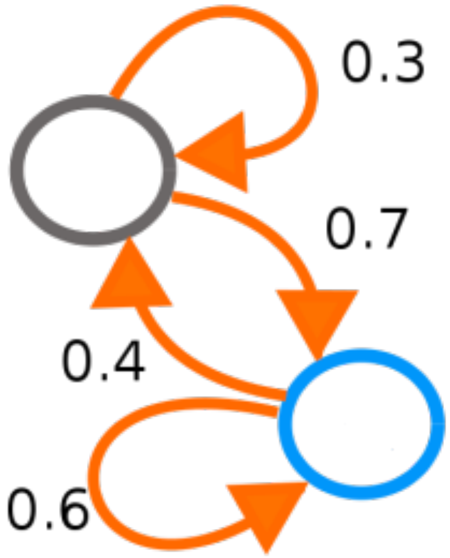
# 15-252
# More Great Ideas in Theoretical Computer Science

## Markov Chains

*April 27th, 2018*




# **Markov Chain**

Andrey Markov (1856 - 1922)

Russian mathematician.

Famous for his work on random processes.

($\Pr[X \geq c \cdot \mathbf{E}[X]] \leq 1/c$ is Markov's Inequality.)

A model for the evolution of a random system.

*The future is independent of the past, given the present.*

# Cool things about Markov Chains

- It is a very general and natural model.

  Applications in:

  computer science, mathematics, biology, physics, chemistry, economics, psychology, music, baseball,...

- The model is simple and neat.

- Cilantro

# The plan

Motivating examples and applications

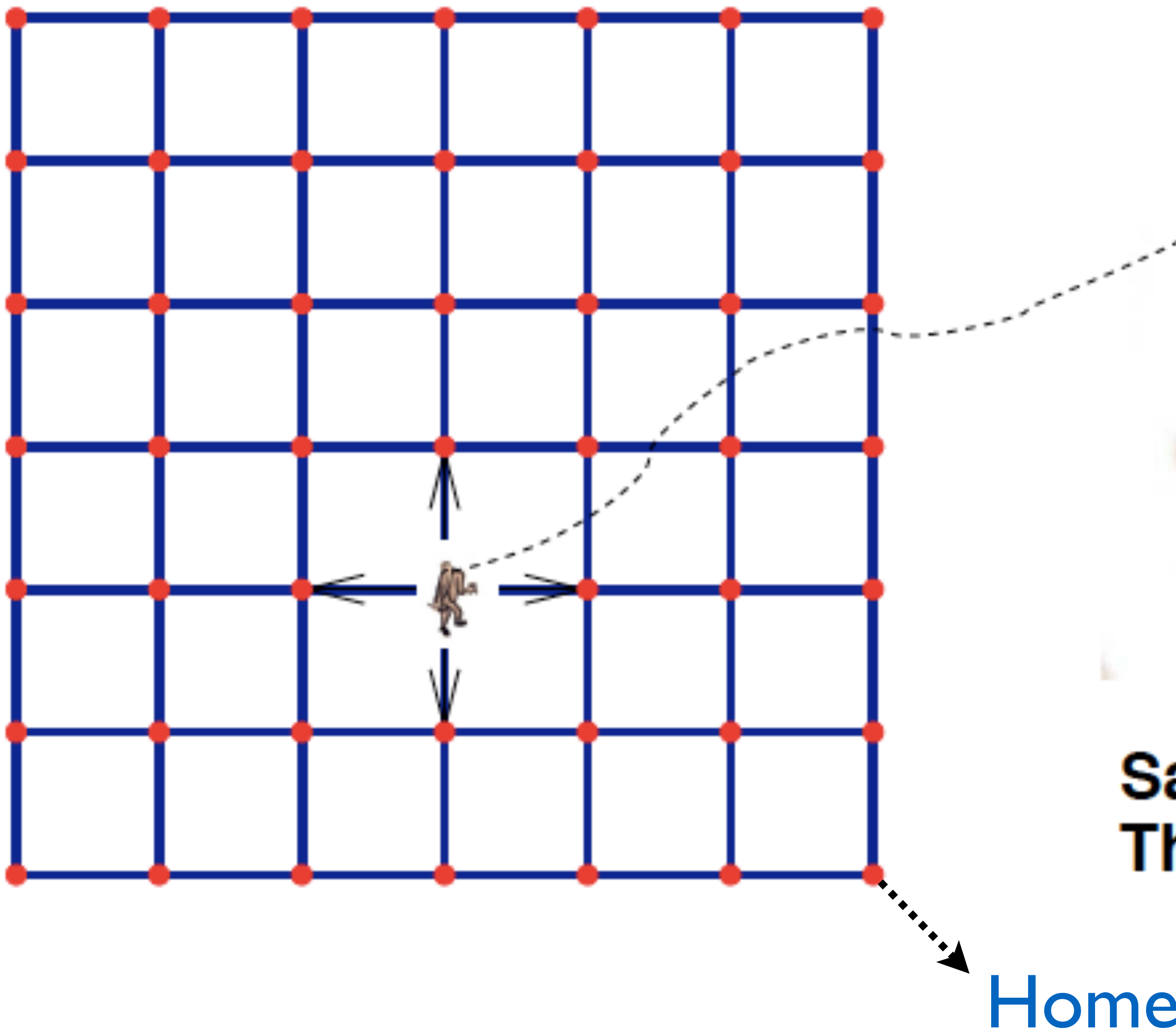
Basic mathematical representation and properties

