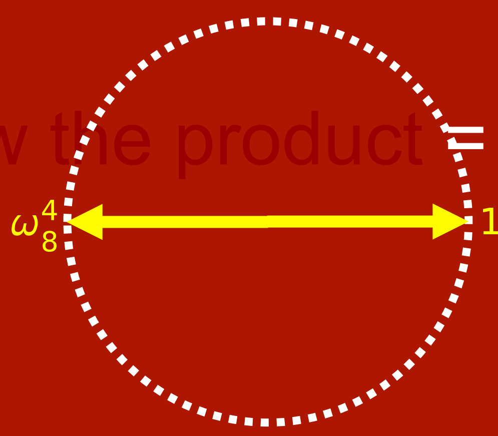
$$\frac{1+\omega^4+1+\omega^4+1+\omega^4+1+\omega^4+}{8}$$

Proof by picture.

average

is 0

We'll show the product =



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

IDFT_8 $\quad$ DFT_8

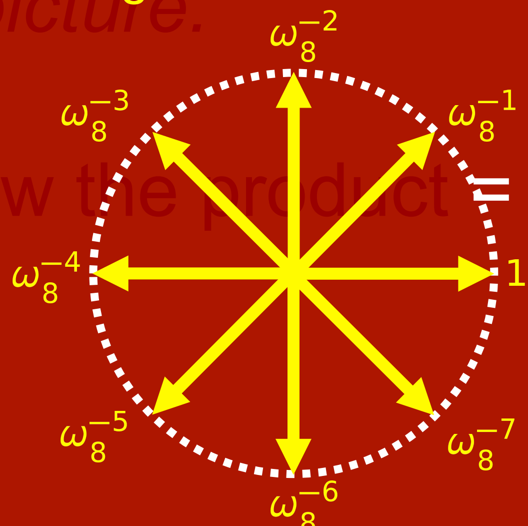
$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$\frac{1 + \omega^{-1} + \omega^{-2} + \omega^{-3} + \omega^{-4} + \omega^{-5} + \omega^{-6} + \omega^{-7}}{8}$$

Proof by picture.

average is 0

We'll show the product =



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

IDFT_8 $\qquad\qquad$ DFT_8

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

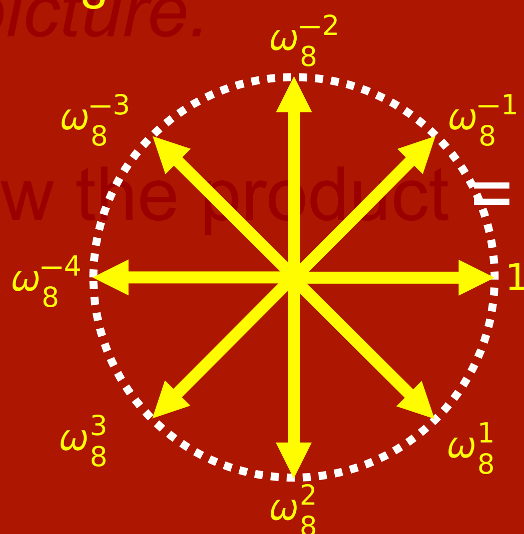
$$\frac{1+\omega^{1}+\omega^{2}+\omega^{3}+\omega^{-4}+\omega^{-3}+\omega^{-2}+\omega^{-1}}{8}$$

Proof by picture.

average

is 0

We'll show the product =



$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{IDFT}_8 \qquad\qquad\qquad\qquad \mathrm{DFT}_8$$

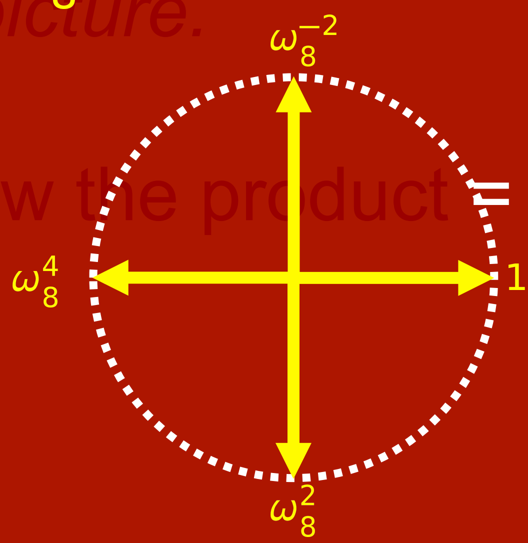
$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$\frac{1+\omega^2+\omega^4+\omega^{-2}+1+\omega^2+\omega^{-4}+\omega^{-2}}{8}$$

