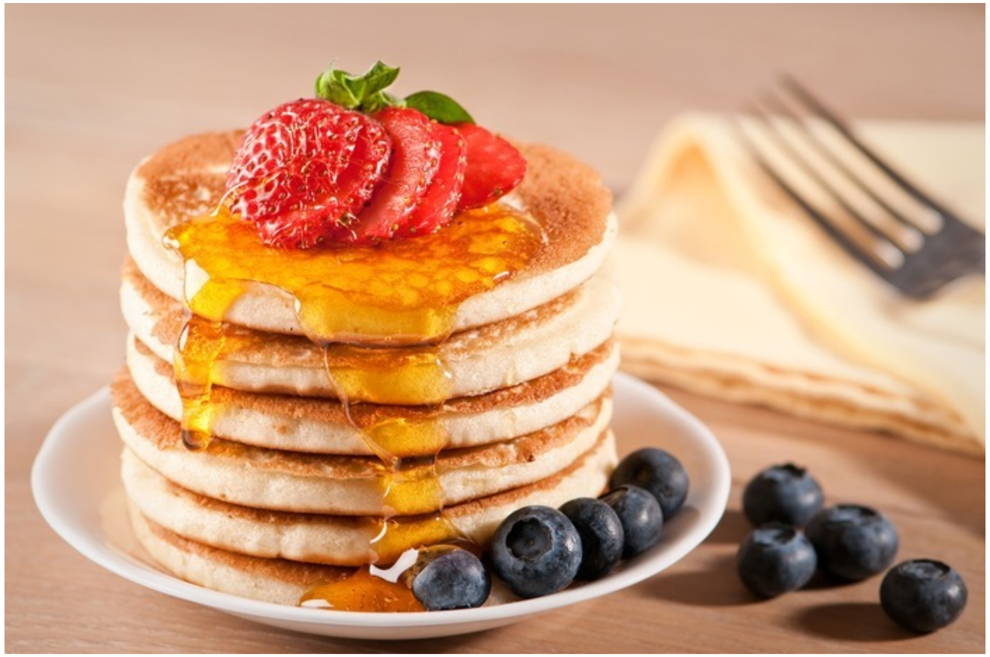
15-252
More Great Ideas in Theoretical Computer Science

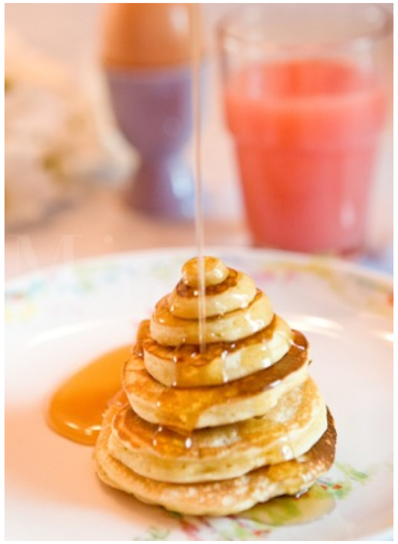
Lecture 1:
Sorting Pancakes



January 20th, 2017

Question

If there are **n** pancakes in total (all in different size), what is the max number of flips that we would ever have to use to sort them?



$P_n =$ the number described above

What is P_n ?

Understanding the question

$$P_n = \max_S \min_A \ \# \text{ flips when sorting } S \text{ by } A$$

$\min_A$: over all strategies/algorithms for sorting

$\max_S$: over all pancake stacks of size **n**

Number of flips necessary to sort the worst stack of size **n**.

Is it always possible to sort the pancakes?

Yes!

A sorting strategy (algorithm):

- Move the largest pancake to the bottom.
- Recurse on the other n-1 pancakes.

Playing around with an example

Introducing notation:

- represent a pancake with a number from 1 to n.
- represent a stack as a permutation of {1,2,…,n}
  e.g. (5 2 3 4 1)

  ↓ top ↓ bottom

Let X = min number of flips to sort (5 2 3 4 1)

What is X ?