A bit more on applications

*The future is independent of the past, given the present.*

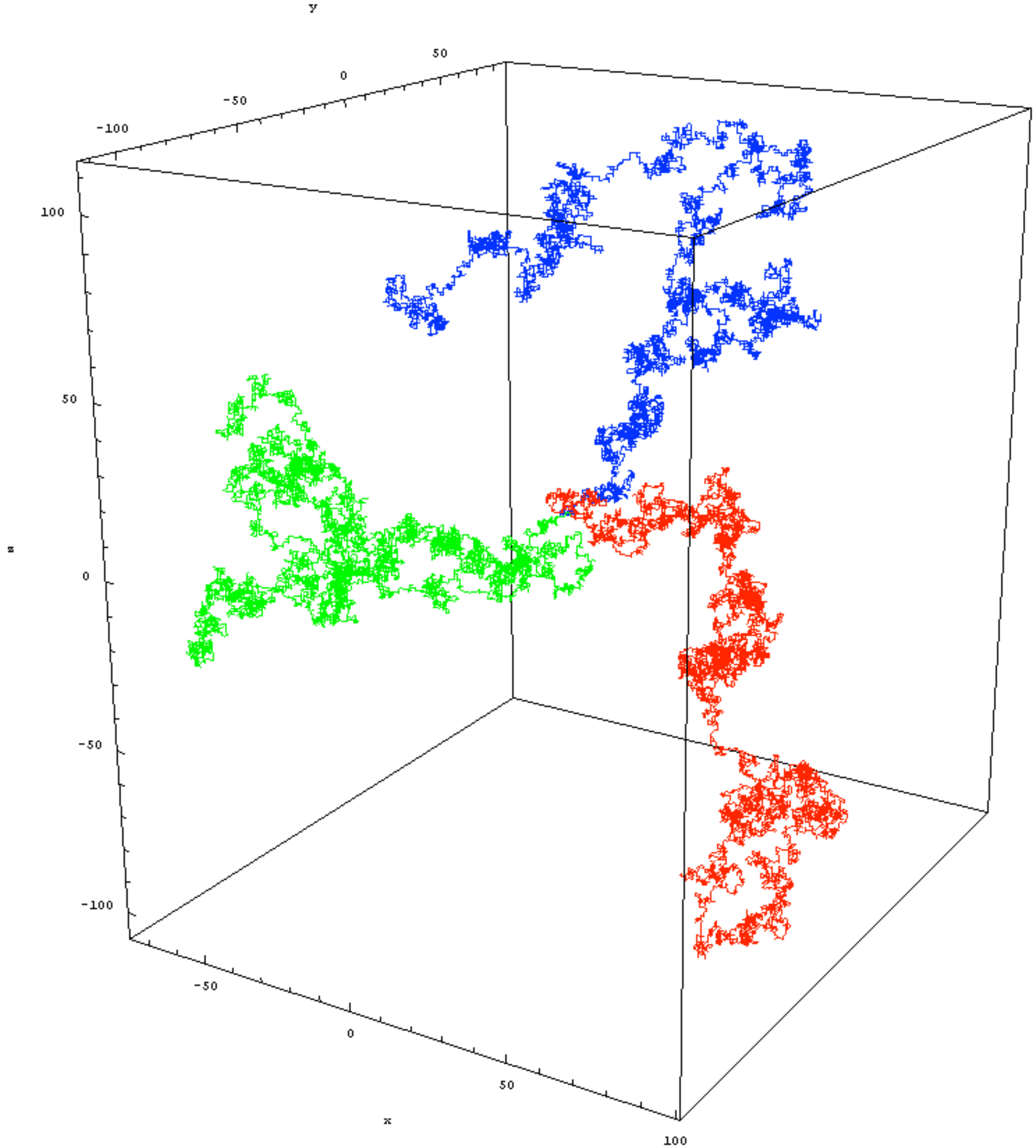
# Some Examples of Markov Chains

# Example:  Drunkard Walk




**Salvador Dali (1922)**
**The Drunkard**

# Example: Diffusion Process



# Example: Weather

A very(!!) simplified model for the weather.

Probabilities on a daily basis:

Pr[sunny to rainy] = 0.1

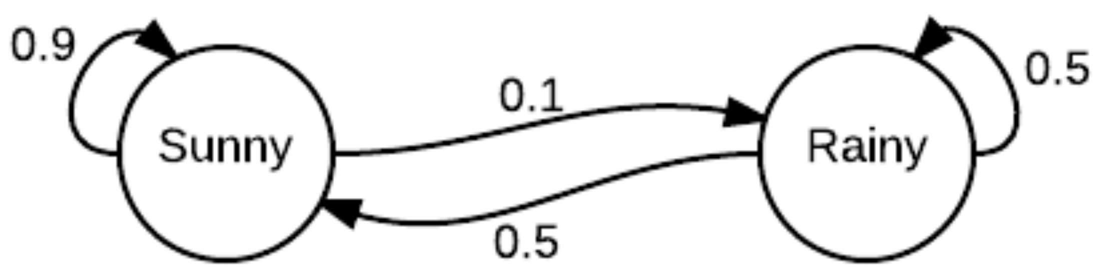
Pr[sunny to sunny] = 0.9

Pr[rainy to rainy] = 0.5

Pr[rainy to sunny] = 0.5

**S = sunny**
**R = rainy**

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{S} & \mathbf{R} \end{array} \\ \begin{array}{c} \mathbf{S} \\ \mathbf{R} \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$$




Encode more information about current state for a more accurate model.

# Example: Life Insurance

Goal of life insurance company:

figure out how much to charge the clients.

Find a model for how long a client will live.

Probabilistic model of health on a monthly basis:

Pr[healthy to sick] = 0.3
Pr[sick to healthy] = 0.8
Pr[sick to death] = 0.1
Pr[healthy to death] = 0.01
Pr[healthy to healthy] = 0.69
Pr[sick to sick] = 0.1
Pr[death to death] = 1

