GTI
Descriptive Complexity

A. Ada & K. Sutner

Carnegie Mellon Universality

Spring 2018

1. Descriptive Complexity

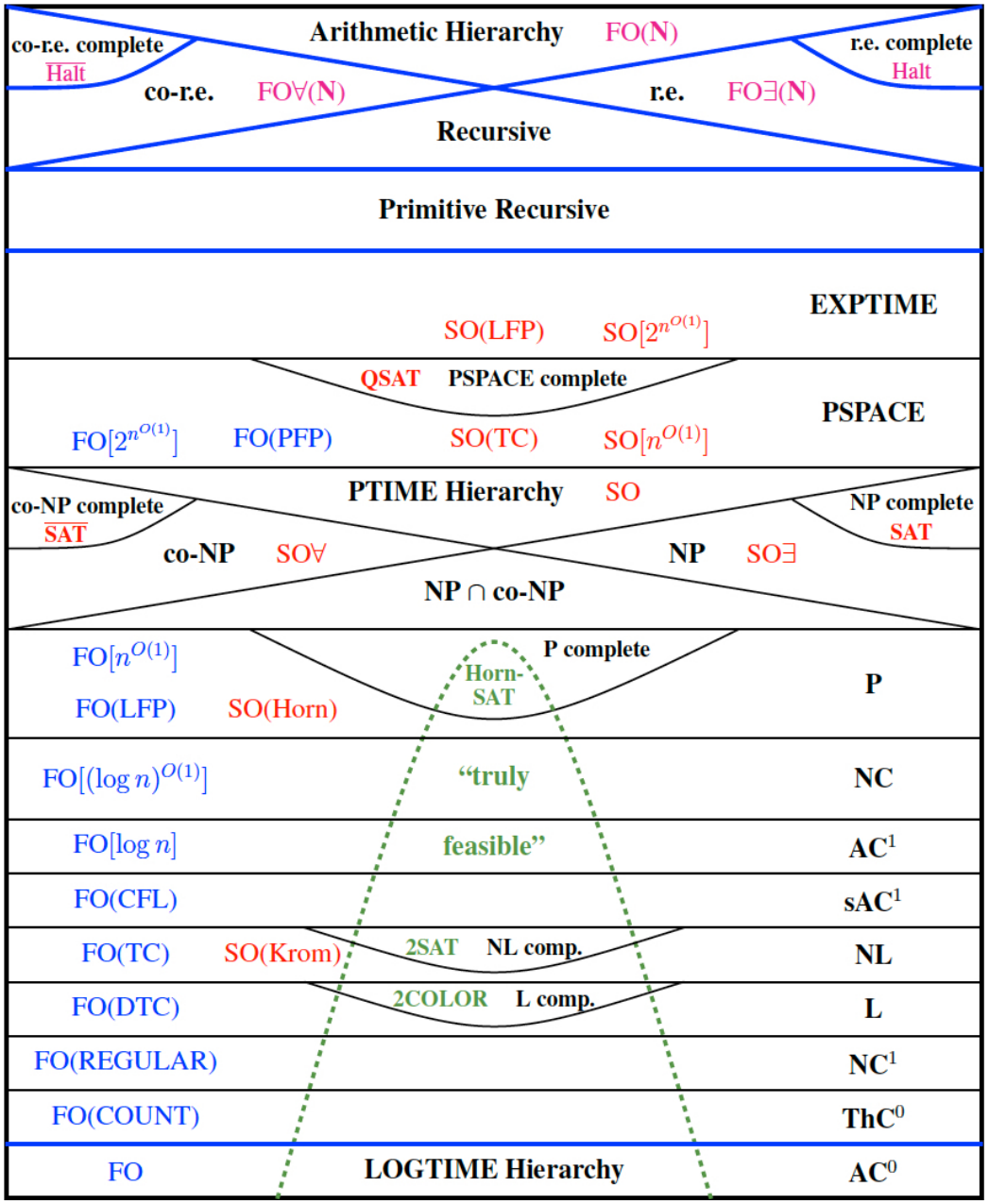
- Words as Structures

- Existential SOL

So far, we have used time complexity (sometimes in conjunction with nondeterminism and randomness) to get results on the complexity of certain computational problems.

Running time (or really: number of logical steps) is a very natural notion, but it annoyingly depends on details of the underlying computational model.

> Turing machines, register machines, random access machines, while programs, recursive functions, Herbrand-Gödel equations, λ-calculus, combinatory logic, Markov algorithms, Post systems ...


