# 1 Group Recognizer (100)

**Background**

There is a very important group called the dihedral group $D_4$ that represents the symmetries of a square (all rigid motions of the plane that move a square back to itself). For example, a clockwise rotation looks like so:
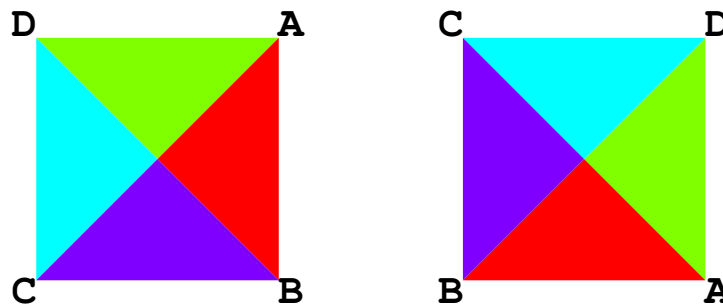


The group has 8 elements and can be written as $\{1, a, a^2, a^3, b, ba, ba^2, ba^3\}$. Here $a$ corresponds to a rotation by 90 degrees, and $b$ is the reflection along the horizontal axis.

Now consider all words over the alphabet $\Sigma = \{a, b\}$. Each word corresponds to a group element in the obvious fashion, so we have a map $\eta : \Sigma^\star \to D_4$ which turns out to be a monoid homomorphism.

**Task**

A. Explain the group elements $\{1, a, a^2, a^3, b, ba, ba^2, ba^3\}$ geometrically.

B. Argue that $a^4 = b^2 = 1$ and $ab = ba^3$.

C. Show that the language $L = \{\, x \in \Sigma^\star \mid \eta(x) = 1 \,\}$ of all words that evaluate to 1 is regular. Construct the smallest possible DFA for $L$.

D. Using your DFA, determine the cardinality of $L \cap \Sigma^n$.