# Example: Life Insurance

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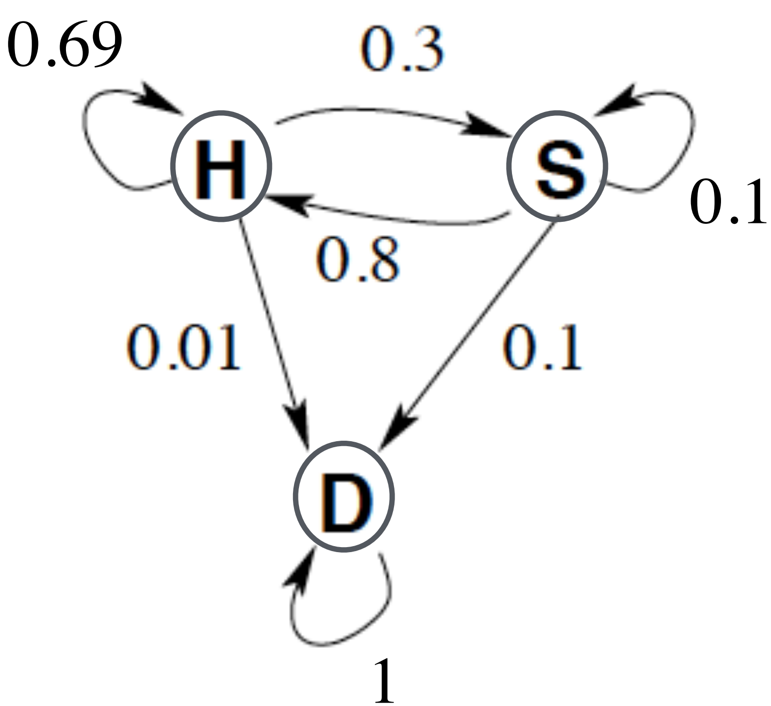
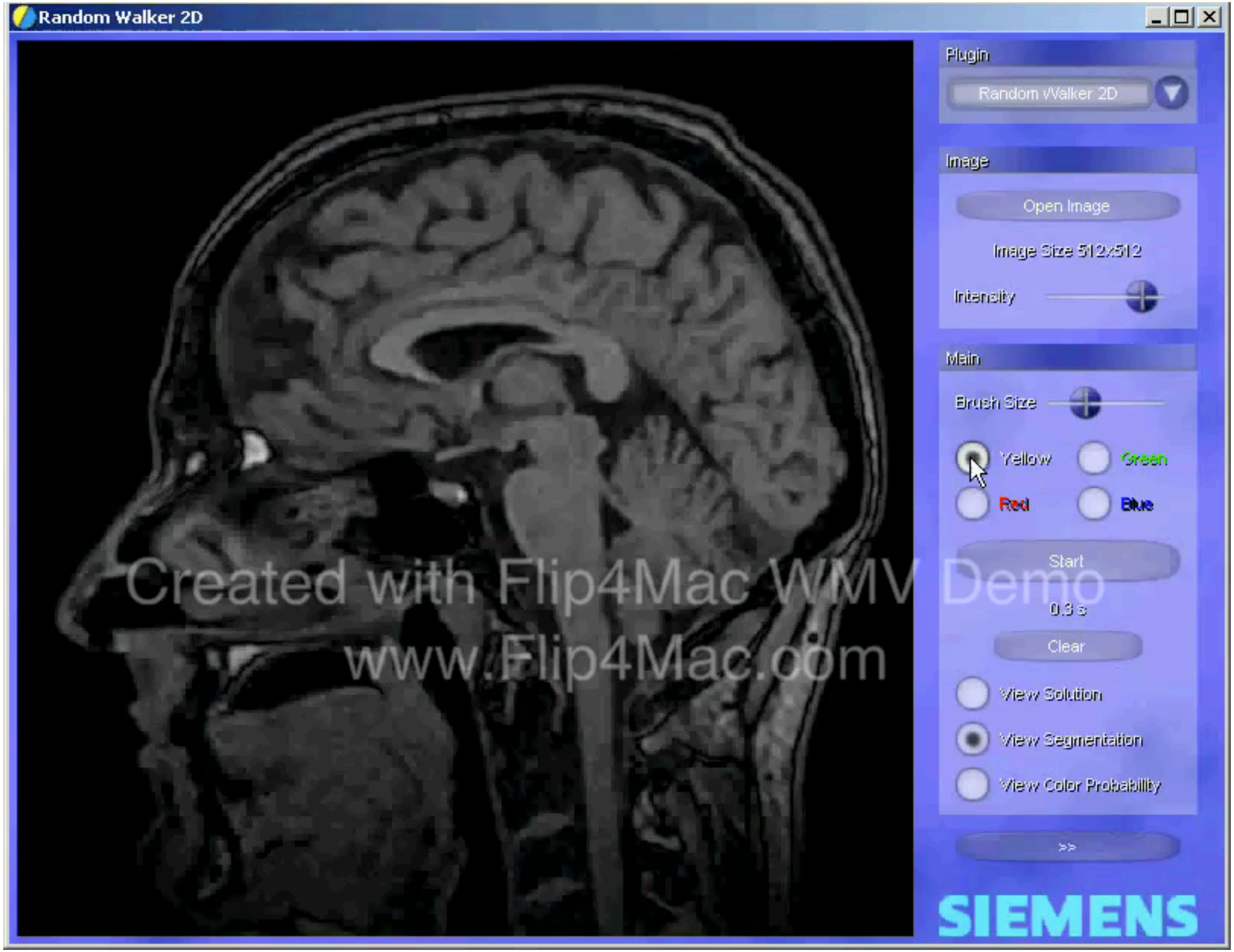



| | H | S | D |
|---|---|---|---|
| H | 0.69 | 0.3 | 0.01 |
| S | 0.8 | 0.1 | 0.1 |
| D | 0 | 0 | 1 |

# Some Applications of Markov Models

# Application: Algorithmic Music Composition

# Application: Image Segmentation



# Application: Automatic Text Generation

Random text generated by a computer
(putting random words together):

"While at a conference a few weeks back, I spent an interesting evening with a grain of salt."