Arithmetic Hierarchy
FO(N)
co-r.e. complete
Halt
co-r.e.
FO∀(N)
r.e. complete
Halt
r.e.
FO∃(N)
Recursive
Primitive Recursive
EXPTIME
SO(LFP)
SO[2^{n^{O(1)}}]
QSAT
PSPACE complete
PSPACE
FO[2^{n^{O(1)}}]
FO(PFP)
SO(TC)
SO[n^{O(1)}]
PTIME Hierarchy
SO
co-NP complete
SAT
co-NP
SO∀
NP complete
SAT
NP
SO∃
NP ∩ co-NP
FO[n^{O(1)}]
FO(LFP)
SO(Horn)
P complete
Horn-SAT
P
FO[(log n)^{O(1)}]
"truly
NC
FO[log n]
feasible"
AC^1
FO(CFL)
sAC^1
FO(TC)
SO(Krom)
2SAT
NL comp.
NL
FO(DTC)
2COLOR
L comp.
L
FO(REGULAR)
NC^1
FO(COUNT)
ThC^0
FO
LOGTIME Hierarchy
AC^0

Truly interesting concepts always have multiple definitions and are fairly robust under minor changes.
If there are no alternative approaches, then we are probably dealing with an artifact.

For example, computability admits at least a dozen substantially different definitions.

One might wonder whether complexity classes like $\mathbb{P}$ or $\mathbb{NP}$ admit some radically different definition that does not just come down to counting steps in some model of computation. In other words: find an alternative way to define these classes that does not depend on accidents of beancounting.

One way to separate oneself from vexing definitional details of machine models is to recast everything in terms of logic.

The main idea is the following: measure the complexity of a problem by the complexity of the logic that is necessary to express it. In other words, write down a careful description of your problem in a as weak a formal system as you can manage, and declare the complexity of the problem to be the complexity of that system.

This is in stark contrast to the standard approach where everything is coded up in Peano arithmetic or Zermelo-Fraenkel set theory (typically using first-order logic): these are both sledge hammers, very convenient and powerful, but not subtle.

A logic or logical system has the following parts:

- a formal language (syntax)
- a class of structures (semantics)
- a notion of proof
- effectiveness requirements

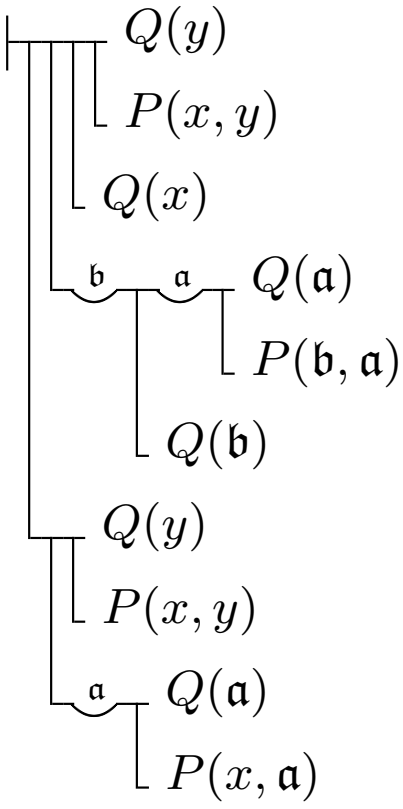
The effectiveness requirements depend a bit on the system in question, minimally we would want that it is decidable whether a string is a formula. Also, it should be decidable whether an object is a valid proof (this says nothing about proof search).

At any rate, we are here mostly interested in syntax and semantics.

- propositional logic
- equational logic
- first-order logic
- higher-order logic

These are all hugely important. Note, though, that higher-order logic tends to drift off into set-theory land: quantifying over sets and functions is a radical step that introduces a host of difficulties.

Nowadays, first-order logic is the general workhorse in math and ToC.



It may seem that syntax questions are trivial, but just take a look at Frege's system in his Begriffsschrift.

$\bot$, $\top$	constants false, true
p, q, r, ...	propositional variables
$\neg$	not
$\wedge$	and, conjunction
$\vee$	or, disjunction
$\Rightarrow$	conditional (implies)

Negation is unary, all the others a binary.

A "structure" here is just an assignment of truth values to variables, an assignment or valuation

$$\sigma : \mathrm{Var} \to \mathbf{2}$$

We have seen that an accepting computation of a polynomial time Turing machine $\mathfrak{M}$ can be translated into a question of whether a certain Boolean formula $\Phi_{\mathfrak{M}}$ has a satisfying truth assignment.

The trick is to use lot and lots of Boolean variables to code up the whole computation.