Playing around with (5 2 3 4 1)

$$? \leq X \leq 4$$

Need an argument for a lower bound.

A strategy/algorithm for sorting gives us an upper bound.

$0 \leq X$?

$1 \leq X$?

$2 \leq X$?

$3 \leq X$?

$4 \leq X$?

Playing around with (5 2 3 4 1)

Proposition: $X = 4$

Proof: We already showed $X \leq 4$.

We now show $X \geq 4$. The proof is by contradiction.

So suppose we can sort the pancakes using 3 or less flips.

Observation: Right before a pancake is placed at the bottom of the stack, it must be at the top.

Claim: The first flip must put 5 on the bottom of the stack.
Proof: If the first flip does not put 5 on the bottom of the stack, then it puts it somewhere in the middle of the stack.
After 3 flips, 5 must be placed at the bottom.
Using the observation above, 2nd flip must send 5 to the top.
Then after 2 flips, we end up with the original stack.
But there is no way to sort the original stack in 1 flip.
The claim follows. $\square$

Playing around with (5 2 3 4 1)

Proposition: $X = 4$

Proof continued:

So we know the first flip must be: $(5\ 2\ 3\ 4\ 1) \longrightarrow (1\ 4\ 3\ 2\ 5)$.

In the remaining 2 flips, we must put 4 next to 5.

Obviously 5 cannot be touched.

So we can ignore 5 and just consider the stack $(1\ 4\ 3\ 2)$.

We need to put 4 at the bottom of this stack in 2 flips.

Again, using the observation stated above,
the next two moves must be:

$$(1\ 4\ 3\ 2) \longrightarrow (4\ 1\ 3\ 2) \longrightarrow (2\ 3\ 1\ 4)$$

This does not lead to a sorted stack,
which is a contradiction since we assumed we could sort the stack in 3 flips. $\square$

Playing around with (5 2 3 4 1)

$$X = 4$$

What does this say about P_n ?

Pick one that you think is true:

- $P_n = 4$
- $P_n \leq 4$
- $P_n \geq 4$
- $P_5 = 4$
- $P_5 \leq 4$
- $P_5 \geq 4$
- None of the above.
- Beats me.

Playing around with (5 2 3 4 1)

$$X = 4$$

What does this say about P_n ?

$$P_5 = \max_S \min_A \; \# \text{ flips when sorting } S \text{ by } A$$

↳ all stacks of size 5 (for S)

all stacks:	(5 2 3 4 1)	(5 4 3 2 1)	(1 2 3 4 5)	(5 4 1 2 3)	$\cdots$
min # flips:	4	1	0	2	

$P_5 =$ max among these numbers

$P_5 =$ min # flips to sort the "hardest" stack

So: $X = 4 \implies P_5 \geq 4$

Playing around with (5 2 3 4 1)

In fact: (will not prove)

$$5 \le P_5 \le 5$$

Find a specific "hard" stack.
Show any method
must use 5 flips.

Find a generic method
that sorts any 5-stack
with 5 flips.

Good progress so far:

- we understand the problem better
- we made some interesting observations

Ok what about P_n for general n ?

P_n for small n

$$P_0 = 0$$

$$P_1 = 0$$

$$P_2 = 1$$

$$P_3 = 3$$

lower bound:

(1 3 2) requires 3 flips.

upper bound:

- bring largest to the bottom in 2 flips
- sort the other 2 in 1 flip (if needed)

A general upper bound: "Bring-to-top" alg.

if n = 1: do nothing

else:

- bring the largest pancake to bottom in 2 flips
- recurse on the remaining n-1 pancakes

A general upper bound: "Bring-to-top" alg.

if n = 1: do nothing

else if n = 2: sort using at most 1 flip

else:

- bring the largest pancake to bottom in 2 flips
- recurse on the remaining n-1 pancakes

$T(n)$ = max # flips for this algorithm

$$T(1) = 0$$

$$T(2) \leq 1$$

$$T(n) \leq \boxed{2} + \boxed{T(n-1)} \qquad \text{for } n \geq 3$$

$$\implies T(n) \leq 2n - 3 \qquad \text{for } n \geq 2$$

A general upper bound: “Bring-to-top” alg.