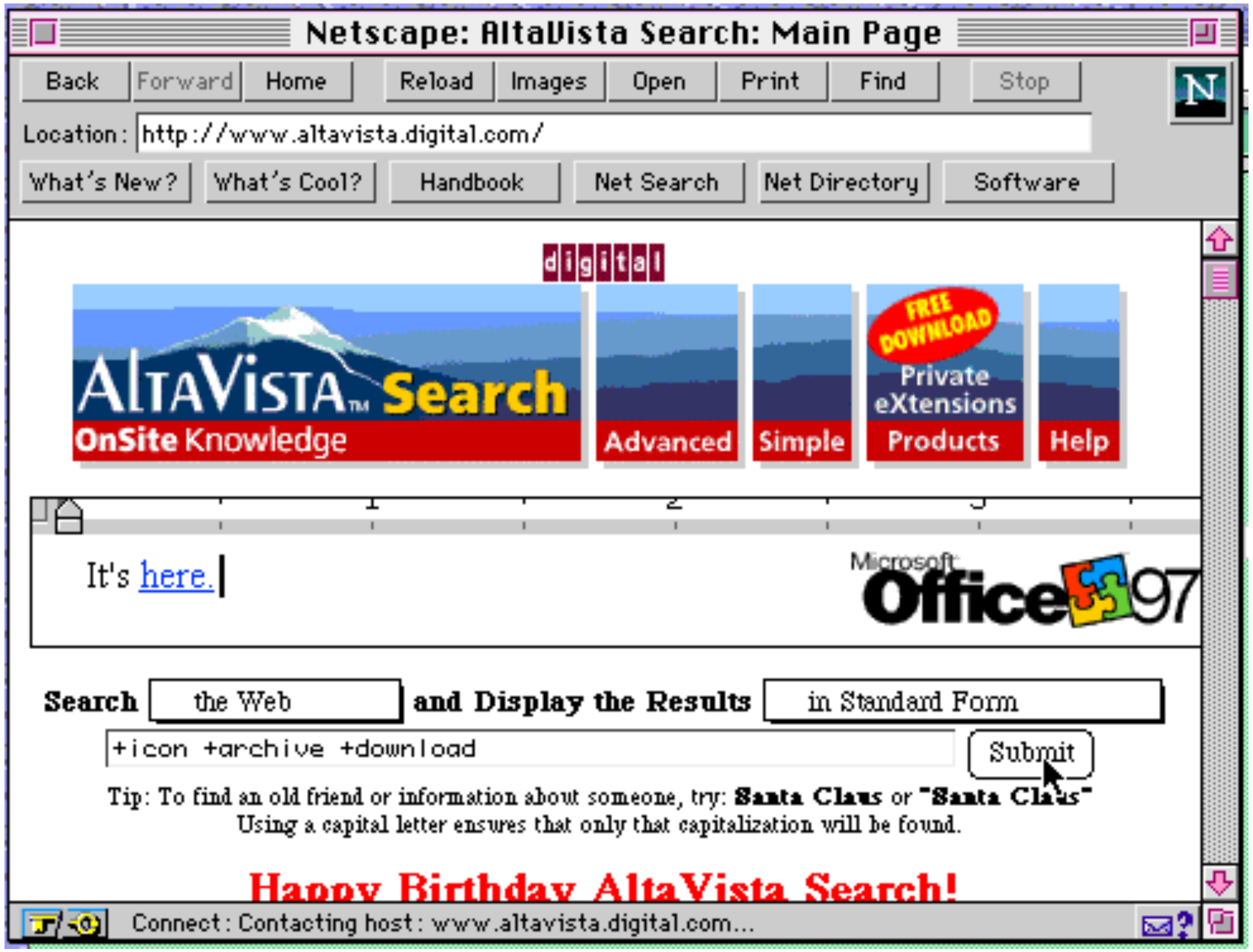
Google: Mark V Shaney

# Application: Speech Recognition

Speech recognition software programs use Markov models to listen to the sound of your voice and convert it into text.

# Application: Google PageRank

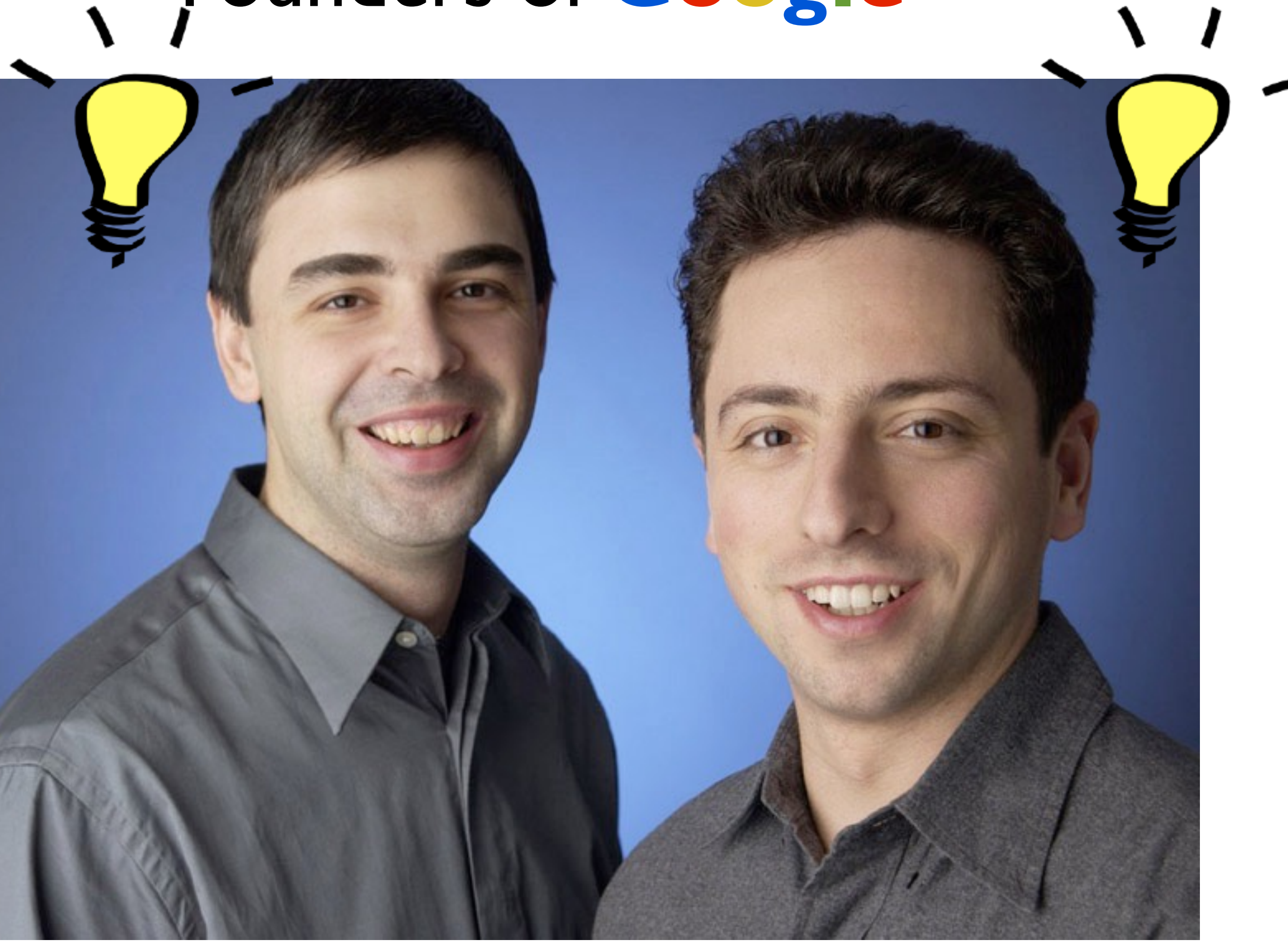
**1997**: Web search was horrible




Sorts webpages by number of occurrences of keyword(s).

# Application: Google PageRank

Founders of Google



Larry Page Sergey Brin

$40Billionaires

# Application: Google PageRank

Jon Kleinberg

Nevanlinna Prize

# Application: Google PageRank

How does Google order the webpages displayed after a search?

**2 important factors:**

- Relevance of the page.

- Reputation of the page.

↓

*The number and reputation of links pointing to that page.*

Reputation is measured using PageRank.

PageRank is calculated using a Markov Chain.