One might wonder whether a more expressive logic would produce other interesting arguments along these lines: translate a machine into an "equivalent" formula.

We'll do this for finite state machines, and then again for Turing machines.

The main problem with propositional logic is that our translation from Turing machines is quite heavy-handed; in particular it has little to do with they way a computation of a TM would be defined ordinarily.

More promising seems a system like first-order logic which serves as the standard workhorse in much of math and CS.

To wit, what is generally considered to be a "math proof" is an argument in FOL. Instead of defining the syntax and proof theory, let's just look at the corresponding structures.

Definition

A (first order) structure is a set together with a collection of functions and relations on that set. The signature of a first order structure is the list of arities of its functions and relations.

In order to interpret a formula we need something like

$$\mathcal{A} = \langle\, A; f_1, f_2, \ldots, R_1, R_2, \ldots \rangle$$

The set A is the carrier set of the structure.

We have $f_i : A^{n_i} \to A$ and $R_i \subseteq A^{m_i}$.

Note that a first order structure is not all that different from a data type. To wit, we are dealing with a

- collection of objects,
- operations on these objects, and
- relations on these objects.

In the case where the carrier set is finite (actually, finite and small) we can in fact represent the whole FO structure by a suitable data structure (for example, explicit lookup tables). For infinite carrier sets, things are a bit more complicated.

Data types (or rather, their values) are manipulated in programs, we are here interested in describing properties of structures using the machinery of FOL.

- Descriptive Complexity

2. Words as Structures

- Existential SOL

We code everything as words over some alphabet.

Wild Idea: Can we think of a single word as a structure?

And, of course, use logic to describe the properties of the word/structure?

This may seem a bit weird, but bear with me. First, we need to fix an appropriate language for our logic.

As always, we want at least propositional logic: logical not, or and, and so forth.

We will have variables x, y, z, ... that range over positions in a word, integers in the range 1 through n where n is the length of the word.

We allow the following basic predicates between variables:

$$x < y \qquad\qquad x = y$$

Of course, we can get, say, $x \geq y$ by Boolean operations.

Most importantly, we write

$$Q_a(x)$$

for "there is a letter a in position x."

We allow quantification for position variables.

$$\exists\, x\, \varphi \qquad \forall\, x\, \varphi$$

For example, the formula

$$\exists\, x, y\, (x < y \wedge Q_a(x) \wedge Q_b(y))$$

intuitively means "somewhere there is an a and somewhere, to the right of it, there is a b."

The formula

$$\forall\, x, y\, (Q_a(x) \wedge Q_b(y) \Rightarrow x < y)$$

intuitively means "all the a's come before all the b's."

We need some notion of truth

$$w \models \varphi$$

where w is a word and φ a sentence in $\mathrm{MSO}[<]$.

We won't give a formal definition, but the basic idea is simple: Let $|w| = n$:

- the variables range over $[n] = \{1, 2, \ldots, n\}$,
- $x < y$ means: position x is to the left of position y ,
- $x = y$: well . . . ,
- for the $Q_a(x)$ predicate we let

$$Q_a(x) \iff w_x = a$$

$$aaacbbb \models \forall\, x\, (Q_a(x) \vee Q_b(x) \vee Q_c(x))$$

$$aaabbb \models \exists\, x, y\, (x < y \wedge Q_a(x) \wedge Q_b(y))$$

$$bbbaaa \not\models \exists\, x, y\, (x < y \wedge Q_a(x) \wedge Q_b(y))$$

$$aaabbb \models \exists\, x, y\, (x < y \wedge \neg\exists\, z\, (x < z \wedge z < y) \wedge Q_a(x) \wedge Q_b(y))$$

$$aaacbbb \not\models \exists\, x, y\, (x < y \wedge \neg\exists\, z\, (x < z \wedge z < y) \wedge Q_a(x) \wedge Q_b(y))$$

$$aaacbbb \models \exists\, x\, (Q_c(x) \Rightarrow \forall\, y\, (x < y \Rightarrow Q_b(y)))$$

Very good, but recall that we not really interested in single words, we want languages, sets of words. No problem, for any sentence φ, we can consider the collection of all words that satisfy φ:

$$\mathcal{L}(\varphi) = \{\, w \in \Sigma^\star \mid w \models \varphi \,\}.$$

So our key idea is that the "complexity" of $\mathcal{L}(\varphi)$ is just the complexity of the formula φ.

More precisely, if we have that the right logic will produce interesting collections of languages.