Theorem: $P_n \leq 2n - 3 \quad \text{for } n \geq 2\,.$

Corollary: $P_3 \leq 3\,.$

Corollary: $P_5 \leq 7\,.$

(So this is a loose upper bound, i.e. not tight.)

A general lower bound

How about a lower bound?

You must argue against **all** possible strategies.

What is the worst initial stack?

A general lower bound

Observation:

Given an initial stack, suppose pancakes i and j are adjacent.

They will remain adjacent if we never insert the spatula in between them.

$(\boxed{5\ 2}\ 3\ 4\ 1)$

So:

If i and j are adjacent and $|i - j| > 1$,
then we **<u>must</u>** insert the spatula in between them.

Definition:

We call i and j a bad pair if

- they are adjacent
- $|i - j| > 1$

A general lower bound

Lemma (Breaking-apart argument):

A stack with b bad pairs needs at least b flips to be sorted.

e.g. $(\boxed{5\ 2}\,3\,\boxed{4\ 1})$ requires at least 2 flips.

In fact, we can conclude it requires 3 flips. Why?

Bottom pancake and plate can also form a bad pair.

A general lower bound

Theorem: $P_n \geq n$ for $n \geq 4.$

Proof:

Take cases on the parity of $n.$

If n is even, the following stack has n bad pairs:

$$(2\ 4\ 6\ \cdots\ n-2\ n\ 1\ 3\ 5\ \cdots\ n-1)$$

If n is odd, the following stack has n bad pairs:

$$(1\ 3\ 5\ \cdots\ n-2\ n\ 2\ 4\ 6\ \cdots\ n-1)$$

By the previous lemma, both need n flips to be sorted.

So $P_n \geq n$ for $n \geq 4.$ $\square$

Where did we use the assumption $n \geq 4$?

So what were we able to prove about P_n ?

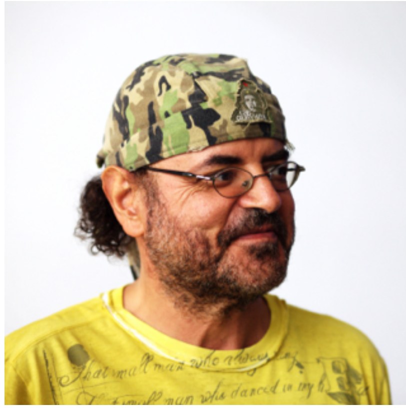
Theorem: $n \leq P_n \leq 2n - 3$ for $n \geq 4.$

Best known bounds for P_n

Jacob Goodman 1975: what we saw

published under pseudonym Harry Dweighter

William Gates and Christos Papadimitriou 1979:



$$\frac{17}{16}n \leq P_n \leq \frac{5}{3}(n+1)$$

Currently best known: $$\frac{15}{14}n \leq P_n \leq \frac{18}{11}n$$

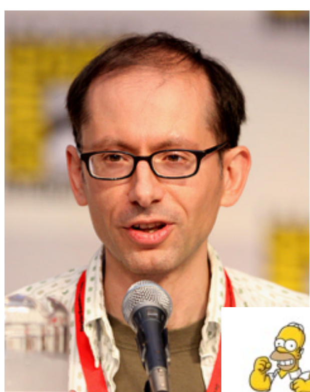
BP$_n$

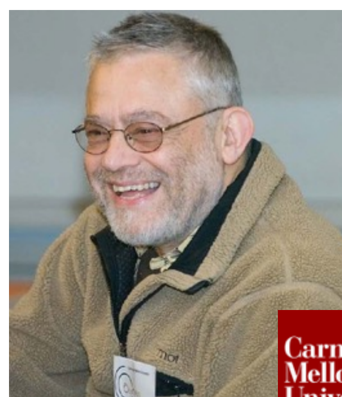
William Gates and Christos Papadimitriou 1979:

Introduced "Burnt pancakes" problem.

