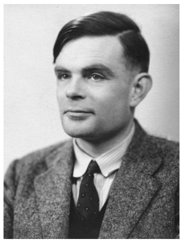
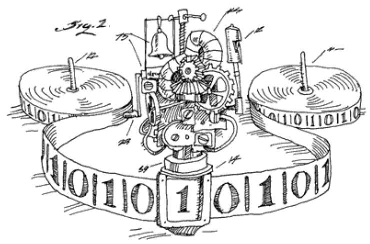
15-251
Great Ideas in
Theoretical Computer Science

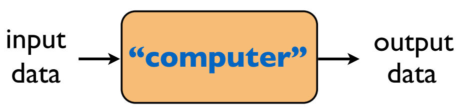
Lecture 5:
Turing's Legacy: Turing Machines





January 30th, 2018

This Week



What is computation?

What is an algorithm?

How can we mathematically define them?

Goal of this lecture:

Define Turing machines.

Understand how they work.

Goal of next lecture:

Explore physical, philosophical, historical questions surrounding Turing machines.

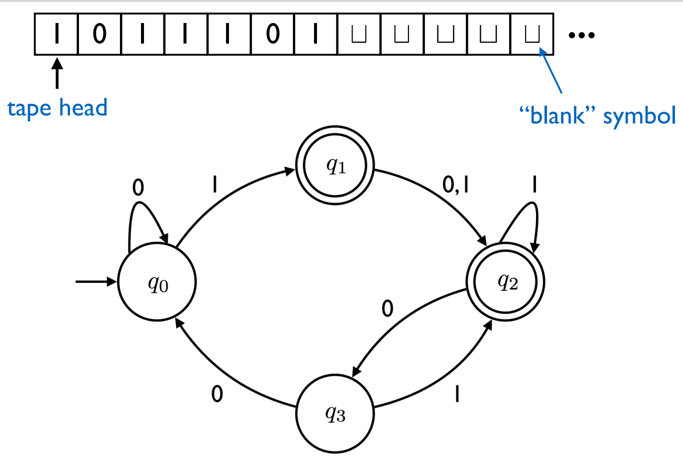
Let's assume two things about our world

1. No "universal" machines exist.

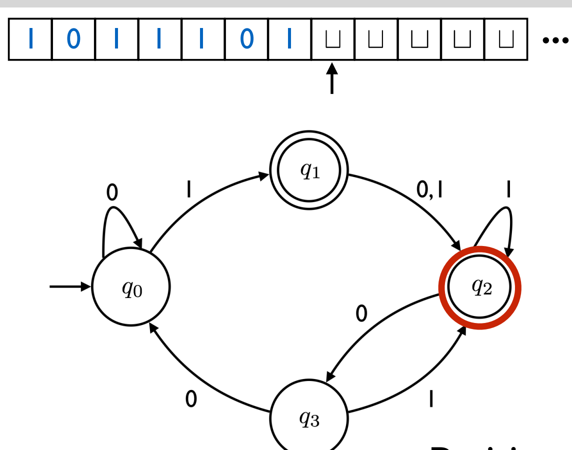
+ | **isPrime** | **Sorting** | **DFA |x| even?**

2. We only have machines to solve decision problems.

DFA: state diagram + input tape



DFA: state diagram + input tape

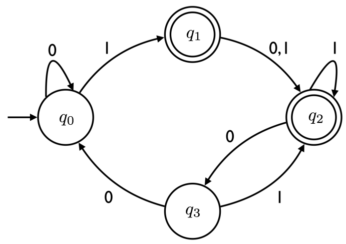


Decision: Accept

DFA as a programming language

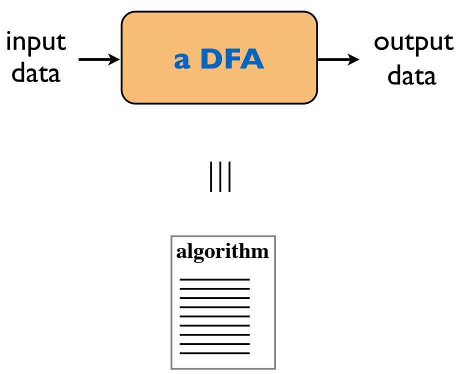
```
def foo(input):
  i = 0;
  STATE 0:
    if (i == input.length): return False;
    letter = input[i];
    i++;
    switch(letter):
      case '0':  go to STATE 0;
      case '1':  go to STATE 1;

  STATE 1:
    if (i == input.length): return True;
    letter = input[i];
    i++;
    switch(letter):
      case '0':  go to STATE 2;
      case '1':  go to STATE 2;
  ...
```