Example

In first-order logic, we can hardwire factors. For example, to obtain a factor abc let

$$\varphi \equiv \exists\, x, y, z\, (y = x + 1 \wedge z = y + 1 \wedge Q_a(x) \wedge Q_b(y) \wedge Q_c(z))$$

Then $w \models \varphi$ iff $w \in \Sigma^\star abc \Sigma^\star$.

You might object to the use of "$y = x + 1$" which is not part of our language. No worries, it's just an abbreviation:

$$y = x + 1 \iff x < y \wedge \forall\, z\, (x < z \Rightarrow y \leq z)$$

This is quite typical: one defines a small language that is easy to handle, and then boosts usability by adding abbreviations.

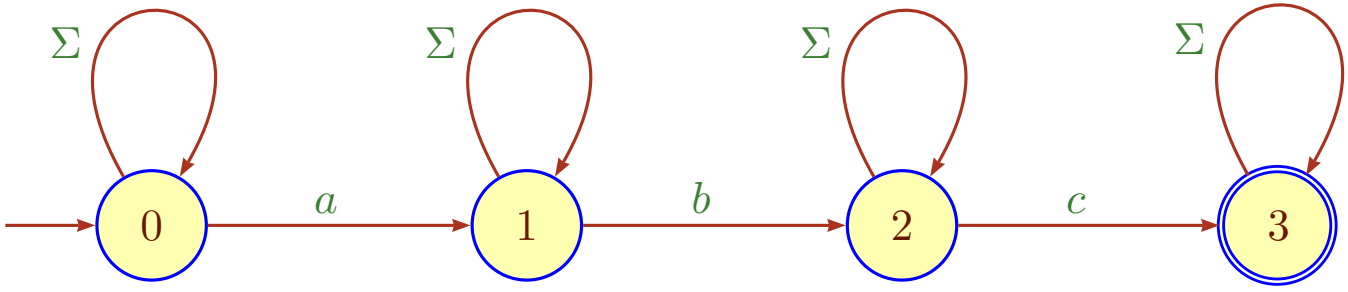
Example

Instead of factors we can similarly get (scattered) subwords by dropping the adjacency condition for the positions:

$$\varphi \equiv \exists\, x, y, z\, (x < y \wedge y < z \wedge Q_a(x) \wedge Q_b(y) \wedge Q_c(z))$$

Then $w \models \varphi$ iff $w \in \Sigma^\star a \Sigma^\star b \Sigma^\star c \Sigma^\star$.

You might feel that this is a complicated formula for a simple concepts, but note that the analogous formula φ_u for a subword u has length $|u|$ and is trivial to construct.



The natural (nondeterministic) automaton is quite similar to the formula.

Example

We can split a word into two parts as in

$$\varphi \equiv \exists\, x \,\forall\, y \,((y \leq x \Rightarrow Q_a(y)) \wedge (y > x \Rightarrow Q_b(y))) \vee \forall\, x \,(Q_b(x))$$

Then $w \models \varphi$ iff $w \in a^\star b^\star$.

Example

Let $\mathsf{first}(x)$ be shorthand for $\forall\, z \,(x \leq z)$, and $\mathsf{last}(x)$ shorthand for $\forall\, z \,(x \geq z)$. Then

$$\varphi \equiv \exists\, x, y \,(\mathsf{first}(x) \wedge Q_a(x) \wedge \mathsf{last}(y) \wedge Q_b(y))$$

Then $w \models \varphi$ iff $w \in a\Sigma^\star b$.

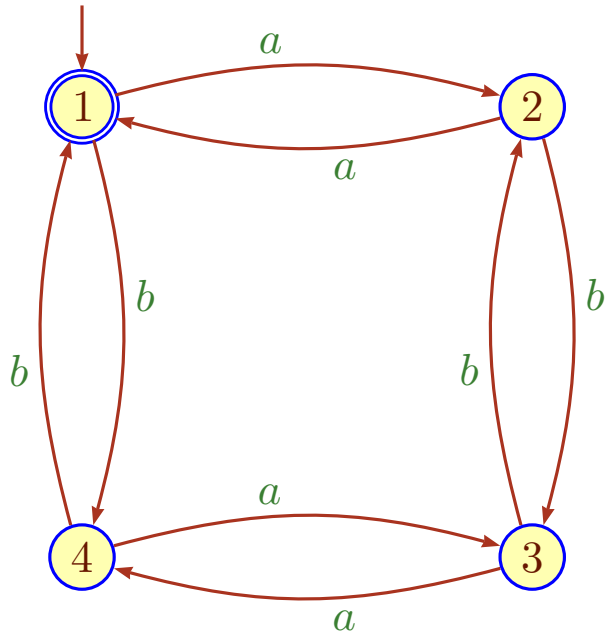
One cannot fail to notice that all the languages $\mathcal{L}(\varphi)$ we have seen so far are in fact regular.