Proof by picture.

average is 0

We'll show the product =



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

IDFT_8 $\quad$ DFT_8

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

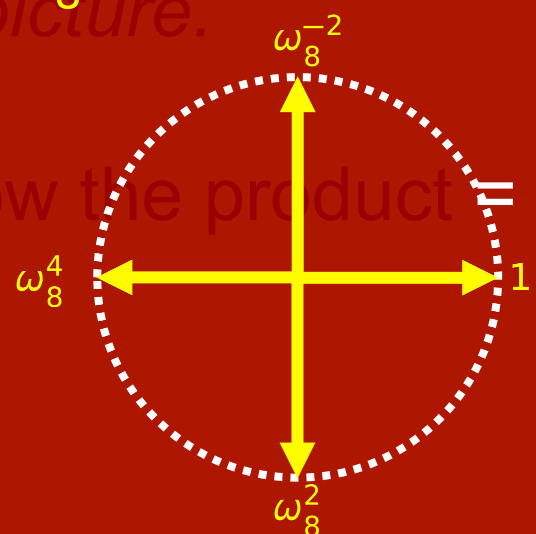
$$\frac{1+\omega^2+\omega^4+\omega^{-2}+1+\omega^2+\omega^{-4}+\omega^{-2}}{8}$$

Proof by picture.

average

is 0

We'll show the product =



$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{IDFT}_8 \qquad\qquad\qquad\qquad \text{DFT}_8$$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

Proof by picture.

Well, looks pretty true.

We'll show the product =

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Proof is an exercise. ☺

Last piece of the puzzle: FFT

$$\mathrm{DFT}_N \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{N-2} \\ a_{N-1} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_{N-2} \\ b_{N-1} \end{bmatrix}$$

Computing this in O(N log N) ops

$$\mathrm{DFT}_N \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{N-2} \\ a_{N-1} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_{N-2} \\ b_{N-1} \end{bmatrix}$$

Terminology:

sequence (the vector a) — transformed sequence (the vector b)

$$
\mathrm{DFT}_N \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{N-2} \\ a_{N-1} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_{N-2} \\ b_{N-1} \end{bmatrix}
$$

Claim: DFT_N reduces to 2 applications of $\mathrm{DFT}_{N/2}$, plus $O(N)$ additional operations.

$\Rightarrow T(N) = 2T(N/2) + O(N) \quad \Rightarrow T(N) = O(N \log N)$

Claim: DFT_8 reduces to 2 applications of DFT_4, plus "O(8)" additional operations.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix}$$

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$

Claim: DFT_8 reduces to 2 applications of DFT_4,
plus "O(8)" additional operations.

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$

Claim: DFT_8 reduces to 2 applications of DFT_4,

plus "O(8)" additional operations.

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$

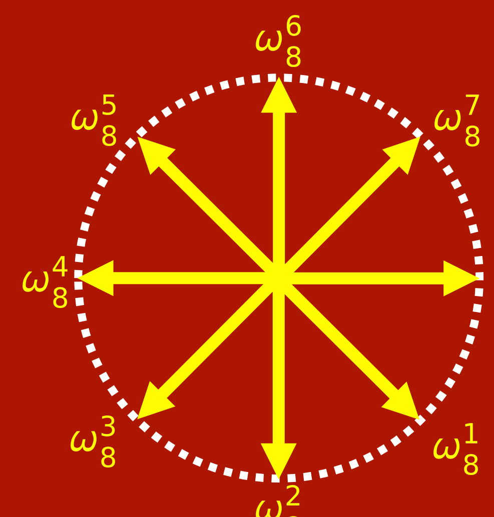
Claim: DFT_8 reduces to 2 applications of DFT_4, plus "$O(8)$" additional operations.

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix}$$

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix}$$

$$= \left(\text{ditto} \right)$$

$$+ a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$



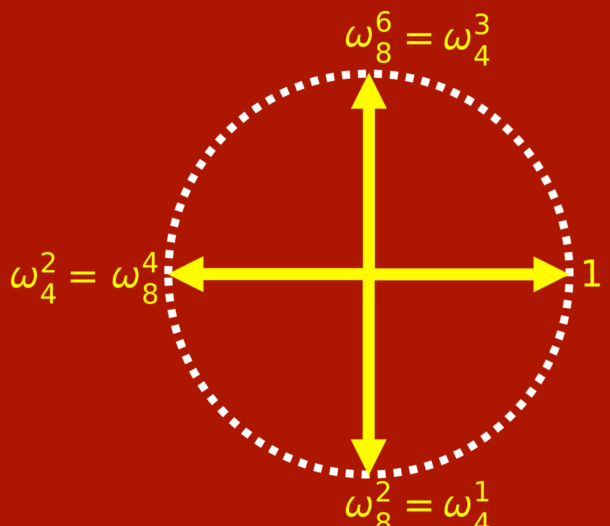
Claim: DFT_8 reduces to 2 applications of DFT_4, plus "$O(8)$" additional operations.