# The plan

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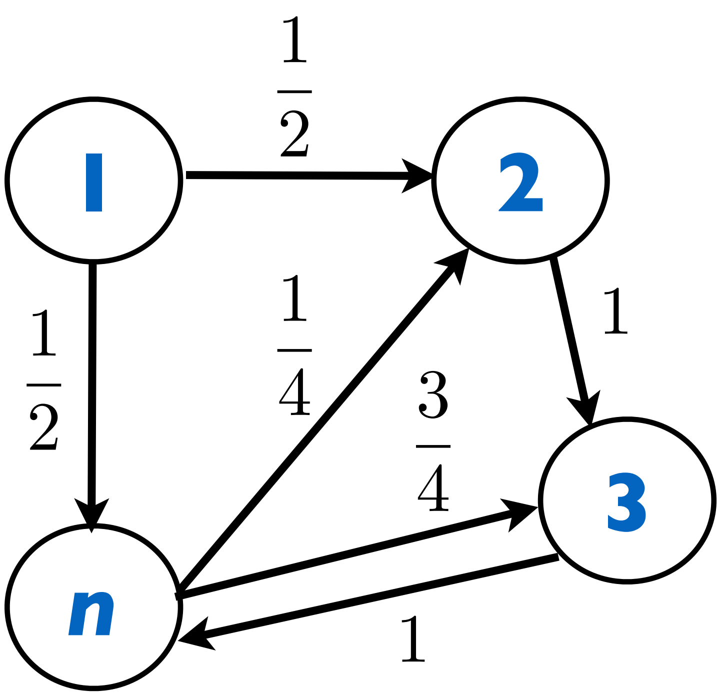
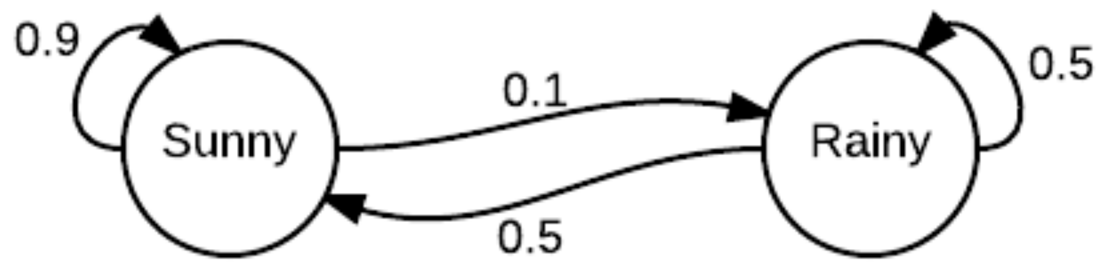
# The Setting

There is a system with *n* possible states/values {1, 2, …, n}.

At each time step, the state changes probabilistically.




*Memoryless*

The next state only depends on the current state.

*Evolution of the system:* random walk on the graph.

# The Definition

A **Markov Chain** is a digraph with $V = \{1, 2, \ldots, n\}$ such that:

- Each edge is labeled with a value in $(0, 1]$ (a probability).
  self-loops allowed

- At each vertex, the probabilities on outgoing edges sum to $1$.

(We usually assume the graph is strongly connected.

i.e. there is a directed path from ***i*** to ***j*** for any ***i*** and ***j*****.**)

The vertices of the graph are called states.

The edges are called transitions.

The label of an edge is a transition probability.

# Notation

**Given some Markov Chain with n states:**

Define

$\pi_t[i] =$ probability of being in state ***i*** after exactly **t** steps.

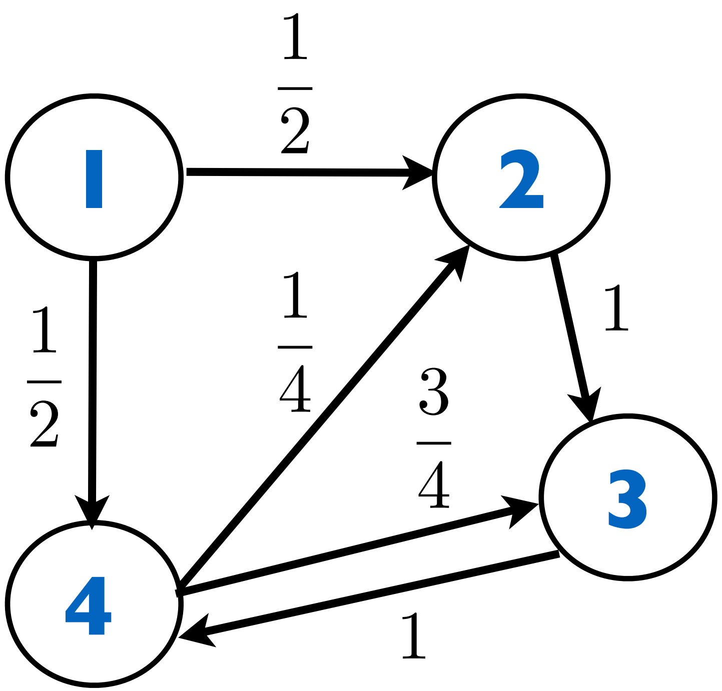
**1 2 n**

$$\pi_t = [p_1 \; p_2 \; \cdots p_n] \qquad \sum_i p_i = 1$$

Note that someone has to provide $\pi_0$.

Once this is known, we get the distributions $\pi_1, \pi_2, \ldots$

# Notation




$$\begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \end{array}$$

Transition Matrix

A Markov Chain with **n** states
can be characterized by the **n** x **n** **transition matrix** $K$

$$\forall i, j \in \{1, 2, \ldots, n\} \quad K[i, j] = \Pr[i \to j \text{ in one step}]$$

Note: rows of $K$ sum to 1.

# Some Fundamental and Natural Questions

What is the probability of being in state ***i*** after ***t*** steps (given some initial state)?

$$\pi_t[i] = ?$$

What is the expected time of reaching state ***i*** when starting at state ***j*** ?

What is the expected time of having visited every state (given some initial state)?

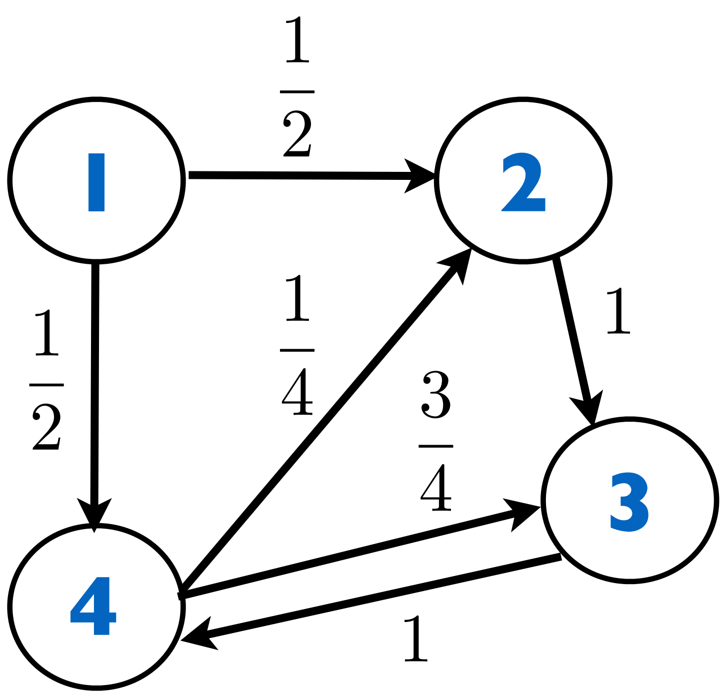
⋮

How do you answer such questions?