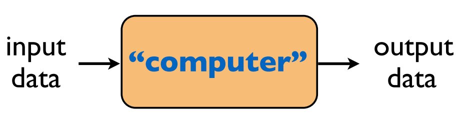
input = | 0 | 1 | 1 | 1 | 1 |



---

machine ≈ algorithm describing it



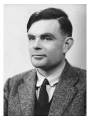
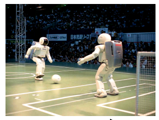


What is computation?

What is an algorithm?

How can we mathematically define them?

The properties we want from the definition:

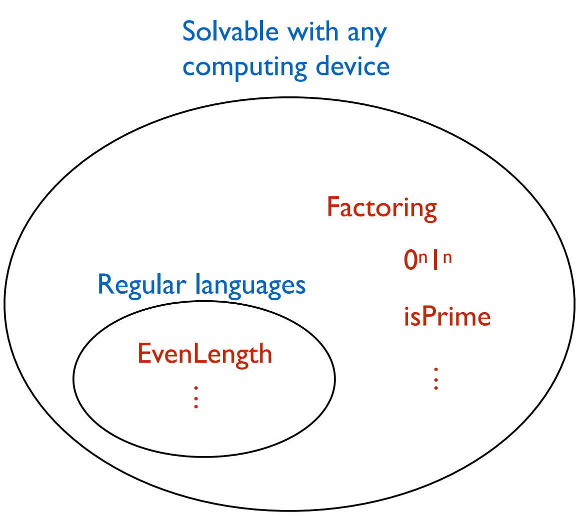
1900 1936 2015

Goal is to reach the definition of a Turing machine.

---

input data → **"computer"** → output data

2 important observations:



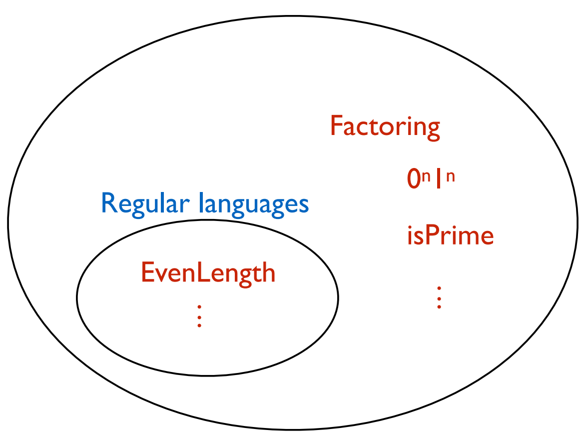
Solving 0^n1^n in Python

```
def foo(input):
   i = 0
   j = len(input) - 1
   while(j >= i):
      if(input[i] != ‘0’ or input[j] != ‘1’):
         return False
      i = i + 1
      j = j - 1
   return True
```

Solving 0^n1^n in C

```
int foo(char input[])
{
   int i = 0, j;
   while(input[j] != NULL)  /* NULL is end-of-string character */
      j++;
   j—;
   while(j >= i)
   {
      if(input[i] != ‘0’ || input[j] != ‘1’)
         return 0;  /* Reject */
      i++;
      j—;
   }
   return 1;  /* Accept */
}
```

Solvable with **Python**???



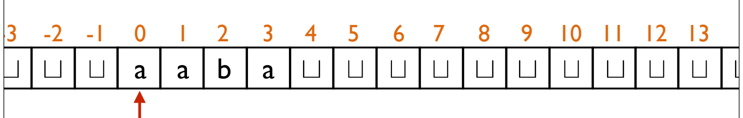
Should we define **computable** to mean
what is computable by a Python function/program?

Downsides as a formal definition?