If you are the kind of person that jumps to conclusion you might suspect that we get exactly the regular language from our little logic.

Alas, consider the language

$$L_{e,e} = \{\, x \in \{a,b\}^\star \mid \#_a x, \#_b x \text{ even} \,\}$$

This language has a trivial 4-state DFA, but building a corresponding formula φ seems impossible. Try.



$\varphi =??????$

This and other examples lead one to suspect that first-order logic is a bit too weak to produce all regular languages.

In logic, if FOL does not work, one turns to second order logic. In our case, it turns out we need only a weak subsystem where second-order quantification is restricted to just two kinds:

- individuals, and
- sets of individuals (relations of type R^1).

Notation:

$$\exists X \qquad \forall X$$

$$x \in X \qquad X(x)$$

Let's ignore words for a moment, and just try to get an idea what kinds of concepts one can express in MSO.

Assuming a total order $\leq$, we can express the assertion that every bounded set has a least upper bound:

$$\forall X \left(\exists z\, X(z) \wedge \exists x \forall z\, (X(z) \Rightarrow z \leq x) \Rightarrow \right.$$
$$\left. \exists x\, (\forall z\, (X(z) \Rightarrow z \leq x) \wedge \forall y\, (\forall z\, (X(z) \Rightarrow z \leq y) \Rightarrow x \leq y))\right)$$

This is the critical property of the standard order on the reals, and cannot be expressed in FOL.

Again assume a total order $\leq$. We can express the assertion that we have a well-order in terms of the least-element principle: every non-empty set has a least element.

$$\forall\, X \left(\exists\, z\, X(z) \Rightarrow \right.$$

$$\left. \exists\, x \left(X(x) \wedge \forall\, z \left(X(z) \Rightarrow x \leq z\right)\right)\right)$$

This is the critical property of the natural numbers with the standard order, and cannot be expressed in FOL.

Lastly, consider a digraph, a single binary edge relation E.

We can express the assertion that there is a path from s to t as follows:

$$\forall X \left(X(s) \wedge \forall x, y \left(X(x) \wedge E(x, y) \Rightarrow X(y) \right) \Rightarrow X(t) \right)$$

Again, FOL is not strong enough to express path existence in general (and thus other concepts like connectivity).

We allow second-order variables X, Y, Z, ... that range over sets of positions in a word.

$$\exists X\, \varphi \qquad \forall X\, \varphi \qquad X(x)$$

Sets of positions are all there is; we do not have variables in our language for, say, binary relations on positions (we do not use full SOL).

This system is called monadic second-order logic (with less-than), written $\mathrm{MSO}[<]$.

In applications, the atomic relation $x < y$ is slightly more useful than $y = x + 1$, but either one would have the same expressiveness.

We have already seen

$$y = x + 1 \iff x < y \wedge \forall\, z\, (x < z \Rightarrow y \leq z)$$

On the other hand write $\mathsf{closed}(X)$ for the formula $\forall\, z\, (X(z) \Rightarrow X(z+1))$. Then

$$x < y \iff x \neq y \wedge \forall\, X\, (X(x) \wedge \mathsf{closed}(X) \Rightarrow X(y))$$

This is sometimes written as $\mathrm{MSO}[<] = \mathrm{MSO}[+1]$.

Suppose we want at least three a's.

$$\exists\, x, y, z\ (x < y < z \wedge Q_a(x) \wedge Q_a(y) \wedge Q_a(z))$$

And at most three a's.

$$\exists\, x, y, z\, \forall\, u\ (Q_a(u) \Rightarrow u = x \vee u = y \vee u = z)$$

Exactly three is now obtained by conjunction, much easier than a product operation on finite state machines.

Example

Write $\mathsf{even}(X)$ to mean that X has even cardinality and consider

$$\varphi \equiv \exists\, X \left(\forall\, x\, (Q_a(x) \iff X(x)) \wedge \mathsf{even}(X)\right)$$

Then $w \models \varphi$ iff the number of a's in w is even.