# Mathematical representation of the evolution

Suppose we start at state 1 and let the system evolve.

How can we mathematically represent the evolution?

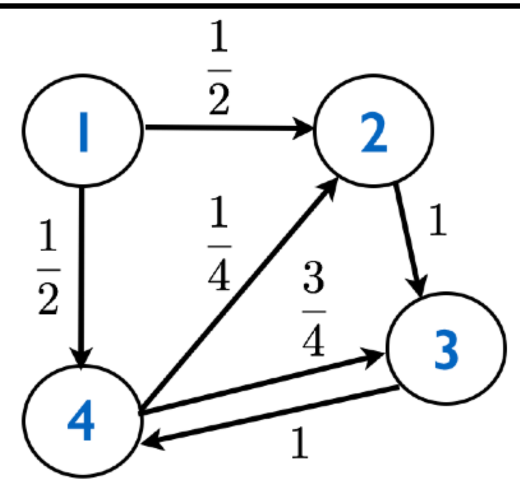



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \end{array}$$

$$\pi_0 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

What is $\pi_1$? By inspection, $\pi_1 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \end{array}$.

# Mathematical representation of the evolution



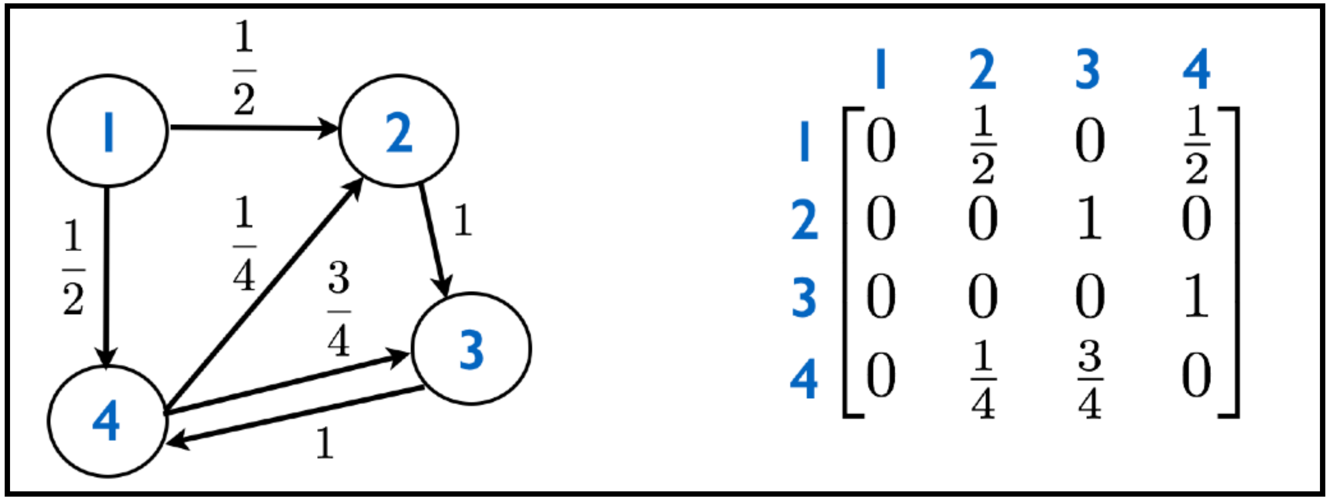

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{array}$$

The probability of states after 1 step:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{\pi_0} \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}}_{K} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}}_{\pi_1}$$

the new state (probabilistic)

# Mathematical representation of the evolution

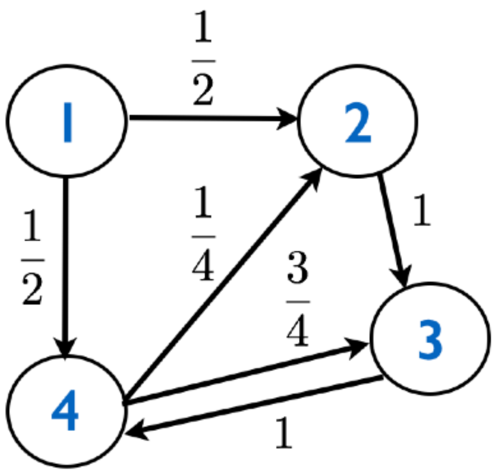



The probability of states after 2 steps:

$$\underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}}_{\pi_1} \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}}_{K} = \underbrace{\begin{bmatrix} 0 & \frac{1}{8} & \frac{7}{8} & 0 \end{bmatrix}}_{\pi_2}$$

the new state (probabilistic)

# Mathematical representation of the evolution




$$
\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \end{array}
$$

$$\pi_1 = \pi_0 \cdot K$$

$$\pi_2 = \pi_1 \cdot K$$

So $$\pi_2 = (\pi_0 \cdot K) \cdot K$$
$$= \pi_0 \cdot K^2$$

# Mathematical representation of the evolution

**In general:**

If the initial probabilistic state is $\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} = \pi_0$

$p_i =$ probability of being in state *i*,

$p_1 + p_2 + \cdots + p_n = 1$ ,

after ***t*** steps, the probabilistic state is:

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} \text{Transition} \\ \text{Matrix} \end{bmatrix}^{t} = \pi_t$$

# Remarkable Property of Markov Chains

**What happens in the long run?**

i.e., can we say anything about $\pi_t$ for large $t$ ?

Suppose the Markov chain is "aperiodic".

Then, as the system evolves, the probabilistic state ***converges*** to a limiting probabilistic state.

As $t \to \infty$, for any $\pi_0 = [p_1 \quad p_2 \quad \cdots \quad p_n]$ :

$$[p_1 \quad p_2 \quad \cdots \quad p_n] \begin{bmatrix} \text{Transition} \\ \text{Matrix} \end{bmatrix}^t \longrightarrow \pi$$

# Remarkable Property of Markov Chains

In other words:

$$\pi_t \to \pi \quad \text{as} \quad t \to \infty\,.$$

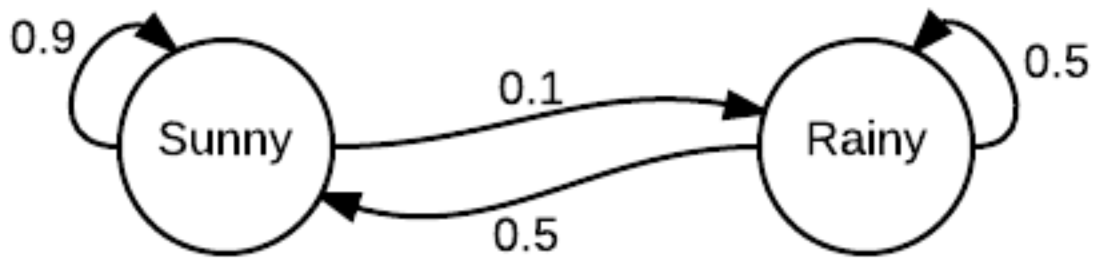
Note:

$$\pi \begin{bmatrix} \text{Transition} \\ \text{Matrix} \\ \\ \end{bmatrix} = \pi$$

↓ stationary/invariant distribution

This $\pi$ is unique.