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix}$$

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix}$$

$$= \left(\quad \text{ditto} \quad \right)$$

$$+ a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$



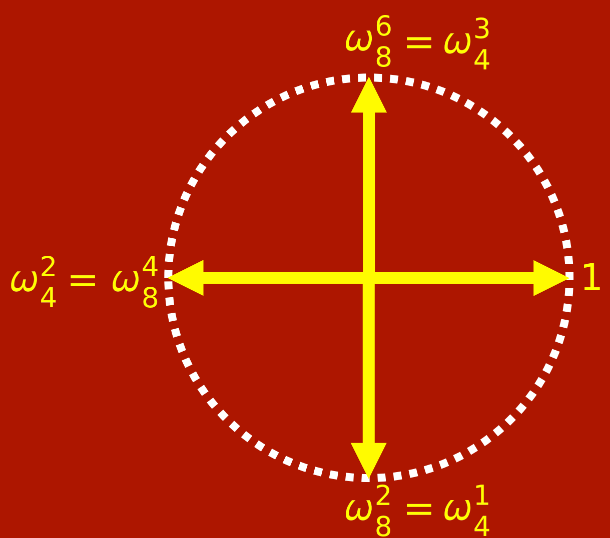
Claim: DFT_8 reduces to 2 applications of DFT_4, plus "O(8)" additional operations.

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix}$$

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega_4^1 \\ \omega_4^2 \\ \omega_4^3 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega_4^2 \\ 1 \\ \omega_4^2 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega_4^3 \\ \omega_4^2 \\ \omega_4^1 \end{bmatrix}$$

$$= \Big(\quad \text{ditto} \quad \Big)$$

$$+ a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$



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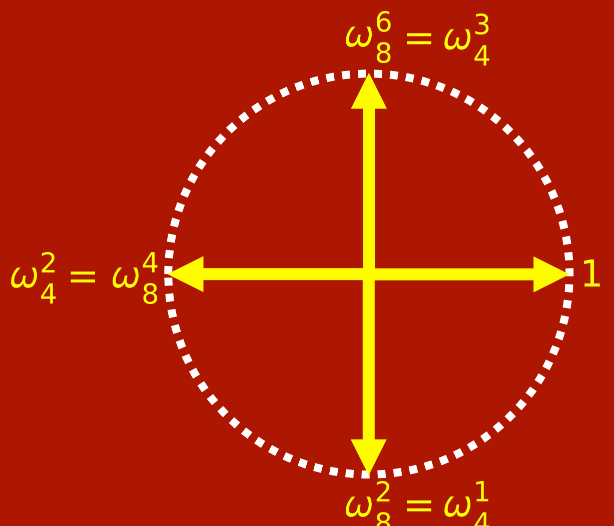
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & 1 & \omega_4^2 \\ 1 & \omega_4^3 & \omega_4^2 & \omega_4^1 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_2 \\ a_4 \\ a_6 \end{bmatrix}$$

$$= \left(\text{ditto} \right)$$

$$+ a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$



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$$= \mathrm{DFT}_4 \cdot \begin{bmatrix} a_0 \\ a_2 \\ a_4 \\ a_6 \end{bmatrix}$$

$$= \left(\quad \text{ditto} \quad \right)$$

$$+ a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$

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Computable with 1 application of DFT_4 to (a_0,a_2,a_4,a_6), and some copying.

$$+ a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$

Now to get this, apply the above to (a_1,a_3,a_5,a_7), and then multiply the j^{th} row by ω^j, for $0 \le j < 7$.

Total: 2 applications of DFT_4, plus "O(8)" more operations.

Summary

- Multiplying two n-bit integers is doable in $O(n)$ time in the Word RAM model
- It reduces to multiplying two *polynomials* of degree $< N$ in $O(N \log N)$ time.
- DFT_N reduces Coefficients Representation to Values Representation over roots of unity.
- FFT_N computes DFT_N (and inverse) in $O(N \log N)$ time.
- DFT_N has myriad uses in CS & Engineering.