15-251: Great Theoretical Ideas In Computer Science

Recitation 7

Match These Definitions

- A **matching** in G is a subset of G's edges which share no vertices.
  A **maximal** matching is one which isn't a subset of any other matching.
  A **maximum** matching is a matching which is at least as large as any possible matching.
  A **perfect** matching is a matching such that every vertex is contained in one of its edges.
- An **alternating path** (with respect to some matching M) is one which alternates between edges in M and edges not in M.
  An **augmenting path** is an alternating path which begins and ends with vertices not matched in M.

Counting Colors 1, 2, 3, ...

Let $G = (V, E)$ be an undirected graph. Let $k \in \mathbb{N}^+$. A *k-coloring* of V is just a map $\chi : V \to C$ where C is a set of cardinality k. (Usually the elements of C are called *colors*. If $k = 3$ then {red, green, blue} is a popular choice. If k is large, we often just call the colors $1, 2, \ldots, k$.) A k-coloring is said to be *legal* for G if every edge in E is *bichromatic*, meaning that its two endpoints have different colors. (I.e., for all $\{u, v\} \in E$ it is required that $\chi(u) \neq \chi(v)$.) Finally, we say that G is *k-colorable* if it has a legal k-coloring.

(a) Suppose G has no cycles of length greater than 251. Prove that G is 251-colorable. Hint: DFS.

(b) Give an example to show that the above is tight, i.e., find a graph G with no cycles of length greater than 251 that is not 250-colorable.

From Colors to Covers

A <u>vertex cover</u> in a graph $G = (V, E)$ is a subset $U \subseteq V$ such that every edge $e \in E$ has at least one of its endpoints in U. We say that a vertex cover is a <u>minimum vertex cover</u> in G if it has the smallest size among all vertex covers in G. Let $p(G)$ denote the size of a minimum vertex cover in G.

Recall that a maximum matching in G is a matching with the largest size among all matchings in G (the size of a matching is the number of edges in the matching). Let $m(G)$ denote the size of a maximum matching in G.

In this problem, we will prove that $p(G) = m(G)$ in bipartite graphs G.

(a) Let G be any graph (not necessarily bipartite). Prove that for any vertex cover U and any matching M in G, $|U| \geq |M|$. (Note that this implies $p(G) \geq m(G)$.)

(b) Construct a graph G where the size of every vertex cover in G is strictly larger than the size of a maximum matching in G (i.e., construct G such that $p(G) > m(G)$). Is your graph G bipartite?

(c) It turns out that in bipartite graphs, $p(G) = m(G)$. By part (a) above, to prove this, you only need to show $p(G) \leq m(G)$ in bipartite graphs. And in order to show this, you can argue that if M is a maximum matching, then one can find a vertex cover U^* such that $|U^*| \leq |M|$.

Fix your bipartite graph $G = (X, Y, E)$ and a maximum matching M in G.

Prove that if M matches every vertex in X, then there is a vertex cover of size $|M|$. So in this case, we can conclude $p(G) = m(G)$.

(d) In this part, we assume M does not match every vertex in X. Let $S \subseteq X$ be the vertices in X unmatched by M. We turn the original graph into a directed graph as follows. Direct the edges of M from Y to X and the remaining unmatched edges from X to Y. Let $D \subseteq X \cup Y$ be all the vertices reachable by a directed path starting from a vertex in S (note that $S \subseteq D$). This construction is very similar to the one in the proof of Hall's theorem.

Prove that $U^* = (X \setminus D) \cup (Y \cap D)$ is a vertex cover in G.

(e) Show that $|U^*| \leq |M|$ by arguing that every vertex of U^* is matched in M by a distinct edge of M. Conclude that $p(G) = m(G)$ in bipartite graphs.