# Remarkable Property of Markov Chains




Stationary distribution is $\begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix}$.

$$\begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

*In the long run, it is Sunny **5/6** of the time,*
*it is Rainy **1/6** of the time.*

# Remarkable Property of Markov Chains

How did I find the stationary distribution?

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}^2 = \begin{bmatrix} 0.86 & 0.14 \\ 0.7 & 0.3 \end{bmatrix}$$

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}^4 = \begin{bmatrix} 0.8376 & 0.1624 \\ 0.812 & 0.188 \end{bmatrix}$$

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}^8 = \begin{bmatrix} 0.833443 & 0.166557 \\ 0.832787 & 0.167213 \end{bmatrix}$$

**Exercise**: Why do the rows converge to $\pi$ ?

# Things to remember

Markov Chains can be characterized by the transition matrix $K$.

$$K[i,j] = \Pr[i \to j \text{ in one step}]$$

What is the probability of being in state ***i*** after *t* steps?

$$\pi_t = \pi_0 \cdot K^t \qquad \pi_t[i] = (\pi_0 \cdot K^t)[i]$$

# Things to remember

**Theorem** (Fundamental Theorem of Markov Chains):

Consider a Markov chain that is strongly connected and aperiodic.

- There is a unique invariant/stationary distriution $\pi$ such that

$$\pi = \pi K.$$

- For any initial distribution $\pi_0$ ,

$$\lim_{t \to \infty} \pi_0 K^t = \pi$$

- Let $T_{ij}$ be the number of steps it takes to reach state $j$ provided we start at state $i$. Then,

$$\mathbf{E}[T_{ii}] = \frac{1}{\pi[i]}.$$

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# How are Markov Chains applied ?

**2 common types of applications:**

1. Build a Markov chain as a statistical model of a real-world process.

   Use the Markov chain to simulate the process.

   e.g. text generation, music composition.

2. Use a measure associated with a Markov chain to approximate a quantity of interest.

   e.g. Google PageRank, image segmentation

# Automatic Text Generation

Generate a superficially real-looking text given a sample document.

**Idea:**

From the sample document, create a Markov chain.

Use a random walk on the Markov chain to generate text.

**Example:**

Collect speeches of Obama, create a Markov chain.

Use a random walk to generate new speeches.

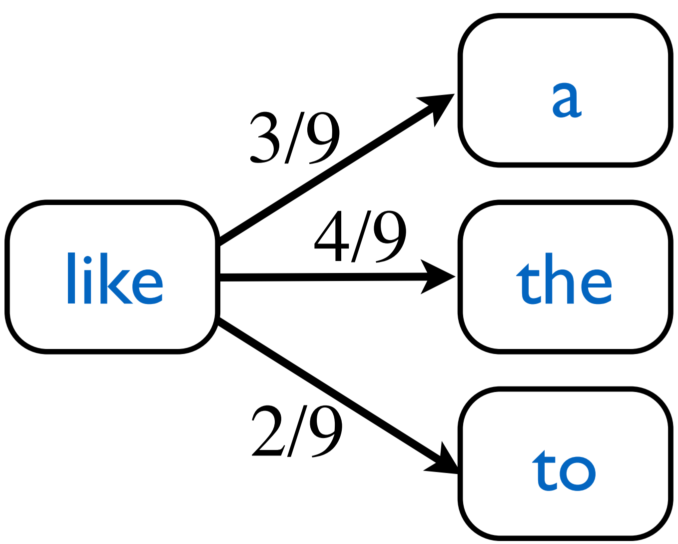
# Automatic Text Generation

## The Markov Chain:

**1**. For each word in the document, create a node/state.

**2**. Put an edge word1 ---> word2
if there is a sentence in which word2 comes after word1.

**3**. Edge probabilities reflect frequency of the pair of words.




| | |
|---|---|
| like a | 3 times |
| like the | 4 times |
| like to | 2 times |

# Automatic Text Generation

"I jumped up. I don't know what's going on so I am coming down with a road to opportunity. I believe we can agree on or do about the major challenges facing our country."

# Automatic Text Generation

**Another use:**

Build a Markov chain based on speeches of Obama.

Build a Markov chain based on speeches of Bush.

Given a new quote, can predict if it is by
Obama or Bush.