We're cheating, of course; we need to show that the predicate $\mathsf{even}(X)$ is definable in our setting. This is tedious but not really hard:

$$\mathsf{even}(X) \iff \exists\, Y, Z\, (X = Y \cup Z \wedge \emptyset = Y \cap Z \wedge \mathsf{alt}(Y, Z))$$

Here $\mathsf{alt}(Y, Z)$ is supposed to express that the elements of Y and Z strictly alternate as in

$$y_1 < z_1 < y_2 < z_2 < \ldots < y_k < z_k$$

$$
\begin{aligned}
X = Y \cup Z &\iff \forall\, u\, (X(u) \Leftrightarrow Y(u) \vee Z(u)) \\
\emptyset = Y \cap Z &\iff \neg\exists\, u\, (Y(u) \wedge Z(u)) \\
\mathsf{alt}(Y, Z) &\iff \exists\, y \in Y\, \forall\, x < y\, (\neg Z(x)) \wedge \\
&\qquad \exists\, z \in Z\, \forall\, x > z\, (\neg Y(x)) \wedge \\
&\qquad \forall\, y \in Y\, \exists\, z \in Z\, (y < z \wedge \forall\, x\, (y < x < z \Rightarrow \neg Y(x) \wedge \neg Z(x))) \\
&\qquad \forall\, z \in Z\, \exists\, y \in Y\, (y < z \wedge \forall\, x\, (y < x < z \Rightarrow \neg Y(x) \wedge \neg Z(x)))
\end{aligned}
$$

Exercise

The alt *formula above does not handle the case where Y and Z are empty; fix this.*

Show that one can check if the number of a's is a multiple of k, for any fixed k.

Definition

A language L is $\mathrm{MSO}[<]$ definable (or simply $\mathrm{MSO}[<]$) if there is some sentence φ such that

$$L = \mathcal{L}(\varphi) = \{\, w \in \Sigma^\star \mid w \models \varphi \,\}.$$

Our examples suggest the following theorem that connects complexity with definability:

Theorem (Buechi 1960, Elgot 1961)

A language is regular if, and only if, it is $\mathrm{MSO}[<]$ *definable.*

Obviously, the proof comes in two parts:

- For every regular language L we need to construct a sentence φ such that $L = \mathcal{L}(\varphi)$.
- For every sentence φ we have to show that the language $\mathcal{L}(\varphi)$ is regular.

We should expect part (1) to be harder since there is no good inductive structure to exploit.

Part (2) is by straightforward induction on φ, but there is the usual technical twist: we need to deal not just with sentences but also with free variables. Since we don't have a formal semantics we will not give details of this construction.

We may safely assume that the regular language L is given by a DFA $M = \langle\, Q, \Sigma, \delta; q_0, F \,\rangle$.

For simplicity assume $Q = [n]$ and $q_0 = 1$.

We have to construct a formula φ such that $w \models \varphi$ iff M accepts w.

Consider a trace of M on input w

$$q_0\, w_1\, q_1\, w_2\, q_2 \ldots q_{m-1}\, w_m\, q_m.$$

Here m can be arbitrarily large.

We can think of states as being associated with the letters of the word as in

$$\begin{array}{cccccc} & w_1 & w_2 & w_3 & \ldots & w_m \\ q_0 & q_1 & q_2 & q_3 & \ldots & q_m \end{array}$$

Thus, position $x = 1, \ldots, m$ in the word is associated with state $\delta(q_0, w_1 \ldots w_x)$.

In order to express this in a $\mathrm{MSO}[<]$ formula, we partition the set of positions $[m]$ into $n = |Q|$ blocks X_1, X_2, ..., X_n such that

$$X_p(x) \iff \delta(q_0, w_1 \dots w_x) = p$$

Some of these blocks may be empty, but note that the number of blocks is always exactly n (which we can express as a formula).

But given state p in position x we can determine the state in position $x + 1$ given w_{x+1} by a table lookup – which table lookup can be hardwired in a formula.

Technically, this is done by a formula

$$\Phi_{p,a} \equiv \forall\, x \left(X_p(x) \wedge Q_a(x+1) \Rightarrow X_{\delta(p,a)}(x+1) \right)$$

meaning "if at position x we are in state p and the next letter is an a, then the state in position $x+1$ is $\delta(p,a)$.

Note that this is not quite right, we really need a non-existing position 0 corresponding to state q_0.

Exercise

Figure out how to fix this little glitch. Also figure out how to express "the last state is final."

Now consider the big conjunction of $\Phi_{p,a}$ where $p \in Q$ and $a \in \Sigma$. Add formulae that pin down the first and last state to arrive at a formula of the form

$$\varphi \equiv \exists\, X_1, \ldots, X_n\, \Psi$$

where Ψ is first-order as indicated above. □

Note that in conjunction with the opposite direction of Büchi's theorem, this result has the surprising consequence that every $\mathrm{MSO}[<]$ formula is equivalent to a $\mathrm{MSO}[<]$ formula containing only one block of existential second-order quantifiers.

