15-251: Great Theoretical Ideas In Computer Science

Recitation 1

Announcements

- **Check the course calendar** for times and locations for all course events!
- **Look at the course notes** on the website for definitions and example proofs.
- Looking for a group? **Use the teammate search on Piazza**. This week there are no GROUP problems, so you still have some time to find a group!
- The **first homework assignment is out** - start early!
- Recitations and solutions will be posted online. Please try to go through everything before attempting the solo problems on the homework! Extra problems are similar in difficulty to the problems we will cover, and provide more practice problems to go through. Bonus problems are a bit more difficult, but are still worth your time!
- If you have not filled out the recitation availability form, contact the professors!

Remember 76-101?

As you prepare to do the writeups next week, keep in mind the following 'style guidelines' :

- Writing good proofs requires as much attention to the principles of English composition as to those of mathematics.
- Seriously, apply your knowledge from 76-101 when you write proofs (as much as your knowledge from 21-127/21-128/15-151).
- Write in complete sentences and pay attention to grammar. Make sure your writing is organized in coherent sections.
- Avoid run-on sentences.

Some Definitions

- Σ is your *alphabet*: non-empty and finite. Elements of Σ are called *symbols* or *characters*.
- Given an alphabet Σ, a finite *string* or *word* over Σ is a finite sequence of symbols, where each symbol is in Σ.
- Σ^* is the set of all strings over Σ of finite length, including the empty string ϵ.
- Any subset $L \subseteq \Sigma^*$ is called a *language* over Σ.
- A *computational problem* is a function $f : \Sigma^* \to \Sigma^*$.
- A *decision problem* is a function $f : \Sigma^* \to \{0, 1\}$.
- There is a one-to-one correspondence between decision problems and languages.

Note that the solutions/proofs below are bad or incorrect. Do not use them as a template.

Clearly false

Prove or disprove: for $n \in \mathbb{N}^+$, any n people all have the same hair color.

Solution: We prove the claim via induction on n.
Base case ($n = 1$): trivial.
Induction hypothesis: Suppose that for some $n \in \mathbb{N}^+$, all groups of n people have the same hair color.
Induction step: Consider a group of $n+1$ people and take the entire group except one. By the IH, these n have the same hair color. Now take another n-sized subgroup and exclude a different person. Again, these n share a hair color. Now note that the two people who were excluded both have the same hair color as the $n-1$ people who were picked twice - by transitivity, they must have the same hair color too. So all $n+1$ people have the same hair color.

Critique this solution.

Arrangement of ♣ and ♢

How many ways are there to arrange $c \geq 0$ ♣s and $d \geq 0$ ♢s so that all ♣s are consecutive?

Solution: You can have any number between 0 and d ♢s, then a string of ♣s; then the remainder of the ♢s. Hence, there are $d+1$ possibilities.

Critique this solution.

Chips in a circle

There is a circle of 15,251 chips, green on one side, red on the other. Initially, all show the green side. In one move, you may take any four consecutive chips and flip them. Is it possible to get all of the chips showing red?

Solution: No it is not possible. Let's assume for contradiction we converted all 15,251 chips to red. But this means in the very last move there must be 4 consecutive green chips and the remaining 15,247 must be red. Repeating this k times for $1 \leq k \leq 3812$, we get three consecutive red chips, with the rest green. But we started from all green, contradiction.

Critique this solution.

Balanced Parentheses

Consider the following recursively defined language L over $\{(,)\}^*$:

- $() \in L$,
- If $x \in L$, then $(x) \in L$. (WRAP)
- If $x, y \in L$, then $xy \in L$. (CONCAT)

We claim that these rules create exactly "the set of balanced strings of parentheses".
But what is that anyway?

(a) How do you reasonably define a "balanced string of parentheses"?

(b) Prove that a string x over $\{(,)\}^*$ is in L if and only if it satisfies your above definition of a "balanced string of parentheses".

Inductio Ad Absurdum (Extra Problem)

It is well known that $\ln 2$ is an irrational number that is equal to the infinite sum

$$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots$$

.

However, Leonhard claims to have a proof that shows otherwise:

He claims that $\ln 2$ is rational and will prove this by showing $\sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}$ is rational for all $n > 0$ via induction.

Base Case: $n = 1$: $\sum_{i=1}^{1} \frac{(-1)^{i+1}}{i} = 1$ is indeed rational.

Induction Hypothesis: Suppose that $\sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}$ is rational for $0 < n < k+1$ for some $k \in \mathbb{N}$.

Induction Step: It now suffices to show that $\sum_{i=1}^{k+1} \frac{(-1)^{i+1}}{i}$ is rational. We have that

$$\sum_{i=1}^{k+1} \frac{(-1)^{i+1}}{i} = \sum_{i=1}^{k} \frac{(-1)^{i+1}}{i} + \frac{(-1)^{k+2}}{k+1}$$

and by induction hypothesis, the first term is rational, and clearly the second term is also rational, and since the sum of two rationals is rational, we are done! ■

Where did Leonhard go wrong?

Induction (Extra Problem)

Prove $n^2 \geq n$ for all non-negative integers n.

Solution: We prove $F_n =$ "$n^2 \geq n$" by induction on n. The base case is $n = 0$: indeed, $0^2 \geq 0$. For the induction step, assume F_k holds for all k. We now show that F_{k+1} holds...

Critique this solution.

Sneaky structures (Bonus Problem)

Suppose that everyone in your recitation knows at least one other person in the recitation. We say that two students are *connected* if there exists a chain of students, each consecutive pair of which know each other, spanning between the two. For example, if Xavier knows Yvonne and Yvonne knows Zachary, then Xavier and Zachary are connected even if they don't know each other. Prove or disprove: every student is connected to every other student.

Solution: We prove the claim via induction on n, the number of students.
Base case ($n = 2$): Since every student knows at least one other, the two students must know each other and are therefore connected.
Induction hypothesis: Suppose for some k that this works for all groups of k students. Induction step: Consider $k+1$. We know the k-people recitation is connected. The $(k+1)$th person cannot know no one, so they are connected to at least one other person in the recitation, who, by the induction hypothesis, is connected to everyone else. Thus, the whole party is still connected.

Critique this solution.