$$\frac{3}{2}n - 1 \leq BP_n \leq 2n + 3$$

David Cohen and Manuel Blum 1995:



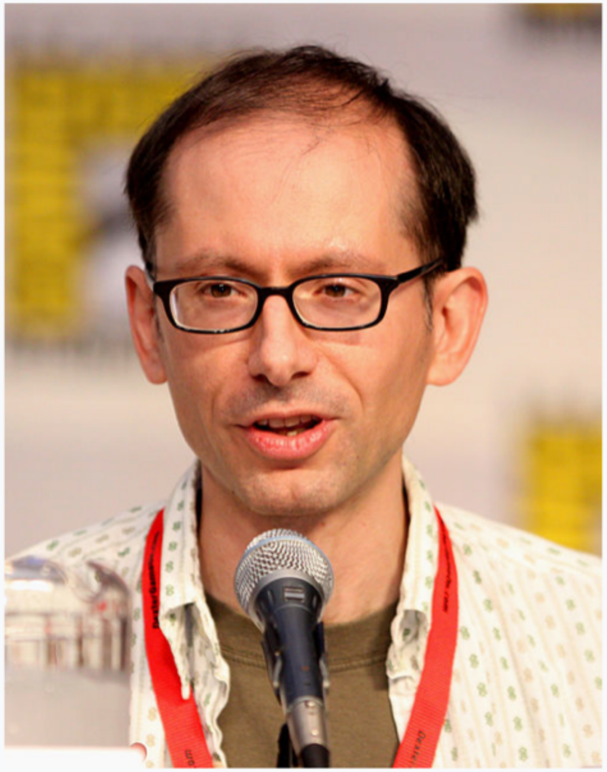


$$\frac{3}{2}n \leq BP_n \leq 2n - 2$$

David Samuel Cohen (born July 13, 1966), better known as **David X. Cohen**, is an American television writer. He has written for *The Simpsons* and served as the head writer and executive producer of *Futurama*.

Contents [hide]

David X. Cohen



Cohen at the 2010 San Diego Comic-Con International.

Born	David Samuel Cohen July 13, 1966 (age 50) New York, New York, United States
Occupation	Television writer
Period	1992–present
Genre	Comedy
Spouse	Patty Cohen
Children	1

Early life [edit]

Cohen was born in New York City. He changed his middle initial around the time *Futurama* debuted due to Writers' Guild policies prohibiting more than one member from having the same name.[1] Both of his parents were biologists, and growing up Cohen had always planned to be a scientist, though he also enjoyed writing and drawing cartoons.[2]

Cohen graduated from Dwight Morrow High School in Englewood, New Jersey, where he wrote the humor column for the high school paper and was a member of the school's state champion mathematics team.[3] From there, Cohen went on to attend Harvard University, graduating with a B.A. in physics, and the University of California, Berkeley, with a M.S. in computer science.[4] At Harvard, he wrote for and served as President of the *Harvard Lampoon*.

Cohen's most notable academic publication concerned the theoretical computer science problem of pancake sorting,[5] which was also the subject of an academic publication by Bill Gates.[6]

Best known bounds for P_n

n	P_n
4	4
5	5
6	7
7	8
8	9
9	10
10	11
11	13
12	14
13	15
14	16
15	17
16	18
17	19
18	20
19	22

$P_{20} = ?$

23 or 24

Why study pancake numbers?

Perhaps surprisingly, it has interesting applications.

- In designing efficient networks that are resilient to failures of links.

  Google: pancake network

- In biology.

  Can think of chromosomes as permutations.

  Interested in mutations in which some portion of the chromosome gets flipped.

Lessons

Simple problems may be hard to solve.

Simple problems may have far-reaching applications.

By studying pancakes, you can be a billionaire.

Analogy with computation

input: initial stack

output: sorted stack

computational problem: (input, output) pairs
pancake sorting problem

computational model: specified by the allowed operations on the input.

algorithm: a precise description of how to obtain the output from the input.

computability: is it always possible to sort the stack?

complexity: how many flips are needed?