<u>So what we want is:</u>

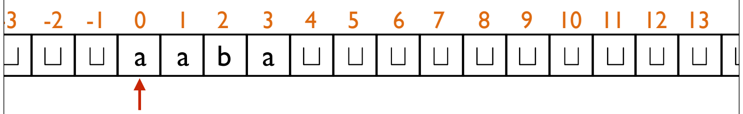
Turing machine description

TM ≈ DFA + infinite tape



Turing machine description

TM ≈ DFA + infinite tape



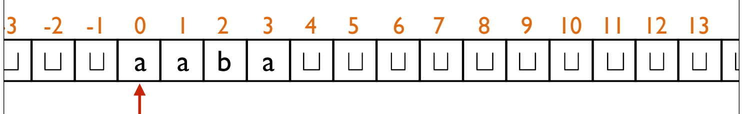
TM could have been defined as a sequence of instructions, where the allowed instructions are:

> Move the head left
> Move the head right
> Write a symbol a (from the alphabet)
> If head is reading symbol a, GOTO step j
> Halt and accept
> Halt and reject

But, we want to keep the definition as simple as possible.

Turing machine description

TM ≈ DFA + infinite tape



So a TM is a sequence of steps (states), each looking like:

STATE 0:
switch(letter under the head):
case ‘a’: **write** ‘b’; **move** Left; **go to** STATE 2;
case ‘b’: **write** ‘⊔’; **move** Right; **go to** STATE 0;
case ‘⊔’: **write** ‘b’; **move** Left; **go to** STATE 1;

Turing machine description

STATE 0:
switch(letter under the head):
case ‘a’: **write** ‘b’; **move** Left; **go to** STATE 2;
case ‘b’: **write** ‘⊔’; **move** Right; **go to** STATE 0;
case ‘⊔’: **write** ‘b’; **move** Left; **go to** STATE 1;

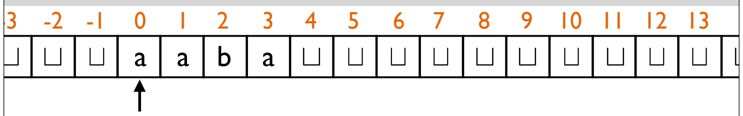
At each step, you have to:

Don’t want to change the symbol:

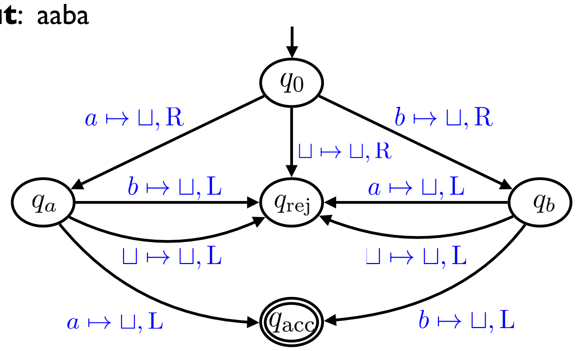
Want to stay put:

Don’t want to change state:

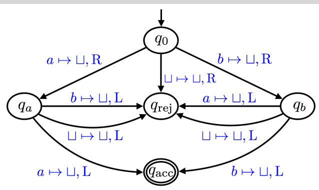
Turing machine official picture



Input: aaba

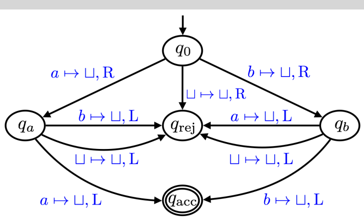


TM as a programming language




```
def M(input):
  i = 0
  STATE 0:
    letter = input[i];
    switch(letter):
      case 'a':  input[i] = ' '; i++; go to STATE a;
      case 'b':  input[i] = ' '; i++; go to STATE b;
      case ' ':  input[i] = ' '; i++; go to STATE rej;
  STATE a:
    letter = input[i];
    switch(letter):
      case 'a':  input[i] = ' '; i--;  go to STATE acc;
      case 'b':  input[i] = ' '; i--;  go to STATE rej;
      case ' ':  input[i] = ' '; i--;  go to STATE rej;

  ⋮
```

Poll



The machine accepts a string x if and only if:

Exercise

Let $\Sigma = \{a, b\}$.

Draw the state diagram of a TM that accepts a string iff it starts and ends with an a.

Formal definition: Turing machine

A Turing machine (TM) M is a 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{rej})$$

where

- Q
- Σ
- Γ
- δ
- $q_0 \in Q$
- $q_{\text{acc}} \in Q$
- $q_{\text{rej}} \in Q$, $q_{\text{rej}} \neq q_{\text{acc}}$

Formal definition: TM accepting a string

A bit more involved to define rigorously.

Not too much though.

See course notes.

DFAs vs TMs

Definition: decidable/computable languages

Let M be a Turing machine.