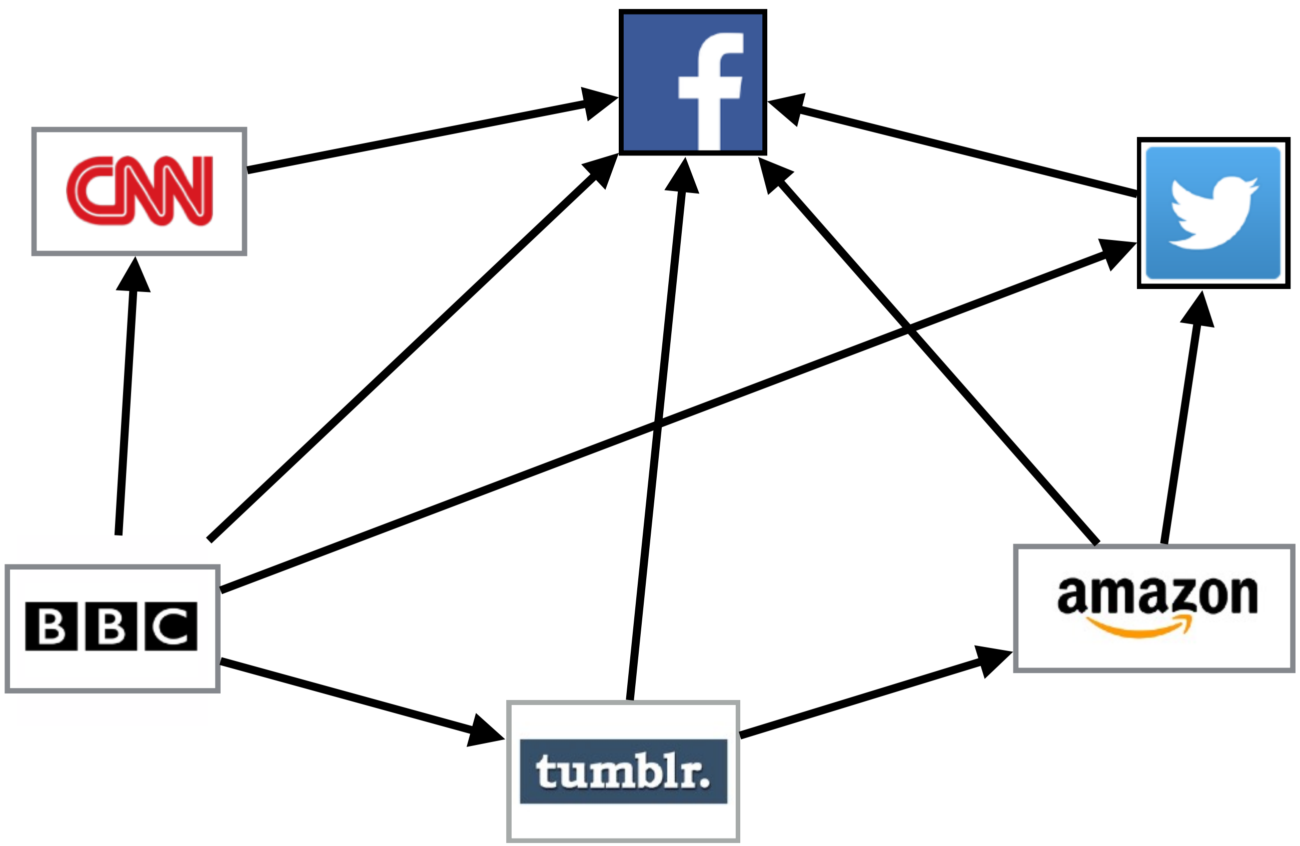
(by testing which Markov model the quote fits best)

# Google PageRank

PageRank is a measure of reputation:

The number and reputation of links pointing to you.

**The Markov Chain:**



# Google PageRank

PageRank is a measure of reputation:

The number and reputation of links pointing to you.

## The Markov Chain:

**1**. Every webpage is a node/state.

**2**. Each hyperlink is an edge:

if webpage **A** has a link to webpage **B**, **A** ---> **B**

**3a**. If **A** has ***m*** outgoing edges, each gets label **1/*m***.

**3b**. If **A** has no outgoing edges, put edge **A** ---> **B** $\forall$ **B** (jump to a random page)

# Google PageRank

A little tweak:

Random surfer jumps to a random page with 15% prob.

**Stationary distribution:**

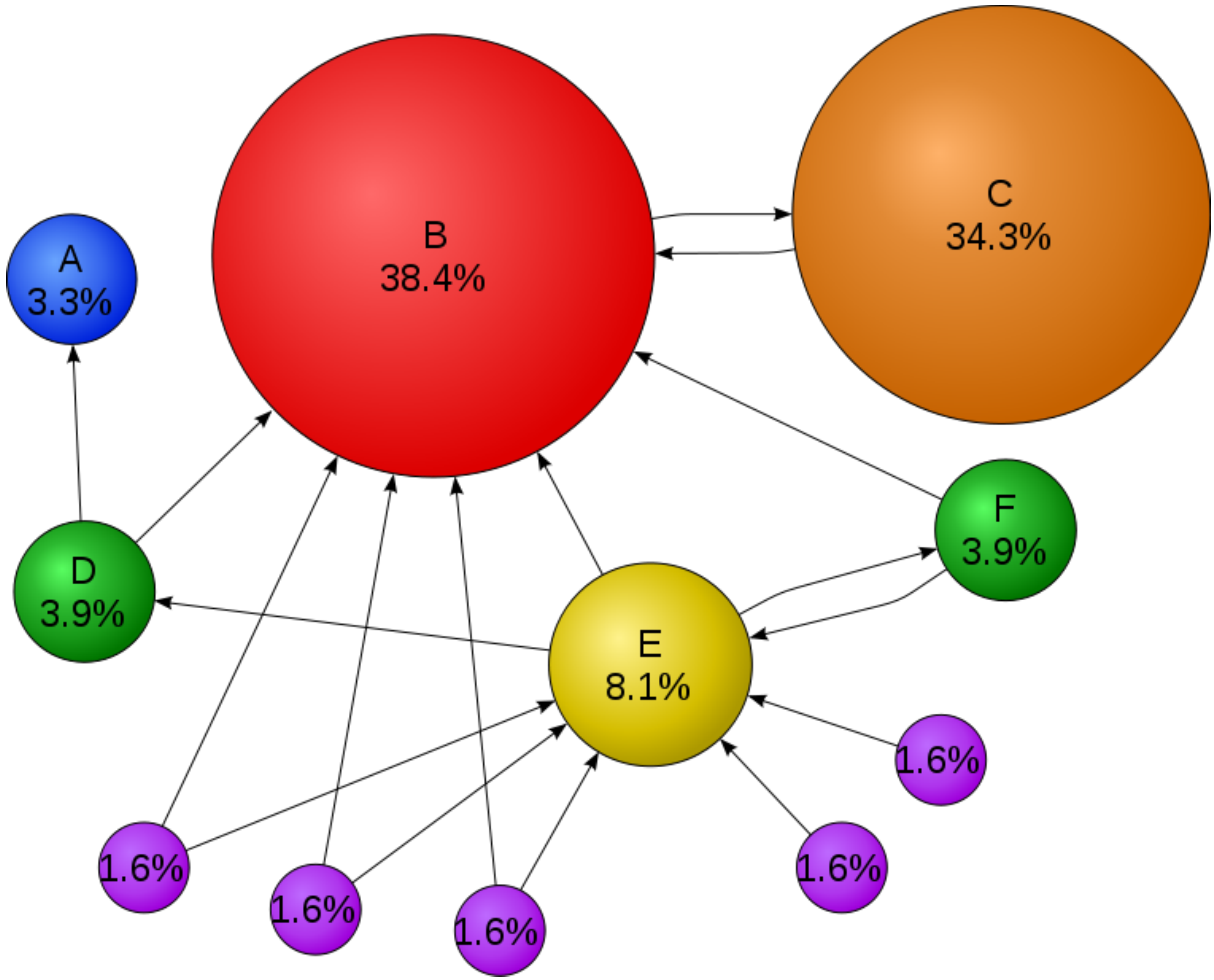
probability of being at webpage **A** in the long run

PageRank of webpage **A**

**=**

The stationary probability of **A**

# Google PageRank



# Google PageRank

**Google**:

*"PageRank continues to be the heart of our software."*

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