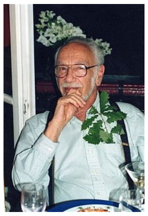
15-251
Great Ideas in Theoretical Computer Science

Lecture 11:
Stable Matchings



February 20th, 2018

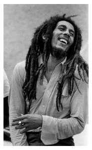
Stable matching problem

2-Sided Markets

A market with 2 distinct groups of participants
each with their own preferences.

2-Sided Markets

1. B
2. A
3. C
4. D



Company A
1. Alice
2. Bob
3. Charlie
4. David

Company B

•
•
•

Company C

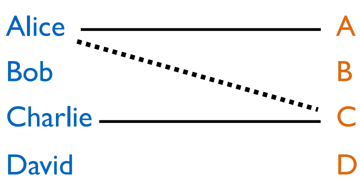
Company D
1. Bob
2. David
3. Alice
4. Charlie

Other examples:

medical residents - hospitals
students - colleges
professors - colleges
:

Aspiration: A Good Centeralized System

What can go wrong?

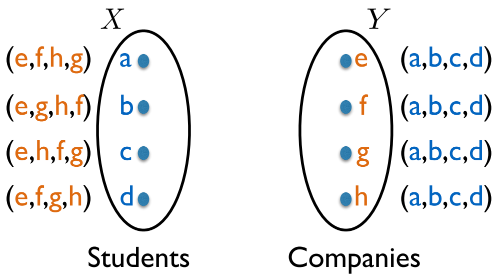


How do you solve a problem like this?

1. Formulate the problem
2. **Ask**: Is there a trivial algorithm?
   Find and analyze.
3. **Ask**: Is there a better algorithm?
   Find and analyze.
4. Maker further observations.

Formalizing the problem

An instance of the problem can be represented as a tuple
$(X,\ Y)$ + *preference list for each element.*

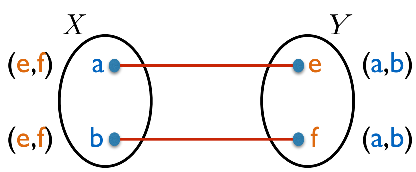


$$|X| = |Y| = n$$

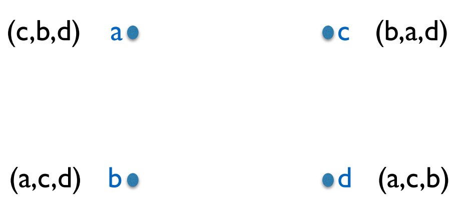
Goal:

Formalizing the problem

What is a stable matching?

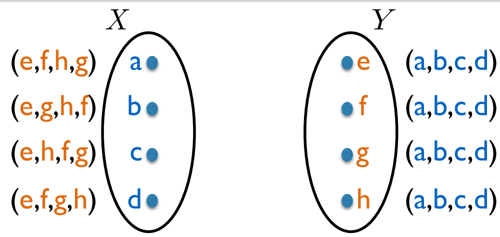


A variant: Roommate problem



Does this have a stable matching?

Stable matching: Is there a trivial algorithm?



Trivial algorithm:

The Gale-Shapley proposal algorithm

While there is a man **m** who is not matched:

- Let **w** be the highest ranked woman in **m**'s list to whom **m** has not proposed yet.
- If **w** is unmatched, or **w** prefers **m** over her current match:
  - Match **m** and **w**.
    (The previous match of **w** is now unmatched.)

Cool, but does it work correctly?

- Does it always terminate?
- Does it always find a stable matching?
  (Does a stable matching always exist?)

Gale-Shapley algorithm analysis

Theorem:
The *Gale-Shapley proposal algorithm* always terminates with a stable matching after at most n^2 iterations.

A *constructive* proof that a stable matching always exists.

3 things to show:

Gale-Shapley algorithm analysis

1. Number of iterations is at most n^2.

Gale-Shapley algorithm analysis

2. The algorithm terminates with a perfect matching.

If we don't have a perfect matching:

A man is not matched

$\Longrightarrow$ All women must be matched

$\Longrightarrow$ All men must be matched.

Contradiction

Gale-Shapley algorithm analysis

3. The matching has no unstable pairs.

"Improvement" Lemma:

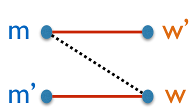
(i) A man can only go down in his preference list.

(ii) A woman can only go up in her preference list.

Unstable pair:

(m,w) unmatched

but they prefer each other.



□

Further questions

Theorem:
The *Gale-Shapley proposal algorithm* always terminates with a stable matching after at most n^2 iterations.

Does the order of how we pick men matter?
Would it lead to different matchings?

Is the algorithm "fair"?
Does this algorithm favor men or women or neither?

Further questions

m and **w** are *valid partners* if there is a stable matching in which they are matched.

best(**m**) = highest ranked valid partner of **m**

Theorem:

Proof of man optimality

Proof:

Further questions

worst(**w**) = lowest ranked valid partner of **w**

Theorem:

Proof of woman pessimality

Proof:

Real-world applications

Variants of the Gale-Shapley algorithm
is used for:

- matching medical students and hospitals
- matching students to high schools (e.g. in New York)
- matching students to universities (e.g. in Hungary)
- matching users to servers

⋮