Exercise

Fill in all the details in the last proof.

Inquisitive minds will want to know what happened to plain first-order logic? It must correspond to some subset of regular, but is there any meaningful characterization of the languages definable by FO formulae?

A language $L \subseteq \Sigma^\star$ is star-free iff it can be generated from $\emptyset$ and the singletons $\{a\}$, $a \in \Sigma$, using only operations union, concatenation and complement (but not Kleene star).

Note well: $a^\star b^\star a^\star$ is star-free.

Theorem

A language $L \subseteq \Sigma^\star$ is $\mathrm{FOL}[<]$ definable if, and only if, L is star-free.

- Descriptive Complexity

- Words as Structures

3. Existential SOL

Regular and star-free are nice, but nowhere near where we want to be. How do we get an alternative description of a complexity class like $\mathbb{NP}$?

We need a stronger logic to get up there. Our goal is to establish the following result.

Theorem (Fagin 1974)

The complexity class $\mathbb{NP}$ corresponds to existential second-order logic.

We will write existential SOL as ∃SO.

∃SO means we are considering formulae of the kind

$$\exists\, X_1, X_2, \ldots, X_k\, \varphi$$

where φ is first-order: there are no second-order quantifiers other than the existential ones up front.

But now the X_i need not be monadic, in particular we will be allowed to quantify over k-ary relations: $\exists\, X^k \ldots$ for any $k \geq 1$.

So far, we have focused on word structures, but it is not hard to generalize to other combinatorial objects such as ugraphs: we need a binary predicate E for edges.

3-Colorability of a ugraph is easily expressed as a $\exists\mathrm{SO}$ formula:

$$\exists\, X, Y, Z \left(\forall\, u\, (X(u) \vee Y(u) \vee Z(u)) \wedge \forall\, u, v\, (E(u,v) \Rightarrow \right.$$
$$\left. \neg(X(u) \wedge X(v)) \wedge \neg(Y(u) \wedge Y(v)) \wedge \neg(Z(u) \wedge Z(v)))\right)$$

Note that this is just the ordinary definition of 3-colorability, written in a formal notation. There is nothing mysterious going on.

Similar descriptions exist for all our $\mathbb{NP}$ problems.

Suppose we have a formula

$$\psi = \exists\, X_1, X_2, \ldots, X_k\, \varphi$$

where φ is first-order.

To check whether φ holds we can guess the subsets X_i of the carrier set A, and then verify in polynomial time that φ holds. More precisely, if X_i has arity k we need to guess a subset of A^k, so we are still within polynomial time. The first-order quantifiers in φ are not a problem either: each just corresponds to a loop over some finite set.

So testing a $\exists\mathrm{SO}$ formula for validity over some structure is in $\mathbb{NP}$.

Suppose $\mathfrak{M}$ is some verifier. We want to express the computation of $\mathfrak{M}$ on some input $\boldsymbol{x}$, given some witness $\boldsymbol{w}$.

For simplicity assume that the running time of $\mathfrak{M}$ on an input of size n is $N = n^k - 1$: this allows us to think of both time and space as being written in k-digit base n numbers.

We can write a configuration in a computation as a word in

$$\Gamma^\star \left(\Gamma \times Q\right) \Gamma^\star$$

of length N where Γ is the tape alphabet of $\mathfrak{M}$ and Q the state set. So the whole computation C of $\mathfrak{M}$ is a $N \times N$ table of letters in Γ, augmented in one place per row by a state.

In an accepting computation C, the first and the last row look like

$$q_0 \; \sqcup \; x_1 \ldots x_n \; \sqcup \; w_1 \ldots w_m \; \sqcup \ldots \sqcup$$

$$q_Y \; \sqcup \; \sqcup \ldots \sqcup$$

Here $\boldsymbol{x}$ is the input, and $\boldsymbol{w}$ the corresponding witness.