We let $L(M)$ denote the set of strings that M accepts.

So, $L(M) = \{x \in \Sigma^* : M(x) \text{ accepts.}\}$

What is the analog of regular languages in this setting?

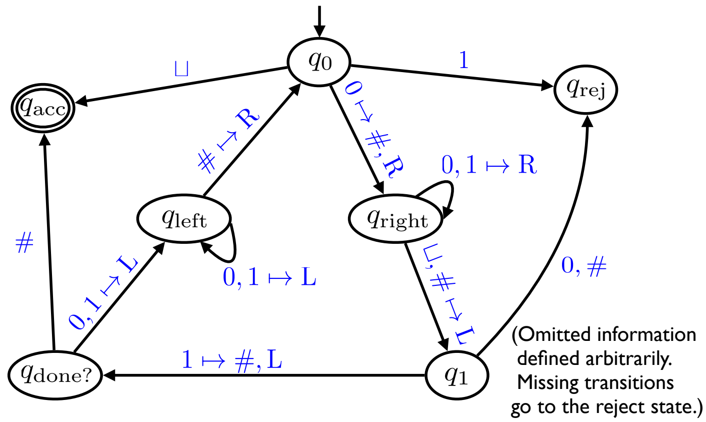
Definition:

Definition:

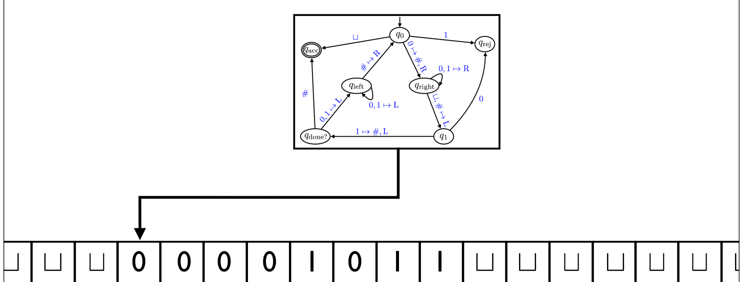
regular languages $\stackrel{?}{=}$ decidable languages

Turing machine that decides 0^n1^n

$\Sigma = \{0, 1\}$ $\Gamma = \{0, 1, \#, \sqcup\}$

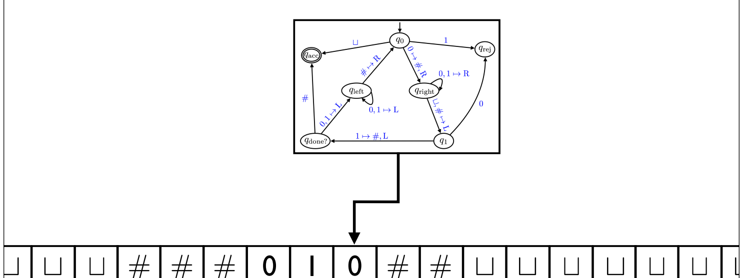


Turing machine that decides 0^n1^n



Input: 00001011

Turing machine that decides 0^n1^n



Input: 00001011 **Decision**: reject

Some TM subroutines and tricks

- Move right (or left) until first $\sqcup$ encountered
- Shift entire input string one cell to the right
- Convert input from

  $x_1 x_2 x_3 \dots x_n$ to $\sqcup x_1 \sqcup x_2 \sqcup x_3 \dots \sqcup x_n$
- Simulate a big Γ by just $\{0, 1, \sqcup\}$
- "Mark" cells. If $\Gamma = \{0, 1, \sqcup\}$, extend it to

  $\Gamma = \{0, 1, 0^\bullet, 1^\bullet, \sqcup\}$
- Copy a stretch of tape between two marked cells into another marked section of the tape

Some TM subroutines and tricks

- Implement basic string and arithmetic operations
- Simulate a TM with 2 tapes and heads
- Implement basic data structures
- Simulate "random access memory"

  ⋮
- Simulate assembly language

  You could prove this rigorously if you wanted to.

So what we want is:

A totally minimal (TM) programming language such that

- it can simulate simple bytecode (and therefore Python, C, Java, SML, etc…)
- it is simple to define and reason about completely mathematically rigorously

A note

You could describe a TM in 3 ways:

Low level description

Medium level description

High level description

Important Question

Is TM the right definition?

Is there a reasonable definition of "algorithm"
that can compute more languages than TM-decidable ones?