Since we are dealing with existential sentences, the witness part comes for free: given input x, we can always write something like

$$\exists\, W\, \Phi(\boldsymbol{x}, W, \ldots)$$

We'll just pretend $\boldsymbol{w}$ is part of the input.

$$
\begin{array}{ccccccccccccccc}
\overset{q_0}{\sqcup} & a & b & c & \sqcup & 0 & 1 & 0 & 1 & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup \\
\sqcup & \overset{p}{a} & b & c & \sqcup & 0 & 1 & 0 & 1 & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup \\
\sqcup & a & \overset{p}{b} & c & \sqcup & 0 & 1 & 0 & 1 & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup \\
\sqcup & a & b & \overset{p}{c} & \sqcup & 0 & 1 & 0 & 1 & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup \\
\sqcup & a & b & c & \overset{p}{\sqcup} & 0 & 1 & 0 & 1 & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup \\
\sqcup & a & b & c & \sqcup & \overset{q}{0} & 1 & 0 & 1 & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup \\
\sqcup & a & b & c & \overset{q_0}{\sqcup} & 0 & 1 & 0 & 1 & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup
\end{array}
$$

A typical initial segment of a computation C of $\mathfrak{M}$.

Recall that time will be expressed as a k-tuple $\boldsymbol{t} = t_0, t_1, \ldots, t_{n-1}$ of elements in the carrier set $\{0, 1, \ldots, n-1\}$; ditto for space.

Let $\gamma = |\Gamma \times Q \cup Q|$.

We use $2k$-ary predicates X_g, $1 \leq g \leq \gamma$, with the intent that

$$X_g(\boldsymbol{s}, \boldsymbol{t}) \iff C(\boldsymbol{s}, \boldsymbol{t}) = g$$

We have for example

$$\forall\, \boldsymbol{s}, \boldsymbol{t}\, \exists\, g\, X_g(\boldsymbol{s}, \boldsymbol{t}) \wedge \forall\, \boldsymbol{s}, \boldsymbol{t}, g, g'\, \left(X_g(\boldsymbol{s}, \boldsymbol{t}) \wedge X_{g'}(\boldsymbol{s}, \boldsymbol{t}) \Rightarrow g = g'\right)$$

We need to make sure that the entries in the table change only according to the rules of the Turing machine: for the most part, row k is copied to row $k+1$, but close to the position of the $\Gamma \times Q$ symbol there may be changes.

In essence, we need express the transition function of $\mathfrak{M}$ as a formula using assertions such as

$$X_g(\boldsymbol{s},\boldsymbol{t}) \wedge X_{g'}(\boldsymbol{s}+1,\boldsymbol{t}) \Rightarrow X_h(\boldsymbol{s},\boldsymbol{t}+1) \wedge X_{h'}(\boldsymbol{s}+1,\boldsymbol{t}+1)$$

In English, this is something like

> If at time t and in position s there is a symbol a and the head is positioned at s and the state is p, then, at time $t+1$, in position s there is symbol a' and the head has moved one to the right, and is looking at the same symbol that was in position $s+1$ at time t.

In the end, the formula will look somewhat like

$$\exists\, W, X_1, \ldots, X_\gamma \,\exists\, \boldsymbol{u} \,\forall\, \boldsymbol{v} \;\ldots \Phi(W, X_1, \ldots, X_\gamma, \boldsymbol{u}, \boldsymbol{v}, \ldots)$$

and this formula will be valid iff the verifier $\mathfrak{M}$ accepts $\boldsymbol{x}$ together with some suitable witness $\boldsymbol{w}$.

The formula is messy, but it is easy to construct given $\mathfrak{M}$ and $\boldsymbol{x}$.

Hence we can translate any problem in $\mathbb{NP}$ into a corresponding $\exists\mathrm{SO}$ formula.

The Büchi/Elgot theorem establishes a connection between regular languages (aka constant space) and $\mathrm{MSO}[<1]$.

Fagin has shown that $\mathbb{NP}$ corresponds to existential SOL.

And one can push further:

We defined $\mathbb{NP}$ as the class obtained by verifiers using a single existential witness:

$$x \in L \iff \exists\, u\ \mathfrak{M}(x,u)\downarrow$$

We could allow more quantifiers as in

$$x \in L \iff \exists\, u \forall\, v \exists\, w\ \ldots \mathfrak{M}(x,u,v,w,\ldots)\downarrow$$

getting a classification $\mathbb{P}$, $\Sigma_1^p = \mathbb{NP}$, $\Pi_1^p =$ co-$\mathbb{NP}$, Σ_2^p, Π_2^p, $\ldots$

The collection of all these problems is known as the polynomial hierarchy and seems to contain problem much harder than just $\mathbb{NP}$.

For example, the question of whether a circuit has an equivalent circuit using at most k gates is Σ_2^p.

Alas, we do not know whether PH is a proper hierarchy.

But this much we do know:

- PH corresponds to SOL.
- PSPACE corresponds to SOL plus a transitive closure operator.